Analytical solutions of fractal and anomalous diffusion models for pressure and rate transient analysis

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ABSTRACT

In this study, we examine the modeling of the naturally fractured reservoirs based on the fractal, Metzler, and Raghavan anomalous diffusion models which are different from Darcy flux law. The governing equations of these models are presented for a vertical well producing at either a constant rate (CR) or constant bottomhole pressure (CBHP) production in an infinite-acting reservoir. New analytical and early- and late-time asymptotic approximate solutions were derived for the finite-wellbore problem for the Raghavan anomalous model. The approximate solutions identify the fractal and anomalous diffusion parameters that influence behaviors of the early- and late-time pressure and rate transient data. The method of Laplace transformation is used to find the solutions in dimensionless forms which are then inverted to the real-time domain using a numerical inversion algorithm. In addition, we compare the solutions for the fractal, Metzler anomalous, and Raghavan anomalous models and delineate the similarities and differences among these solutions. We show the effect of fractal parameters on diffusion models, the relationship between the fractal model, and Raghavan and Metzler anomalous models, in terms of dimensionless pressure and dimensionless pressure derivative responses in the reservoir. We show that the “dimensionless” pressure defined by Raghavan is not a dimensionless quantity unless the Euclidean dimension \( d = 2 \) (or for a fractal structure with the mass fractal dimension \( d_f = 2 \)). To the best of our knowledge, in the literature, there exists no study providing a detailed review and comparing the applicability of each diffusion model on real field well-test data as well as providing new analytical solutions and early- and late-time asymptotic approximate solutions for the anomalous diffusion for both the CR and CBHP cases for the finite-wellbore problem. Hence, this study fills this gap. In addition, we apply the three diffusion models to an interference well test data from the Kizildere geothermal reservoir in Turkey by using the ensemble smoother with multiple data assimilation (ES-MDA) method. The results show that the fractal model in terms of the goodness of fit (root-mean-square error – RMSE) best matches the field test data.

1. Introduction

Reservoir modeling is crucial and essential to optimize the production life of a hydrothermal, geothermal, oil, or gas reservoir in the subsurface energy industry. The well-test analysis is frequently applied after the drilling process is completed or some of the fluid is produced for the estimation of reservoir hydraulic parameters, such as average reservoir pressure, skin factor, permeability, porosity, etc. However, the analysis and interpretation of the well-test data from dynamic pressure and/or rate data may not lead to unique results. That is, there may be more than one probable answer due to errors in pressure/rate data measurements, uncertainty in the underlying physical model, and nonlinear relationship between the response and the model parameters even if the model chosen for the data may be correct (Kuchuk et al., 2010).

Up to the early 50s, reservoir modeling was based on the homogeneous reservoir. Hence, the perfect history matching could not be obtained in the fractured (heterogeneous) reservoirs. In 1963, Warren and Root modeled the naturally fractured reservoirs represented by the double porosity model (fracture and matrix). That study affects the fractal and anomalous diffusion model fundamentally. Gefen et al. (1983) depicted the mean square displacement (MSD) as

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where $d_f$ is called the Fickian value, and equal to $2 + \theta$, where $\theta$ is called the conductivity index (topology of the network), and $Z$ is a diffusion coefficient. Before their study, the Fickian value is accepted as 2. However, this situation is acceptable if the reservoir is homogeneous. Gefen et al. (1983) asserted that the conductivity index may not be equal to 2 in the case of anomalous diffusion. In other words, the topology of the network may be smaller or bigger than zero in the fractal network $\eta \sim r^{-\alpha}$,

$$\eta \sim r^{-\alpha},$$

where $\eta$ is called the fractal network. Eq. (2) shows that the diffusivity decreases with a distance if the conductivity index is bigger than zero. This situation is called the anomalous diffusion effect. To put it another way, if the fractal network consists of a high tortuous path ($\theta > 0$), the diffusion slower than normal diffusion is called subdiffusion. On the other hand, when the fractal network is made up of a less tortuous path, the diffusion process is called superdiffusion (Balescu, 1998).

The analytical solutions for diffusion on fractal objects were studied by O’Shaughnessy and Procaccia (1985a, 1985b). The main purpose of their study was to understand the generalization of the diffusion equation for Euclidean lattices if the lattices are non-integer dimensions. Consequently, they justified that scaling of conductivity and anomalous diffusion can be derived from the topology of the network ($\theta$).

In the early 90s, the concept of a naturally fractured reservoir (NFR) aroused great interest in the oil industry. Chang and Yortsos (1990) were the first to propose to use of fracts for describing the pressure transient behavior of fractured reservoirs or in general for a rock where its fabric creates transport of fluids that cannot be defined by conventional conservation laws. They expanded the O’Shaughnessy and Procaccia (1985a, 1985b) by developing the fracture network parameter applying fractal distribution. Their study is based on two parts: pressure transients without matrix participation and pressure transients with matrix participation since their fractal model consisted of disconnected matrix blocks; however, these matrix blocks may contact the fracture network. Chang and Yortsos (1990) derived a solution considering a finite wellbore condition and defined the governing equation which is based on single-phase flow and the fractal object embedded into a Euclidean matrix, that is,

$$\frac{1}{m} \frac{\partial \tilde{p}(r, t)}{\partial t} = \frac{1}{\eta^{\beta-1}} \frac{\partial}{\partial r} \left( \eta^{\beta} \frac{\partial \tilde{p}(r, t)}{\partial r} \right),$$

where $d_f$ is the fractal dimension, and $\beta = d_f - \theta - 1$, where $\theta$ is related to the so-called spectral exponent of fractal topology parameter, and $m$ is a local structural property of fractal network similar to the conventional permeability and expresses connectivity and flow conductance (Chang and Yortsos, 1990). Throughout we use the SI unit system, and the units of all symbols are given in the Nomenclature section.

Later, Beier (1990, 1994) improved the Chang and Yortsos solution by introducing porosity using fractal distribution in addition to permeability and applied it to real field data by assuming a line source well with radial symmetry. Furthermore, Beier (1990) noticed that the fractal dimension ($d_f$) should be 2 or slightly less in oil-in-place calculations. Acuna et al. (1995) applied the Chang and Yortsos model to tests conducted in the Geysers geothermal field and acquired reasonable outcomes. Consequently, they showed that change in wellbore pressure has a power-law relationship with time. Acuna and Yortsos (1995) emphasized the importance of the box-counting method which has been used to find the fractal properties of real networks. Flamenco-Lopez and Camacho-Velazquez (2003) applied the fractal model for a vertical well to determine the fractal parameters. They derived an analytical solution of transient and pseudo-steady-state flow periods for matrix blocks and fractal fracture contribution. In addition, they proved that it is impossible to identify four parameters of fractal parameters from a single pressure transient test. Onur et al. (2003) applied the fractal model without matrix participation to analyze the active and interference well test data obtained in Kızıldere geothermal field, Turkey. They found that the pressure behavior observed at the wells can be indicative of a fractal reservoir with a fractal dimension of $0.64 \leq d_f \leq 1.3$. Metzler et al. (1994) presented an anomalous diffusion-based model which is different from Chang and Yortsos’s model because Metzler et al. claimed that the anomalous model diffusivity equation should include a fractional derivative. In other words, they believed that the diffusion process in fractal reservoirs is history-dependent. Hence, the Chang-Yortsos fractal model may not explain the historical effect without the temporal fractional derivative. The new diffusion equation proposed by Metzler et al. and expressed by Camacho-Velázquez et al. (2008) is given by

$$\left( \frac{c \mu \eta^{\beta}}{m} \right) \frac{\partial \tilde{p}(r, t)}{\partial t} = \frac{1}{\eta^{\beta-1}} \frac{\partial}{\partial r} \left( \eta^{\beta} \frac{\partial \tilde{p}(r, t)}{\partial r} \right).$$

where $\alpha$ is the anomalous diffusion exponent. The partial derivative in the left-hand-side of Eq. (4a) is the fractional derivative which is interpreted in the Caputo (1967) sense. It should be noted that Eq. (4a) is obtained from the dimensionless form given by Eq. (13) of the Camacho-Velázquez et al. (2008) paper by using the dimensionless variables of pressure, time, and radial distance defined in that paper. Raghavan (2011) expresses Eq. (4a) in a slightly different way, given by

$$\frac{1}{\eta^{\beta-1}} \frac{\partial}{\partial t} \left( \eta^{\beta} \frac{\partial \tilde{p}(r, t)}{\partial r} \right).$$

where $\tilde{p}$ may be referred to as the diffusivity of the coefficient of anomalous diffusion. He defined it as $\tilde{p} = m/\left[ \Gamma(1 - \alpha) \Gamma(\alpha) \right]$. It has a dimension of $L^{\alpha-2} T^{-\alpha}$, $L$ represents length and $T$ represents time. It is worth noting that the Metzler model was used by Camacho-Velázquez et al. (2008) for studying the production rate decline under the CBHP production case and Park (2000) for studying the pressure transient behavior for the CR production case. Raghavan (2011) introduced a parameter denoted by $A_\alpha$ that is governed by the properties of the fractal structure. However, its existence is not explicitly discussed in the works of Metzler et al. Camacho et al., and Park et al. It has a dimension of $\tilde{p} L^{1-\alpha}$ and makes Eq. (4b) dimensionally consistent. Raghavan (2011) argues that the way that $A_\alpha$ is taken in the diffusivity equations used by Camacho et al. and Park et al. that is, $A_\alpha = (\mu \rho c_f)^{1-2}/m^{\alpha-1}$, has no basis for the fractal structure as it contains parameters such as wellbore radius which is not related to the fractal structure of the medium. Further discussion of the parameter $A_\alpha$ can be found in Raghavan (2011). However, as also pointed out by Acuna et al. (1995), in a fractal medium, all properties of any region of size $r$ are scale-dependent, following a power law. Hence, permeability and porosity in a fractal network follow a power law relationship and hence can be expressed in terms of any lower cutoff scale above which the fractal behavior, which could be the wellbore radius $r_w$ representing the sandface (see Appendix A of this paper). Another point to note is the inverse of the diffusion coefficient defined by Raghavan (2011), $\tilde{p}^{-1} = \left( \frac{\tilde{p}}{m} \right)^{1-\alpha} \Gamma(1 - \alpha) \Gamma(\alpha)$. Comparing Eqs. (4a) and (4b), we see that Raghavan’s version of the Metzler equation contains an additional constant $\Gamma(1 - \alpha)$. It is not clear to us if this constant should appear in the Mertzel model because the anomalous diffusion equation given by Camacho-Velázquez et al. (2008) does not contain this term. In this study, we will use the version of the Metzler et al. anomalous diffusion model as given by Camacho-Velázquez et al. (2008).

Camacho-Velázquez et al. (2008) compared the fractal model (OP) and Metzler anomalous model (MGN) for the NFR. They applied the fractal model and Metzler anomalous model on a vertical well in a closed reservoir and an infinite reservoir. The models are investigated based on a dimensionless variable in Laplace space for the constant wellbore pressure condition and the finite wellbore case. Camacho-Velázquez et al. (2012) studied the actual field cases to find mass fractal dimension.
Raghavan (2011) introduced a new anomalous diffusion model. The main starting point of his work is the continuity equation. Raghavan claimed that Chang and Yortsos and Metzler et al. cannot derive the continuity equation by using the conventional form of Darcy’s law because they utilized the convoluted form of the continuity equation. Because of this idea, Raghavan proposed a new flux law since he thought that he should change the flux law or conservation equation. He proposed the following diffusivity equation in the radial geometry:

\[
\frac{1}{r^{\mu-1} \phi_c} \frac{\partial}{\partial t} \left( r^{\mu-1} \phi_c \frac{\partial p(r,t)}{\partial r} \right) = \phi_c \frac{d^\alpha p(r,t)}{dr^\alpha},
\]

where \( \phi_c = k_o / \mu \) and \( \alpha < 1 \). Here, it is worth noting that \( k_o \) is not the conventional permeability, referred to as the phenomenological coefficient, and it has a dimension of \( L^{2-\alpha} T^{-1} \). He introduced new dimensionless time, pressure, and flow rate equations. Consequently, he derived analytical solutions for CBHP production for a line-source well and CBHP production for a finite radius well.

Raghavan (2012) extended his study by applying Caputo fractional operator (Caputo, 1967). The main idea of this study is to model multiple fractures in heterogeneous reservoirs. Then, Raghavan and Chen (2013a) derived asymptotic solutions in Laplace space for a single vertical fracture model. Analytical solutions are presented for the constant terminal rate case and the constant terminal pressure case. Raghavan and Chen (2013b) combined the analytical solution for transient linear flow under subdiffusion with a finite conductivity fracture. To put it another way, they introduced an analytical solution to describe the behavior of linear flow under transient conditions, in which subdiffusion is the dominant mechanism in the conductivity fractures. The authors showed that using the anomalous diffusion approach is beneficial for assessing fractal systems. Then, Chen and Raghavan (2015) changed a new flux equation that can be used for both subdiffusion (time-fractional derivative) and superdiffusion (spatial fractional derivative).

The Raghavan (1991) model has also been applied to other reservoir models. For example, Ozcan et al. (2014) applied to the trilinear flow, which is different from the discrete fracture model and dual-porosity media. Their results suggested that the trilinear anomalous diffusion model can be useful in estimating reservoir performance, pressure-transient, and rate-transient analysis. However, we note that the scaled pressures and times defined in their Appendix to write an anomalous diffusion equation are not dimensionless.

Albinali and Ozkan (2016a) utilized the trilinear flow design to perform flow modeling in fractured unconventional reservoirs using the basic concepts of the anomalous diffusion model suggested by Raghavan (1991). They studied transient flow and single-phase flow conditions and combined a subdiffusion flow model within natural fractures with time-fractional flux law and classical material balance. The model was tested in heterogeneous and complex reservoirs with various sensitivity analyses. Later, Albinali and Ozkan (2016b) developed an anomalous diffusion model based on the dual porosity idealization of stimulated reservoir volume (SRV) and applied it to heterogeneous nanoscale porous media. The authors made flow models with the anomalous diffusion model and normal diffusion approaches for Eagle Ford and Niobrara Shale Plays. They found that the anomalous diffusion model provided more information than the normal diffusion model. However, they recommend conducting more detailed studies using petrophysical parameters to confirm their results. In another paper by Albinali et al. (2016), they discussed the utility and construction of 1D analytical and numerical anomalous diffusion models for heterogeneous, nanoporous media, which is commonly encountered in oil and gas production from tight, unconventional reservoirs with fractured horizontal wells. We should note that the scaled pressure and time variables given as dimensionless variables in the appendices of Albinali et al. (2016), Albinali and Ozkan (2016a), and Albinali and Ozkan (2016b) are not dimensionless.

Raghavan et al. (2016) studied a linear problem with anomalous diffusion (superdiffusion) and obtained a solution for the flux condition at the interface using Laplace transformation. Their study showed that it is easy to detect superdiffusion in hydraulic fracture space because of smaller interference times and practical responses, but there are no significant differences between purely superdiffusive and classical diffusion for long-time declines rate. Later, Raghavan and Chen (2017) examined how the heterogeneous geology of fractured rock can impact the flow in shale reservoirs. They created a model that accounts for subdiffusion as the transport mechanism and includes a horizontal well with multiple hydraulic fractures in a rectangular formation. They obtained a numerical solution for the model and presented asymptotic solutions that confirmed the expected characteristics. They also presented solutions for single and multiple fractures and defined long-term behaviors for production at constant pressure. Raghavan and Chen (2018) analyzed the production data from wells in Wolfcamp shale to identify heterogeneities in the fracture and matrix systems using pressure and rate time data. They assumed that subdiffusion dominates the flow in both fracture and matrix systems. Their study found that the heterogeneities in the fracture and matrix systems affect the dynamic performance of the reservoir and that matrix supports long-time productivity, contrary to previous literature. The study suggests that it is possible to obtain meaningful measures of heterogeneity in both the fracture and matrix systems in shale reservoirs.

Munoz Vargas et al. (2018) described a new methodology for identifying, validating, and analyzing naturally fractured reservoirs (NFR) using fractal parameters, such as fractal dimension and connectivity index. The methodology involves analyzing well pressure transient response and well logs data to calculate the fractal parameters, as well as permeability and skin factor.

Alcántara-López et al. (2022) discuss the challenges in modeling fluid flow in naturally fractured reservoirs, which do not always follow Darcy’s law due to the complexity of the fracture network. To capture the sub-diffusion process, various tools such as fractal geometry and fractional calculus have been implemented. The authors suggest a numerically solved spatial fractional Darcy’s equation for infinite naturally fractured reservoirs that preserves dimensional balance.

Romero and Camacho-Velázquez (2022) described a 3D numerical model to study gas flow in hydraulically fractured horizontal wells in anisotropic, heterogeneous naturally fractured reservoirs with triple porosity. The model incorporates geomechanics effects, anisotropy, anomalous diffusion, slip and viscous flow, Knudsen diffusion, kerogen adsorption/desorption, and fractality effects. Anisotropic and heterogeneous fractal distributions around each of the hydraulic fractures serve as a representation of the original natural microfracture network’s improved petrophysical features. Analytical solutions are used to validate the model’s accuracy, and the results show that the production behavior is causally related to the anisotropic fractal exponents and the order of the fractional derivatives, which stand for the density and connectivity of natural fractures within the SRV as well as the degree of anomalous diffusion. The suggested model illustrates the viability of combining all these attributes for reservoir simulation to produce more realistic production scenarios.

As can be seen from the extensive literature review given above, many studies have considered various fractal and anomalous diffusion models to understand the pressure and rate transient behavior from disordered porous media such as naturally fractured conventional and unconventional reservoirs and well systems. However, above, there exist a confusing understanding and definitions of the variables used in these three models appearing in the literature. In this work, our objective is to provide a clear comparison of the fractal model of Chang and Yortsos (1990) and the anomalous diffusion models of Metzler et al. (1994) and Raghavan (2011) by considering the 1D radial flow of slightly compressible fluid toward a fully penetrating vertical well producing in an infinite-acting fractured medium without a matrix contribution. It should be noted that there is no single study that

(d\(r\)), random walk dynamic exponent (or Fickian diffusion value, \(d_r\)), porosity, and permeability by applying the Metzler anomalous method.
compares these three models. So, we aim to fill this gap with this study, though we do this comparison on a relatively simple reservoir and well system. We note that Raghavan (2011) presented a work that compares these three methods on a theoretical basis. However, we show that his dimensionless variables assume that the Euclidean dimension $d$ is equal to 2. Moreover, Raghavan (2011) did not present the analytical solutions for a finite-wellbore model based on his anomalous diffusion approach. He assumes a line-source wellbore model. In this paper, not only do we present the analytical solutions for the finite-wellbore CR and CBHP production cases but also its asymptotic approximations at early and late times that apply for all values of $d$ and $\theta$ such that $d \leq \theta + 1$ or $d > \theta + 1$. It is also worth noting that Raghavan (1990)’s late-time asymptotic approximate solutions for the CR and CBHP are only valid for the case $d \leq \theta + 1$.

In summary, this paper makes a significant contribution to the field of pressure and rate transient analysis by providing a comprehensive comparison of the analytical solutions of fractal and anomalous diffusion models by examining the strengths and limitations of each approach, our study provides valuable insights into the most effective methods for interpreting and predicting reservoir behavior. Specifically, we address a critical research gap by investigating the effect of fractal parameters, such as fractal dimension and conductivity index, on the performance of diffusion models. We believe this would be useful to the readers to better understand the complex interplay between these parameters and to optimize the selection of analytical or numerical methods for a given data set. To facilitate this investigation, we apply our analysis to a real-world data set using the ensemble smoother with the ES-MDA method (Emerick and Reynolds, 2013). Our results demonstrate that by carefully considering the impact of fractal parameters on diffusion models, we can greatly enhance the accuracy and reliability of pressure and rate transient analysis. Overall, this study represents an important step forward in our ability to interpret and predict reservoir behavior and helps us to decide which diffusion models perform better to interpret the pressure transient data analysis.

This paper is organized as follows. Firstly, we define the fractal, anomalous diffusion mechanism, and fractal and anomalous diffusion models. Then, analytical solutions of fractal and anomalous models are presented for both CR and CBHP production conditions. After, some comparisons between fractal and anomalous diffusion models are presented to understand the behavior and relationship of diffusion models. Finally, an interference well test from the Kizildere geothermal field in Turkey is analyzed by using fractal and anomalous diffusion models by utilizing the ensemble smoother with the ES-MDA method (Emerick and Reynolds, 2013).

2. Overview of fractal and anomalous diffusion models

Fractal is first used in the literature by Mandelbrot (1983) and originates from the Latin “fractus”. Fractal means an irregular surface like a broken stone and could be a regular or non-regular geometric shape. Complex patterns are observed in the non-regular geometric shape. However, even if a fractal consists of complex patterns, there are always similar patterns and self-similarity.

There are three fundamental characteristic properties of fractals: self-similarity, power-law expression, and non-integer dimension. In other words, self-similarity means the pattern is repeated across different scales, and fractals have a positive non-integer dimension. Thus, the dimension of a fractal is 1.8 or 2.4, etc. The best examples are fern leaves and Sierpinski Carpet, as shown in Fig. 1.

The position of the particles has a relationship between the mean-square displacement (MSD) of the particle and time. For the heterogeneous media, MSD is explained by

$$r^2 \sim \tau^d.$$  \hspace{1cm} (6)

![Fig. 1. Fern leaves and Sierpinski Carpet (Hardy and Beier 1994).](image)

If the MSD of a particle and time has a linear relationship ($\alpha = 1$), normal diffusion exists in the matrix. This circumstance is valid for homogeneous porous media. On the other side of the coin, if $\alpha$ is not equal to 1, the diffusion is called anomalous. To put it another way, the diffusion process is either faster or slower. In the case of a strange diffusion process (Balescu, 1998), diffusion is described as subdiffusion ($\alpha < 1$), or superdiffusion ($\alpha > 1$). In other words, slower diffusion (subdiffusion) happens when an impediment or obstacle affects the flux, and a faster diffusion process (superdiffusion) is observed if flux moves in the highly conductive and well-connected paths such as in naturally or hydraulically fractured media since the particles change their position in a smaller time scale or jump longer than normal displacement.

2.1. Fractal model

The fractal model is only applicable for subdiffusion where the flow is slower than normal diffusion. The main idea of this model is to define both permeability and porosity (fracture network parameters) as fractals. Chang and Yortsos (1990), Beier (1994), Acuna et al. (1995), and (Zeybek, 2000) characterized permeability and porosity at the wellbore radius by

$$k(r_w) = k_0 \left( \frac{r}{r_w} \right)^{d_f - d},$$ \hspace{1cm} (7)

and

$$\phi(r_w) = \phi_0 \left( \frac{r}{r_w} \right)^{d_f - d}.$$ \hspace{1cm} (8)

The fractal model is based on the following diffusivity equation (Zeybek, 2000):

$$\frac{\partial p}{\partial r} = \frac{k(r_w) r_w^2}{\phi(r_w)c_s \mu} \frac{1}{r_w^{d+\alpha-1}} \frac{d}{dr} \left( r^{d_f - d - 1} \frac{\partial p}{\partial r} \right).$$ \hspace{1cm} (9)

The derivation of Eq. (9) has been presented by Zeybek (2000) and given in Appendix A in this paper. Note that Eq. (9) assumes a fractal system embedded in two-dimensional Euclidean space, i.e., $d = 2$. In addition, Chang and Yortsos (1990) introduced the dimensionless radius, time, and pressure for the CR production case

$$r_D = \frac{r}{r_w},$$ \hspace{1cm} (10)
\[ t_0 = \frac{m}{c_{\mu} \mu^2} t = \frac{k(r_c)}{q(r_c) c_{\mu} \mu^2} t, \]  
(11)

and

\[ p_t = \frac{a V_t}{q_{L} \mu B_{\mu}} [p - p(r,t)] = \frac{2 a k(r_c) h}{q_{L} \mu B_{\mu}} [p - p(r,t)], \]  
(12)

where \( m \) is a fracture-network parameter, \( a \) is a site-density parameter, and \( V_t \) is the site volume(s) in the fractal network where flow storage occurs. Consequently, the diffusivity equation for the fractal model in a dimensionless form as

\[ \frac{\partial p_t}{\partial t} = \frac{1}{r_0^{\theta - 1}} \frac{\partial}{\partial r_0} \left( r_0^{\theta - 1} \frac{\partial p_t}{\partial r_0} \right), \]  
(13)

where \( \theta = d_f - \Theta - 1 \).

2.2. Metzler anomalous model

Metzler et al. (1994) asserted that the diffusivity in the fractal cannot be explained by the fractal model because the fractal model breaks down to obtain the history of anomalous diffusion. Therefore, they focused on anomalous diffusion by utilizing fractional calculus so that the history and nonlocality of transport can be interpreted (Park et al., 2000). Based on Eq. (4a), Metzler et al. (1994) introduced the time-fractional derivative to the governing equation, and presented the following diffusivity equation in the dimensionless form:

\[ \frac{\partial^\alpha p_t}{\partial t^\alpha} = \frac{1}{r_0^{\theta - 1}} \frac{\partial}{\partial r_0} \left( r_0^{\theta - 1} \frac{\partial^{\alpha - 1} p_t}{\partial r_0^\alpha} \right), \]  
(14)

As can be seen from Eq. (14), Metzler’s anomalous diffusion equation contains a fractional time derivative on the left-hand side of Eq. (14). For the fractional derivative, two types are commonly used: the Riemann-Liouville definition and the Caputo definition (Li and Liu, 2018). In this work, we use the Caputo definition to represent the fractional derivative. Letting \( n - 1 < a < n \), where \( n \) is a positive integer, the Caputo fractional derivatives (from \( t = 0 \)) of a function \( q(x,t) \), where \( x \) is the vector of spatial coordinates (Li and Liu, 2018)

\[ \frac{d^\alpha q(x,t)}{dx^\alpha} = \frac{1}{\Gamma(n-a)} \int_0^t \frac{d^{\alpha-1} q(x,s)}{dx^{\alpha-1}} ds, t > 0, \]  
(15)

A few remarks are in order related to the integer \( n \) to be used for the fractional derivative in Eq. (14). First, we note that the anomalous diffusion exponent is \( \alpha = \frac{2}{2-n} \). In the d-dimensional Euclidean case, where \( d_f = d \) and \( \theta = 0 \), all properties become scale-independent and coincide with the point values or the conventional porosity and permeability. The fractal parameter, \( \theta \), is an exponent related to the topology of the network and characterizes the diffusion process. If \( \theta = 0 \), the connectivity is high, and diffusivity \( q \) is constant. This is the case for the Euclidean systems. A fractal network usually contains highly tortuous paths, i.e., \( \theta > 0 \). In general, \( \theta \) increases with higher tortuosity and poorer connectivity. Chang and Yortsos (1990) report that percolation clusters in \( d = 2 \) have \( d_f \approx 1.9 \), and \( \theta = 0.8 \). The numerical fracture networks of Acuna and Yortsos (1995) indicated \( 0 < \theta < 0.5 \). In 3D percolation networks, it is known that \( \theta = 1.784 \) (Ishchenko, 1992). Unfortunately, there is generally no known unique relationship between the fractal dimension \( d_f \) and the fractal exponent \( \theta \) for arbitrary networks. In light of this information regarding \( \theta \) and \( a = 2/(2 + \theta) \), we can state that \( \alpha \) takes values between 0 and 1. Hence, we can take \( n = 1 \) for the Caputo fractional derivative (Eq. (15)).

2.3. The Raghavan anomalous model

Raghavan (2011) objected to both the fractal and Metzler anomalous models since he claimed that the fractal model is not able to define the naturally fractured reservoir without a time-fractional derivative, and the Metzler anomalous model cannot be derived by starting with the conventional form of Darcy’s law. Hence, he emphasized that either the flux term or continuity equation should be modified. Raghavan presented a new anomalous model for subdiffusion in a dimensional form as

\[ \frac{1}{\lambda^{\alpha-1}} \frac{\partial}{\partial \lambda} \left[ \lambda^{\alpha-1} \frac{\partial q(\lambda, \xi)}{\partial \lambda} \right] = \frac{\partial^\alpha q(\lambda, \xi)}{\partial \lambda^\alpha}, \]  
(16)

where \( \lambda(\xi) = \lambda_s \xi^{-\Theta} = \frac{B}{\xi} \), Raghavan (2011) obtained the above diffusivity equation by using the “flux” term based on the concept of a continuous-time random walk (CTRW), i.e.,

\[ q(x, t) = -\frac{k_s}{\mu} \frac{\partial^{\alpha-1}}{\partial t^{\alpha-1}} (\nabla \Phi(x, t)), \]  
(17)

and applied it to the conservation equation. Note the dimension of \( q \) is \( L^{3-\alpha}/T \). As mentioned previously, \( k_s \) in Eq. (17) is not the conventional permeability (referred to as the phenomenological coefficient), and it has a dimension of \( L^{3-\alpha}/T^{1-\alpha} \). It is also worth noting that Raghavan (2011) states that the solutions for a fractal structure may be obtained from Eq. (16) by replacing \( d \) by \( d_f \).

3. Analytical solutions of fractal and anomalous models

In this section, the analytical solutions of the fractal and anomalous diffusion models described in the previous section are presented in the Laplace space for CR and CBHP cases for no skin and wellbore storage effects.

3.1. CR and CBHP solutions for the fractal model

The diffusivity equation for the fractal model is given in Eq. (13), and the initial condition is

\[ p(t_0, t_0) = 0. \]  
(18)

The inner boundary condition for a finite wellbore case is defined as

\[ \left( \frac{\partial q(x,t)}{\partial r_0} \right)_{t_0=1} = -1. \]  
(19)

The outer boundary condition is considered as

\[ \lim_{t_0 \to \infty} p(t_0, t_0) = 0. \]  
(20)

Applying Laplace transform to the diffusivity equation for the fractal model and utilizing the initial condition and boundary conditions provides a general solution for the CR condition for the fractal model in Laplace space (Ozkan, 2020)

\[ \tilde{p}(t_0, s) = \frac{t_0^d}{\Gamma(1-d/2)} K_{d/2} \left( \frac{s^{1/2}}{2^{d/2} \mu} \right) \]  
(21)

where \( \tilde{p} \) represents the Laplace space solution of the dimensionless pressure, \( s \) is the Laplace space variable with respect to the dimensionless time, \( t_0 = \frac{1}{\mu B_{\mu}} \) and \( K_l \) represents the modified Bessel’s functions of the second kind order \( l \) (Abramowitz and Stegun, 1972). Moreover, for the CBHP case, the inner boundary condition and dimensionless pressure must be changed as,

\[ p(t_0 = 1, t_0) = 1, \]  
(22)

and

\[ p(t_0 = 1, t_0) = 0. \]  
(23)
\[ p_0 = \frac{p_i - p(t, r)}{p_{i0} - p_{i0}} = \frac{\Delta p(r, t)}{\Delta p_{i0}}. \] (23)

As a result, the solution of the fractal model for CBHP is given (Ozkan, 2020)

\[ p_{0i}(r, s) = \frac{d_{w0}}{sK_i \left[ \frac{r_i^n}{\bar{r}^n} \right]}. \] (24)

### 3.2. CR and CBHP solutions for the Metzler anomalous model

Metzler et al. (1994) used a different diffusivity equation than the fractal model. Eq. (14) has a time-fractional derivative. Therefore, the Laplace transform of fractional calculus is applied. Consequently, the Laplace transform of Eq. (14) with respect to time \( \Delta t \) is given (Ozkan, 2020)

\[ \Delta t \int_0^t \delta(t - \tau) \, dt \]

The same initial condition and boundary conditions for the CR case are used in this part. After applying the Laplace transform, the general solution of the Metzler anomalous model for CR is (Ozkan, 2020)

\[ p_{0i}(r, s) = \frac{d_{w0}}{sK_i \left[ \frac{r_i^n}{\bar{r}^n} \right]}. \] (26)

In addition, the general solution of theMetzler anomalous model for the CBHP case is obtained using the same initial condition and outer boundary condition. On the other hand, the inner boundary condition is changed like the fractal model. As a result, the solution for CBHP is given (Ozkan, 2020)

\[ p_{0i}(r, s) = \frac{d_{w0}}{sK_i \left[ \frac{r_i^n}{\bar{r}^n} \right]}. \] (27)

Note that the same order \( \nu = \frac{1}{d} \) is obtained not only at the CR but also at the CBHP conditions for both the fractal and Metzler anomalous models.

### 3.3. CR and CBHP solutions for the Raghavan anomalous model

The governing equation of the Raghavan anomalous diffusion model (Eq. (16)) is first solved in dimensional form by applying the initial condition, inner boundary condition, and outer boundary condition for a fully penetrating vertical well. Note that this is the first study presenting the analytical solution for the finite-wellbores problem for the CR production case. The initial condition is given by

\[ \Delta p(r, t = 0) = 0. \] (28)

The inner boundary condition for the finite wellbore case and the outer boundary condition are given by, respectively,

\[ \lim_{r \to r_w} \left( \rho^{1-d} \frac{\partial \Delta p}{\partial r} \right) = - \frac{q_w B}{\rho_d \alpha \lambda} \] (29)

and

\[ \lim_{r \to r_0} \Delta p(r, t) = 0. \] (30)

The solution of the Raghavan anomalous diffusion model for the finite-wellbores CR production case is obtained by applying the Laplace transform with respect to time \( \tau \) (Ozkan, 2020). We provide the basic steps used to derive Eq. (31) in Appendix B for the readers who may not have access to Ozkan’s MSC thesis:

\[ \Delta t \int_0^t \delta(t - \tau) \, dt \]

where \( \nu = \frac{\lambda}{\beta} - 1 = \frac{1}{d+2} \). Please note that the values of \( \beta \) and \( \nu \) for the Raghavan model are different from the fractal and Metzler models. The \( \beta \) for the Raghavan model is \( d - \theta - 1 \), whereas \( d_{w0} \) and \( \theta \) for the fractal and Metzler models. Furthermore, the order \( \nu \) in the modified Bessel functions of the second kind in the fractal and Metzler models is \( (1 - \beta)/(\theta + 2) \).

Raghavan (2011) introduced the following dimensionless pressure and dimensionless time, respectively, given by

\[ p_0(r, t_0) = \frac{d_{w0} h}{\rho_d \alpha \lambda} \Delta p(r, t), \] (32)

where

\[ d_0 = \left( \frac{1}{\eta \delta_0} \right)^{1-\lambda} \left( \frac{1}{\lambda} \right), \] (33)

where \( \eta_0 = \lambda_0/\rho_0 \), to convert the solution given by Eq. (31) from dimensional form to dimensionless form. A dimensional analysis performed on the right-hand side of Eq. (32) shows that the “dimensionless” pressure \( p_0 \) is not a dimensionless quantity unless the Euclidean dimension \( d = 2 \) (or for a fractal structure with the mass fractal dimension \( d_f = 2 \)). It has a dimension of \( L^{d-2} \).

It is not straightforward to express Eq. (16) in dimensionless form using the dimensionless pressure and time defined by Eqs. (32) and (33) because the chain rule may not apply to the fractional derivative appearing on the right-hand side of Eq. (16). In fact, that is why we solve the dimensional form of the initial boundary value problem (IBVP) based on Eq. (16) in Appendix B for the CR and Appendix C for CBHP cases. Then, we convert the dimensional solution in Laplace space to a “dimensionless pressure” solution (only for \( d = d_f = 2 \)) in Laplace space using the definition of \( p_0 \) while the Laplace variable is still with respect to real-time \( t \). We then invert this “dimensionless” pressure solution to the real-time domain \( \tau \) numerically using the Stehfest (1970) algorithm. Once \( p_0 \) vs \( t \) data were generated, we use Eq. (34) to generate \( p_0 \) vs \( t_0 \) data for the Raghavan (2011) anomalous model. The same approach has also been followed by Raghavan (2011), Raghavan and Chen (2013a, b), and Chen and Raghavan (2015).

As a result, the dimensionless pressure in Laplace space as a function of the Laplace variable \( s \) with respect to real-time \( t \) for the CR case may be given as (see Appendix B or Ozkan, 2020)

\[ p_0(r, s) = \frac{d_{w0} h K_i \left( \frac{r_i}{\bar{r}} \sqrt{u} \right)}{\rho_d \alpha \lambda \bar{r}} \] (35)

where \( u \) is the scaled Laplace variable given by Eq. B18.

Similarly, we obtain the dimensionless pressure solution for the CBHP condition given by (see Appendix C)

\[ p_{0i}(r, s) = \frac{d_{w0} h K_i \left( \frac{r_i}{\bar{r}} \sqrt{u} \right)}{s K_i \left( \sqrt{u} \right)}, \] (36)

where \( p_0 \) is defined by Eq. (23).

It is important to note that dimensionless pressure (for the CR case) and time defined for the Chang and Yortsos (1990) fractal and Metzler et al. (1994) (see Eqs. (11) and (12)) is different from the dimensionless...
pressure (for the CR case) and time for the Raghavan anomalous diffusion model (see Eqs. (32) and (34)).

4. Comparison of fractal and anomalous models

Here, we delineate the differences in their pressure and rate transient behaviors of the fractal and anomalous diffusion models.

4.1. Relationship between the fractal and Metzler anomalous models

Note that Eqs. (13) and (14) are quite similar. The only difference is the time-fractional derivative. Hence, Camacho-Velázquez et al. (2008) realized that the diffusivity equation of the fractal and Metzler anomalous models are the same in the case of a = 1. In this part, fractal and anomalous model solutions for CR and CBHP conditions are examined on the graphs to demonstrate the similarity. This part includes two case studies: at the wellbore (r = r_w) and at the reservoir (r = 100r_w). It is noted that the Stehfest algorithm (Stehfest, 1970) with N_{Stef} = 12 is used to take the inverse Laplace transform of the dimensionless pressure solution.

Figs. 2 and 3 compare the dimensionless pressure and dimensionless pressure derivative (d\Phi_0/\alpha d\theta_0) as defined by Bourdet et al., 1989 responses of the fractal and Metzler models for the cases of \theta = 0 and \theta = 0 and r_D = 100, respectively. Note that for both cases, the anomalous diffusion exponent a = 1. As can be seen, the dimensionless pressure and pressure derivative responses of the fractal model are the same as the Metzler anomalous model for the CR solution. As expected, the fractal and Metzler anomalous models behave the same regardless of the fractal dimension. Furthermore, Figs. 2 and 3 demonstrate that the dimensionless pressure and dimensionless pressure derivative responses decrease as the fractal dimension, \theta, increases.

In summary, the results of Figs. 2 and 3 show that, if the topology of the network is zero, the fractal and Metzler anomalous models behave the same. So, this confirms Camacho-Velázquez et al. (2008)’s claim for the CR condition. Next, we investigate their claim for the CBHP production condition. Fig. 4 illustrates the dimensionless pressure and dimensionless pressure derivative responses in the reservoir. There is a perfect match for both fractal and Metzler anomalous models for CBHP production while fractal dimension changes. This confirms Camacho-Velázquez et al. (2008) claim for CR production both models yield identical responses when a = 1. Next, the dimensionless sandface flow rate solutions given by Eq. (37) and 38 for each model are examined for CBHP production, which are given, respectively, by (Ozkan, 2020)

\[ \Phi_0 = -r_w \frac{\partial \Phi_0(r_w, \theta)}{\partial \theta_0} = \tilde{r}_D \frac{K_{s^1}}{\sqrt{K_s \nu_s^{\alpha}}} \left( \frac{\nu_s^{\alpha}}{\Theta_0} \right) \]

and

\[ \Phi_0 = -r_w \frac{\partial \Phi_0(r_w, \theta)}{\partial \theta_0} = \tilde{r}_D \frac{K_{s^1}}{\sqrt{K_s \nu_s^{\alpha}}} \left( \frac{\nu_s^{\alpha}}{\Theta_0} \right) \]

As can be seen from Fig. 5, the fractal and Metzler anomalous models possess the same (sandface) flow rate results for the CBHP production case when \theta_0 = 1, \theta = 0 and a = 1, as expected from Eq. (37) and 38. In short, the fractal model is the same as the Metzler anomalous model if the topology of the network equals zero (homogeneous reservoir).

4.2. Effect of the fractal parameters for each diffusion model

In this section, we study the effect of various parameters on the pressure responses of the various fractal and anomalous diffusion models.

Fig. 2. The dimensionless pressure and dimensionless pressure derivative responses for the fractal and Metzler anomalous models at CR production as a function of the fractal dimension, a = 1 and r_D = 1.
Fig. 3. The dimensionless pressure and dimensionless pressure derive responses for the fractal and Metzler anomalous models at CR production as a function of the fractal dimension, $\alpha = 1$ and $r_D = 100$.

Fig. 4. The dimensionless pressure and dimensionless pressure derive responses for the fractal and Metzler anomalous models for CBHP as a function of fractal dimension, $\alpha = 1$ and $r_D = 100$. 
Fig. 5. The dimensionless sandface rate for the fractal and Metzler anomalous models for CBHP as a function of fractal dimension, $\alpha = 1$ and $r_D = 1$.

Fig. 6. Dimensionless pressure and pressure-derivative responses for the fractal, Metzler Anomalous, and Raghavan anomalous models, CR production, $\Theta = 0.25$, $d = 2$, $d_f = 2$, and $r_D = 1$. 
4.2.1. Topology of the network

The topology of the network determines if diffusion is either sub-diffusion or superdiffusion. Therefore, it is essential to figure out the conductivity index. In this part, the effect of the topology of the network on diffusion models is observed in three cases: \( \theta = 0.25, 0.50, \) and \( 0.75 \) at the wellbore. Both dimensionless pressure and dimensionless pressure derivative responses are examined by taking the inverse Laplace transform \( (N_{\text{diff}} = 12) \). Even though the dimensionless time is the same for both the fractal and Metzler anomalous models, the Raghavan anomalous model utilizes Eq. (34).

Fig. 6 through 8 compare the dimensionless pressure and derivative solutions for the case where \( d_f = 2 \) for the fractal and Metzler anomalous models and \( d = 2 \) for the Raghavan anomalous model for three different values of the topology network (or conductivity) index; \( \theta = 0.25, 0.50, \) and \( 0.75 \), respectively. For all cases in Figs. 6–8, the dimensionless distance \( r_0 = 1 \). As can be seen from Figs. 6–8, as the topology of the network increases, the difference between the models is more pronounced. In addition, each model generates a steeper straight line for both dimensionless pressure and dimensionless pressure derivative as the conductivity index increases. For example, this result is expected from the early-time and late-time asymptotic approximate equations derived in Appendix B for the Raghavan anomalous diffusion model. Note that the log-log plots of \( p_0 \) and \( dp_0/d \ln t_0 \) at the active well shown in Figs. 6–8 for the Raghavan model display straight lines with slope equal to \( (\theta + 1)/2 \) at early times, while they display straight lines with slope equal to \( (\theta + 2)/2 - d/(\theta + 2) \) at late times. For the fractal model, the late-time asymptotic approximate equation for the dimensionless pressure \( p_0 \) is available; see Chang and Yortsos (1990), and this equation shows that the log-log plots of \( p_0 \) vs \( t_0 \) at \( r_0 = 1 \) for the fractal model display straight lines having a slope equal to \( 1 - d_f/(\theta + 2) \). On the other hand, for the Metzler model, the log-log plots of \( p_0 \) vs \( t_0 \) at \( r_0 = 1 \) at late times display straight lines having a slope equal to \( [2 - 2d_f/(\theta + 2)]/\theta \). It is worth noting that the Raghavan anomalous model generates the largest dimensionless pressure and dimensionless pressure derivative responses in magnitude at late times than the two other models. The Metzler anomalous model produces the least corresponding responses in magnitude at late times. The topology of the network has a significant effect on the responses of each model.

4.2.2. Fractal dimension

The fractal dimension is only applicable for the fractal and Metzler anomalous models because the Raghavan anomalous model uses \( d \) which is an integer number (taking values 1, 2, or 3) instead of \( d_f \) which is a non-integer number. As mentioned previously, Raghavan (2011) states that the solutions for a fractal structure may be obtained from Eq. (16) by replacing \( d \) by \( d_f \).

In this part, the effects of three different fractal dimensions \( (d_f = 1.50, 1.75, \) and \( 2.00) \) on the dimensionless pressure and pressure derivative solutions are examined with respect to two different conductivity indices \( (\theta = 0.25 \) and \( 0.50) \), as shown in Figs. 9 and 10. As the fractal dimension increases, each model generates less dimensionless pressure for the constant conductivity index at the wellbore. On the other hand, the fractal and Metzler anomalous models produce much higher dimensionless pressure, and steeper lines both at the early time and late time while the topology of the network increases. Without considering the topology of the network, the fractal model always generates higher dimensionless pressure than the Metzler anomalous model. In addition, the dimensionless pressure difference between the fractal and Metzler anomalous models rises significantly as the fractal dimension increases from 1.5 to 2.0. On the other side of the coin, similar consequences are applicable for the dimensionless pressure derivatives of the fractal and Metzler anomalous diffusion models. In conclusion, the higher the fractal dimension the lesser dimensionless pressure and dimensionless pressure derivative responses are regarding of the diffusion model.

![Fig. 7. Dimensionless pressure and pressure-derivative solutions for the fractal, Metzler, and Raghavan anomalous models, CR solution, \( \theta = 0.50, d = 2, d_f = 2, \) and \( r_0 = 1 \).](image-url)
Fig. 8. Dimensionless pressure and pressure-derivative solutions for the fractal, Metzler, and Raghavan anomalous models, CR production, $\Theta = 0.75$, $d = 2$, $d_f = 2$, and $r_D = 1$.

Fig. 9. Effect of the fractal dimension $d_f$ on the dimensionless pressure solutions generated from the fractal and Metzler anomalous models, CR production, $\Theta = 0.25$, and $r_D = 1$. 
Fig. 10. Effect of the fractal dimension $d_f$ on the dimensionless pressure solutions generated from the fractal and Metzler anomalous models, CR production, $\Theta = 0.50$, and $r_D = 1$.

Fig. 11. Dimensionless pressure and derivative solutions for the fractal, Metzler, and Raghavan anomalous models, CR production, $\Theta = 0.25$, $d_f = 2$, $d = 2$, and $r_D = 10$. 
4.3. Comparison of dimensionless pressure and dimensionless pressure derivative response in the reservoir for each model

For the results to be presented in this part, we take the values of the fractal dimension equal to $d_f = 2$ for the fractal and Metzler models and $d = 2$ for the Raghavan model, and the conductivity index is taken as $\theta = 0.25$. Figs. 11 and 12 present the dimensionless pressure and dimensionless pressure derivative responses for each model for the dimensionless radius equal to $r_D = 10$ and 100. As mentioned previously, the dimensionless pressure and dimensionless times for the Raghavan model are not the same for the fractal and Metzler models. That is why the dimensionless pressure and Bourdet derivative responses for the Raghavan (2011) model are shifted to the left in the time scale (compare the dimensionless times used for the Raghavan model (Eq. (34)) and fractal and Metzler models (Eq. (11)).

As Figs. 11 and 12 clearly show, the Raghavan anomalous diffusion model generates much higher dimensionless pressure and dimensionless pressure derivative than other models both early and late times. As the dimensionless radius decreases, the difference between Raghavan anomalous model and others reduces dramatically. This is due to the difference in dimensionless times used for the Raghavan (2011) and fractal and Metzler models. The fractal model predicts slightly larger responses than the Metzler anomalous model.

4.4. Verification of the value of $d$ for the Raghavan (2011) anomalous diffusion model

Raghavan (2011) compared the dimensionless production rate of his model with the Metzler anomalous model for the CBHP condition. He used the dimensionless rate of the Metzler anomalous model defined by the Camacho-Velázquez et al. (2008) study. The results shown in Fig. 13 were generated by using the following equations (see Appendix C or Raghavan, 2011):

$$q_D(r_D, s) = d_w r_D \left( \frac{1}{2} \right) \left( \frac{r_D^d}{r_D^d} \right) \left( \frac{1}{s^2} \right) K_{r_D} \left[ \frac{2\delta_f \left( \frac{r_D}{s} \right)^{1/2}}{K_r} \right].$$

(39)
The following equation is derived by Camacho-Velázquez et al. (2008) for the Metzler anomalous model at the wellbore,

$$q_w(s) = d_o \left( \frac{\eta}{\eta + d_e^2} \right)^{1+b} \frac{s^{2\nu}}{2\nu} \left[ \frac{2\nu^2 (s^{2\nu})^2}{s^{2\nu}} \right].$$ \hspace{1cm} (40)

In Fig. 13, \( \gamma \) is equivalent to \( \alpha \), the anomalous diffusion exponent used in this study, and their values are taken from Camacho-Velázquez et al. (2008)’s study. The values of \( \gamma = 0.909 \) and 0.714 are applicable if the conductivity index is \( \theta = 0.20 \) and 0.80, respectively. Raghavan (2011) stated that Fig. 13 is produced by using \( d = 2 \) and \( d_f = 1.65 \).

It is worth noting that we could not produce the same results given in Fig. 13 by Raghavan (2011). Fig. 14 shows the Camacho-Velázquez et al. (2008) vs Raghavan (2011) production rate solution for CBHP when \( d = 2 \) and \( d_f = 1.65 \). As can be seen from Fig. 14, the Raghavan anomalous model gives a larger flow rate at the wellbore in the case of \( d = 2 \) as compared to the Camacho-Velázquez et al. (2008) solution. Therefore, to match Raghavan’s results, we change the \( d \) value and we found that if \( d \) and \( d_f \) are taken as the same value equal to 1.65, then we can generate similar responses as shown in Fig. 15 to those shown in Fig. 13 taken from Raghavan (2011). Using an \( d \) value, not an integer may seem to be contradictory to the results of Raghavan (2011) because he states that \( d \) must be an integer.

5. The ensemble smoother with multiple data assimilation (ES-MDA)

In the next section, we will history match a field pressure data set from the Kızılordere geothermal field in Turkey (Onur et al., 2003) by using the ES-MDA method developed by Emerick and Reynolds (2013). Here, we briefly discuss the method of ES-MDA which is one of the popular history-matching methods. The main idea of the ES-MDA method is to assimilate data in multiple steps and iterate to find the parameters that minimize the least-squares differences between observed and model data. As ES-MDA involves the assimilation of the observed data \( N_a \) times, \( m_i^f \) and \( m_i^a \) are used to denote the forecast and analysis of the \( i \)th realization at the \( i \)th assimilation of the observed data (ith data assimilation step) for \( i = 1, 2, \ldots, N_a \) where \( N_a \) denotes the total number of data assimilation steps. Similarly, \( d_i^f = d_i^f(m_i^f) \) denotes the predicted (forecast) data obtained from the forward model (any of the three anomalous diffusion models) evaluated at \( m_i^f \).

Starting from \( m_i^0 = m_i^a \), before each data assimilation step, we set \( m_i^{f,i} = m_i^{a,i-1} \); then the update or analysis step at the \( i \)th data assimilation step (ith iteration) of ES-MDA is given by the following equation (Rafiee and Reynolds, 2017):

$$m_i^{a,i} = m_i^{f,i} + C_i^{md} [C_i^{dd} + \alpha C_i^{cd}]^{-1} (d_i^{w,i} - d_i^f),$$ \hspace{1cm} (42)

where \( j = 1, 2, \ldots, N_i \) and \( i = 1, 2, \ldots, N_a \). Note that \( m_i^j \) represents the vector of the model parameters, while \( d_i^{w,i} \) is an unconditional realization of the vector of the observed data from a normal distribution with mean zero and covariance matrix \( C_0 \), given by a diagonal matrix.

![Fig. 14. Dimensionless well production rate responses generated from the solutions of Camacho et al. and Raghavan, CBHP production, \( d = 2 \) and \( d_f = 1.65 \).](image-url)
where \( \sigma \) is the standard deviation of errors in observed data. In Eq. (42), \( a_i \) is the inflation factor, we have chosen as \( a_i = N_a \) for \( i = 1, 2, \ldots, N_a \), for our applications. In addition, \( \mathbf{C}_{MD} \) is a cross-covariance matrix and \( \mathbf{C}_{ID} \) is an auto-covariance matrix, given by, respectively,

\[
\mathbf{C}_{MD} = \mathbf{C} \left( \Delta \mathbf{D} \right)^T, \tag{43}
\]

and

\[
\mathbf{C}_{ID} = \Delta \mathbf{D} \left( \Delta \mathbf{D} \right)^T. \tag{44}
\]

Note that \( \Delta \mathbf{M} \) and \( \Delta \mathbf{D} \) are defined as

\[
\Delta \mathbf{M} = \frac{1}{\sqrt{N_e} - 1} \left[ \mathbf{m}_{i}' - \bar{\mathbf{m}}', \ldots, \mathbf{m}_{N_e}' - \bar{\mathbf{m}}' \right], \tag{45}
\]

and

\[
\Delta \mathbf{D} = \frac{1}{\sqrt{N_e} - 1} \left[ \mathbf{d}_{i}' - \bar{\mathbf{d}}', \ldots, \mathbf{d}_{N_e}' - \bar{\mathbf{d}}' \right]. \tag{46}
\]

where \( \mathbf{m}_{i}' = \frac{1}{N_e} \sum_{j=1}^{N_e} \mathbf{m}_{ij}' \) and \( \mathbf{d}_{i}' = \frac{1}{N_e} \sum_{j=1}^{N_e} \mathbf{d}_{ij}' \). For further details on the ES-MDA method, we refer the readers to Emerick and Reynolds (2013), Rafiee and Reynolds (2017), and Silva et al. (2021).

6. Field data application to Kizildere geothermal interference well test data

Here, we present the application of the three anomalous diffusion models to analyze interference test data from the Kizildere geothermal field in Turkey (Onur et al., 2003). The Kizildere geothermal field is in Saraykoy, Denizli, Turkey, and is a liquid-dominated geothermal field.

The observed temperature range is between 195 and 240 °C. The interference test is run between KD-21 (observation well) and KD-22 (active well). The active well is produced at a constant rate of 87 metric tons/h of hot water. The fluid property, well, and reservoir data are given in Table 1. The data are taken from Onur et al. (2003).

The ES-MDA method is used to estimate two reservoir characteristics; \( k h \) and \( \phi \), where \( k \) and \( \phi \) represent the permeability and porosity defined by \( k(r) = k_{\text{ref}}(r/r_a)^{d_1-\sigma} \) and \( \phi(r) = \phi_{\text{ref}}(r/r_a)^{d_2-\sigma} \) for the fractal and Metzler models, and \( k_\alpha(r)h \) and \( \phi(r)\phi_c(h) \) for the Raghavan model, where \( k_\alpha(r) \) is the distance between the active and observation wells. Although the fractal and Metzler anomalous models have the same dimension of \( L^{d_1} \) for \( k_\alpha \), the parameter \( k_\alpha(r)h \) for the Raghavan anomalous model includes the dimensions of \( L^{2+d_2} \), because of the definition of "permeability" or the phenomenological coefficient \( k_\alpha \) (which has a dimension of \( L^{2+d_2} \)) used for the Raghavan model. We consider 100 realizations of these two model parameters and 8 data assimilation steps in matching the interference data set by the ES-MDA method.

The data are examined in two different ways. Two different conductivity indexes and fractal dimension values are used as input into the ES-MDA. Babadagli et al. (1997) provided the fractal dimension from outcrop and satellite studies using the box-counting method. Onur et al. (2003) used the topology of the network and fractal dimension values of

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Flow rate ((q_{\text{in}}B))</td>
<td>87 t/h</td>
</tr>
<tr>
<td>Reservoir thickness ((b))</td>
<td>350 m</td>
</tr>
<tr>
<td>Fluid Viscosity ((\mu))</td>
<td>1.5x10^{-4} Pa s</td>
</tr>
<tr>
<td>Wellbore radius ((r_a))</td>
<td>0.108 m</td>
</tr>
<tr>
<td>Distance between wells ((r))</td>
<td>200 m</td>
</tr>
<tr>
<td>Bottomhole Temperature ((T))</td>
<td>205 °C</td>
</tr>
</tbody>
</table>

Table 1

KD-21 and KD-22 fluid property, well, and reservoir input data.

Fig. 15. Dimensionless well production rate responses generated from the solutions of Camacho et al. and Raghavan, CBHP production, \( d = d_i = 1.65 \).
\[ \theta = 2.33 \text{ and } 0.5, \text{ and } d_f = 1.3 \text{ and } 0.75, \text{ respectively, based on their analysis of the data by the fractal model.} \]

6.1. Case 1; \( \theta = 2.33 \) and \( d_f = 1.3 \)

The topology of the network is assumed \( 2.33 \) and the fractal dimension is found to be \( 1.3 \) because the slope is equal to \( 1 - \delta = 0.7 \) in Fig. 16; see Onur et al. (2003) for the analysis. As the Raghavan anomalous model does not consider a fractal dimension and uses an integer value \( d \) which is assumed to be 2 here. The ES-MDA method was used to find \( k(r)_h \) and \( \phi(r)_c \) for the fractal and Metzler models and \( k_{\alpha}(r)_h \) and \( \phi(r)_c \) for the Raghavan model. The value of \( \alpha \) for this case for all models is the same and equal to 0.462. The values of \( \beta \) and \( \nu \) for the fractal and Metzler models are \( 2.03 \) and 0.7, respectively, whereas \( 1.33 \) and \( 0.538 \) for the Raghavan model, respectively.

The model parameters and their lower and upper bounds considered during the data assimilation for this case are given in Table 2. Table 3 summarizes the estimated model parameters and their associated absolute 95% confidence intervals as plus/minus and percentages (given in parentheses) obtained from the history matching for three models. The history-matched observed data for the three models are shown in Figs. 17–19. According to the RMS values reported in Table 3 and history matches of observed data shown in Figs. 17–19, the best match is obtained by the fractal model since it has the lowest RMS. Based on the RMS values, the confidence intervals obtained for the parameters, and the history matches obtained for the observed data, we can state that the fractal model best represents the field interference pressure test data.

The Raghavan (2011) model yields the highest RMS value, which is three times larger than that obtained for the fractal model, and the largest confidence interval for “flow capacity.” It is interesting to note

### Table 2
Lower and upper bounds of the unknown parameters used for the fractal, Metzler, and Raghavan models to history match of KD-21 interference pressure data; Kizildere geothermal field; Case 1.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Fractal Model</th>
<th>Metzler Anomalous Model</th>
<th>Raghavan Anomalous Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lower bound</td>
<td>Upper bound</td>
<td>Lower bound</td>
<td>Upper bound</td>
</tr>
<tr>
<td>( k(r)_h, \text{ m}^3 )</td>
<td>( 2.9 \times 10^{-11} )</td>
<td>( 1.4 \times 10^{-10} )</td>
<td>( 2.0 \times 10^{-10} )</td>
</tr>
<tr>
<td>( \phi(r)_c, \text{ m/Pa} )</td>
<td>( 4 \times 10^{-8} )</td>
<td>( 6 \times 10^{-8} )</td>
<td>( 7 \times 10^{-9} )</td>
</tr>
<tr>
<td>( k_{\alpha}(r)_h, \text{ m}^2 \text{s}^{1-\alpha} )</td>
<td>NA</td>
<td>NA</td>
<td>NA</td>
</tr>
</tbody>
</table>

### Table 3
Summary of results for history matching of KD-21 interference pressure data; Kizildere geothermal field; Case 1.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Fractal Model</th>
<th>Metzler Anomalous Model</th>
<th>Raghavan Anomalous Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lower bound</td>
<td>Upper bound</td>
<td>Lower bound</td>
<td>Upper bound</td>
</tr>
<tr>
<td>( k(r)_h, \text{ m}^3 )</td>
<td>( 8.67 \times 10^{-11} )</td>
<td>( 3.64 \times 10^{-8} )</td>
<td>NA</td>
</tr>
<tr>
<td>( \pm 5.29 \times 10^{-12} ) (( \pm 6.1% ))</td>
<td>( 8.04 \times 10^{-10} ) (( \pm 2.2% ))</td>
<td>NA</td>
<td>2.617</td>
</tr>
<tr>
<td>( \phi(r)_c, \text{ m/Pa} )</td>
<td>( 4.76 \times 10^{-13} )</td>
<td>( 1.55 \times 10^{-10} )</td>
<td>( 5.38 \times 10^{-12} ) (( \pm 3.5% ))</td>
</tr>
<tr>
<td>( \pm 4.6 \times 10^{-14} ) (( \pm 7.3% ))</td>
<td>( 4.10 \times 10^{-10} ) (( \pm 0.8% ))</td>
<td>( 1.02 \times 10^{-10} ) (( \pm 12.8% ))</td>
<td>3.838</td>
</tr>
</tbody>
</table>
Fig. 17. Prior and posterior history match of KD-21 interference data, Kizildere geothermal field, the fractal model Case 1; unassimilated data (left) and assimilated data (right).

Fig. 18. Prior and posterior history match of KD-21 interference data, Kizildere geothermal field, the Metzler anomalous model Case 1; unassimilated data (left) and assimilated data (right).

Fig. 19. Prior and posterior history match of KD-21 interference data, Kizildere geothermal field, the Raghavan anomalous model Case 1; unassimilated data (left) and assimilated data (right).
that the fractal and Raghavan anomalous models yield nearly the same values for $\varphi_c h$. However, the value of $k_r(r)/h$ for the Raghavan model cannot be compared with the values of $k(r)/h$ estimated from the fractal and Metzler models.

Onur et al. (2003) obtained $k_h$ and $\varphi_c h$ values of $8.38 \times 10^{-11}$ m$^3$±$5.55 \times 10^{-11}$ (66%) and $3.00 \times 10^{-8}$ m$^3$/Pa±$7.50 \times 10^{-9}$ (25%), respectively, using the fractal model with the same fractal dimension and conductivity index by matching the pressure data with the Levenberg-Marquart method using a single initial guess of the parameters. The RMS value they obtained is 1.25 in this case. Although their RMS value is slightly higher than our results, the confidence intervals for the estimated parameters are much higher than those we estimated from the ensemble of the posterior model parameters predicted by the ES-MDA method, though the ensemble means of the model parameters predicted by this study using ES-MDA and the nonlinear regression analysis used by Onur et al. (2003).

6.2. Case 2; $\theta = 0.5$ and $d_t = 0.75$

Another conductivity index and fractal dimension are examined in this part to check the behavior of each model. Those values which ensure the same slope of 0.7 (see Fig. 16) are again taken from Onur et al. (2003). The model parameters and their lower and upper bounds considered during the data assimilation for this case are given in Table 4. Based on the matches of observed data compared in Figs. 20–22, the RMS values and the confidence intervals presented in Table 5 for each model, like Case 1, we can state that the fractal model best represents the formation where the interference test data are acquired. The RMS value of the fractal model for Case 2 is almost the same as for Case 1. On the other hand, the Raghavan anomalous model performs much better for Case 2. There are almost 10-fold differences in the values of $k(r)/h$ and $\varphi(r)c_h$ obtained from the fractal and Metzler anomalous models. To conclude, it is observed that the topology of the network and fractal dimension affects the diffusion model performance.

Using the fractal model with the same fractal dimension and conductivity index of this case, Onur et al. (2003) found $k(r)/h$ and $\varphi(r)c_h$ values of $1.44 \times 10^{-10}$ m$^3$±$9.55 \times 10^{-11}$ (66%) and $1.74 \times 10^{-8}$ m$^3$/Pa±$4.50 \times 10^{-9}$ (26%), respectively, by using a nonlinear regression analysis with a single set of initial guesses for the model parameters. Their RMS value is 1.25 kPa, which is similar to the value of RMS we obtained from the ES-MDA method for the fractal model. As in Case 1, the 95% confidence intervals obtained for the parameters by the ES-MDA are smaller than those obtained by Onur et al. (2003), though the estimated parameters from the ES-MDA agree well with those obtained by Onur et al. (2003).

Here, we would like to make a few remarks regarding the confidence intervals computed from the estimated model parameters by the ES-MDA and the nonlinear regression analysis used by Onur et al. (2003). First, we should emphasize the 95% confidence intervals given here for both approaches are approximate because that they assume that the estimated model parameters have normal (Gaussian) distributions symmetric around the mean and that the model (pressure equations based on fractal or anomalous diffusion) behave linearly or is approximately linear in a neighborhood of the minimum, which is not true for the models considered in the paper. If these assumptions do not hold, then the approximate confidence intervals computed based on these assumptions can be inaccurate. However, in the general nonlinear model, the estimated parameters may not have a Gaussian distribution and the model may not behave linearly around the minimum. Thus, we do not know how to compute confidence intervals in the general case. There are more accurate methods of computing approximate confidence regions and intervals that do not assume that the model is linear, e.g., likelihood ratio confidence intervals (Carvalho, 1993). This method requires multiple regressions as in the case of the ES-MDA method we used to history match the pressure data. In general, Carvalho found that the likelihood confidence intervals do not centerline around the mean of the estimated parameters and that the conventional confidence intervals (based on the assumptions stated above are usually conservative, i.e., the conventional confidence intervals accommodate the likelihood ratio confidence intervals. Further details can be found in Carvalho (1993).

7. Summary and conclusions

In this study, we compared the analytical solutions of the fractal model and the Metzler and Raghavan anomalous diffusion models for both CR and CBHP cases and delineated the differences in their pressure and rate transient behaviors. New analytical and early- and late-time asymptotic approximate solutions were derived for the finite-wellbore problem for the Raghavan anomalous model for both the CR and CBHP cases. The approximate solutions identify the fractal and anomalous diffusion parameters that influence behaviors of the early- and late-time pressure and rate transient data. In addition, we applied the models to an interference well test data from the Kızıldere geothermal reservoir in Turkey by using the ES-MDA method. The following conclusions can be withdrawn from the results of this study:

- Although the fractal model and Metzler anomalous model are derived from the same flux law and continuity equation, their solutions are different since the Metzler anomalous model includes a time-fractional exponent.
- The Raghavan anomalous model produces excessively higher dimensionless pressure and dimensionless pressure derivative responses than the other two models for the same set of parameters. This occurs because of the different flux law definitions and dimensionless variables used by Raghavan (2011).
- We show that the “dimensionless” pressure defined by Raghavan (2011) is not a dimensionless quantity unless the Euclidean dimension $d = 2$ (or for a fractal structure with the mass fractal dimension $d_f = 2$).
- The same fractal parameters affect each model differently. That is, as the topology of the network increases remarkably, the dimensionless pressure and pressure derivative responses for CR production for each diffusion model increase in magnitude. As the fractal dimension decreases, the dimensionless pressure and logarithmic pressure derivative responses for the CR production case for the fractal and Metzler anomalous models increase crucially when the conductivity index is constant.
- For the CR case of the Raghavan model, the log-log plots of $\Delta p$ and $\Delta \Delta p/d \ln t$ vs. time $t$ at the active well and observation wells display straight lines with slope equal to $(1/\theta+1)/(\theta+2)$ at early times regardless of the values of $\theta$ and $d_t$. On the other hand, at late times,
Fig. 20. Prior and posterior history match of KD-21 interference data, Kizildere geothermal field, the fractal model Case 2; unassimilated data (left) and assimilated data (right).

Fig. 21. Prior and posterior history match of KD-21 interference data, Kizildere geothermal field, the Metzler anomalous model Case 2; unassimilated data (left) and assimilated data (right).

Fig. 22. Prior and posterior history match of KD-21 interference data, Kizildere geothermal field, the Raghavan anomalous model Case 2; unassimilated data (left) and assimilated data (right).
they display straight lines with slopes equal to $1 - 2d/(\theta + 2)^2$ if $d \leq \theta + 1$ or $\theta/(\theta + 2) d > \theta + 1$.

- For the CBHP case of the Raghavan model, the log-log plots of sandface rate $q$ vs. time $t$ display straight lines with a slope equal to $-(\theta + 1)/d$ at early times regardless of the values of $d$ and $\theta$. On the other hand, at late times, they display straight lines with a slope equal to $-1 + 2d/(\theta + 2)^2$ if $d \leq \theta + 1$ or $-\theta/(\theta + 2) d > \theta + 1$.

- Among the three diffusion models, it is found that the fractal model gives the best representation of the interference test data measured in the Kizildere geothermal system.

- Moreover, the field data analysis shows that knowing the fractal dimension and topology of the network is highly vital for estimating the flow capacity and the storativity of the system, though the estimated flow capacities and storativity by history matching may not yield unique results.

The following recommendations for further research may be helpful:

- In this study, we neglected wellbore storage and skin effects at both the active and observation wells. The constant rate solutions can easily be extended to include wellbore storage and skin effects at both wells by using superposition.

- The more general cases of variable rate and BHP cases can be developed by using the CR and CBHP solutions given in this study and by the method of superposition.

- A geothermal field with a highly conductive and well-connected channel is a good candidate for a new field of research. The capacity to infer superdiffusion in both active and observation wells would be improved by this.

**Credit author statement**

Yasin OZKAN: Conceptualization, formal analysis, investigation, methodology, resources, software, visualization, writing the original and revised manuscript. Mustafa ONUR: Supervision, conceptualization, formal analysis, investigation, methodology, resources, review, and editing.

**Declaration of competing interest**

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests:

**Data availability**

Data will be made available on request.

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**Nomenclature**

- $B$: formation volume factor, $[L^3/L^3]$
- $c_i$: compressibility, $[LM^{-1}T^2]$
- $d$: Euclidean dimension
- $d_f$: fractal dimension
- $d_{fr}$: Fickian value
- $G$: geometry factor ($G = A, 2\pi h$, and $4\pi$ for rectilinear, cylindrical, and spherical symmetry, respectively), $L^3 d$
- $k$: permeability, $[L^2]$
- $k_d$: phenomenological coefficient, see Eq. (17), $[L^{\mu - 2} T^{1 - \eta}]$
- $m$: fracture network parameter, $[L^{\mu - 2}]$
- $n$: nonnegative integer
- $p$: pressure, $[L^{-1}MT^{-2}]$
- $p_d$: dimensionless pressure
- $p_i$: initial pressure, $[L^{-1}MT^{-2}]$
- $p_{wf}$: flowing bottomhole pressure, $[L^{-1}MT^{-2}]$
- $q_c$: surface flow rate, $[L^3 T^{-1}]$
- $q$: “flux” given by Eq. (17), $[L^{2 + \theta} T^{-1}] q_d$: dimensionless flow rate
- $q_{sd}$: dimensionless flow rate at the wellbore
- $r$: radial distance, $[L]$
- $r_{pd}$: dimensionless radius
- $r_w$: wellbore radius, $[L]$
- $r_0$: reference radius, $[L]$
- $s$: Laplace transform variable with respect to real-time $t$ $[T^{-1}]$
Appendix ADerivation of the Fractal Diffusion Equation (Eq. 9)

For naturally fractured reservoirs having a fractal fracture network of dimension \( d_f \), embedded in a Euclidean matrix of dimension \( d \), permeability and porosity at a radial distance \( r \) are defined, respectively, by (Chang and Yortsos, 1990; Beier, 1990; Acuna et al., 1995)

\[
k(r) = \frac{aV_r}{G} \mu^{d-d-\alpha} = k_0 \left( \frac{r}{r_0} \right)^{d-d-\alpha}, \quad (A1)
\]

and

\[
\varphi(r) = \frac{aV_w}{G} \mu^{d-\beta} = \varphi_0 \left( \frac{r}{r_0} \right)^{d-\beta}. \quad (A2)
\]

As is clear from Eqs. A1 and A2, the fractal reservoir model is based on power law scalings of permeability and porosity. Because of Eqs. A1 and A2, the diffusivity coefficient of a fractal reservoir is variable, that is, \( \eta \propto r^{d-\alpha} \). In Eqs. A1 and A2, \( k_0 \) and \( \varphi_0 \) are the reference permeability and porosity at \( r_0 \), respectively, and \( r_0 \) is a lower cutoff scale above which the fractal behavior is defined. Then, like Beier (1990), we can define the permeability and porosity at the “sandface” by using Eqs. A1 and A2, respectively, by

\[
k(r_w) = k_0 \left( \frac{r_w}{r_0} \right)^{d-d-\alpha}, \quad (A3)
\]

and

\[
\varphi(r_w) = \varphi_0 \left( \frac{r_w}{r_0} \right)^{d-\beta}. \quad (A4)
\]

Then, we can express Eqs. A1 and A2 in terms of the permeability and porosity at the “sandface” by using the power scaling of Eqs. A1 and A2, respectively, as

\[
k(r) = k(r_w) \left( \frac{r}{r_w} \right)^{d-d-\alpha}, \quad (A5)
\]

and

\[
\varphi(r) = \varphi(r_w) \left( \frac{r}{r_w} \right)^{d-\beta}. \quad (A6)
\]

Note that Eqs. A5 and A6 are useful in that we can express permeability and porosity variation in the radial direction using wellbore radius and the properties at the sandface.

Using the power law scaling of Eqs. A1 and A2, the fractal diffusion equation (Eq. 3) for a slightly compressible fluid of constant viscosity for a fractal network with a local structural property \( m \left( \frac{k_0 r}{\mu^{d-d-\alpha}} \right) = k_0 r_0 \left( \frac{r}{r_0} \right)^{d-d-\alpha} \) embedded in a 2D radial matrix system (\( d = 2 \)) can be expressed as

\[
\varphi(r) \frac{\partial \varphi(r, t)}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} \left( r k(r) \frac{\partial \varphi(r, t)}{\partial r} \right). \quad (A7)
\]

Replacing \( k(r) \) and \( \varphi(r) \) in Eq. (A7) by the right-hand side of Eqs. A5 and A6, respectively, and rearranging the resulting equation gives Eq. (9) in the main text.


Here, we provide the pressure change solution in Laplace space for a vertical well producing slightly compressible fluid of constant viscosity in an infinite-acting fracture network embedded into a 2D matrix that can be described by the Raghavan (2011) anomalous diffusion model. The initial boundary value problem (IBVP) for this case can be described by the following partial differential equation (PDE), initial condition (IC), the inner wellbore boundary condition (IWBC) and outer reservoir boundary (ORBC) condition, respectively,
\[ \text{PDE: } \frac{1}{r^{d-1}} \frac{\partial}{\partial r} \left( r^{2-d} \frac{\partial \Delta p}{\partial r} \right) = \frac{q_c v}{\nu \phi_c} \frac{\partial \Delta p(r,t)}{\partial r} \quad (B1) \]

\[ \text{IC: } \Delta p(r, t=0) = 0 \quad (B2) \]

\[ \text{IWBC: } \left( r^{2-d} \frac{\partial \Delta p}{\partial r} \right)_{r=r_w} = -\frac{q_w B}{\nu \phi_c H} \quad (B3) \]

\[ \text{ORBC: } \lim_{r \to \infty} \Delta p(r, t) = 0 \quad (B4) \]

Taking Laplace transform of Eq. (1) with respect to time \( t \) and then using the IC gives

\[ \frac{d}{dr} \left( r^{d-1} \frac{\partial \Delta \tilde{p}}{\partial r} \right) - \frac{\mu}{\eta} \frac{1}{\nu} \Delta \tilde{p}(r, s) = 0 \quad (B5) \]

where \( \Delta \tilde{p} \) represents the Laplace transform of \( \Delta p \) and is defined by

\[ \Delta \tilde{p}(r, s) = \mathcal{L} \{ \Delta p(r, t) \} = \int_0^\infty e^{-st} \Delta p(r, t) dt \quad (B6) \]

and for simplicity, we define \( \beta \) and \( \eta_0 \), respectively, as

\[ \beta = d - 1 - \theta \quad (B7) \]

and

\[ \eta_0 = \frac{\lambda_0}{\eta} \quad (B8) \]

Note that \( \beta \) depending on the values of \( d \) and \( \theta \) can in general assume either positive or negative values. If \( d = 2 \) and \( 0 \leq \theta < 1 \), then \( \beta \) is always positive but less than unity.

The general solution of the ordinary differential given by Eq. (B5) can be given as (Arpaci, 1966)

\[ \Delta \tilde{p}(r, s) = r^{d-\frac{\beta}{\theta+2}} \left[ egin{array}{l} c_1 I_\kappa \left( \sqrt{\frac{s}{\eta}} \sqrt{\frac{\mu}{\nu}} \right) + c_2 K_\kappa \left( \sqrt{\frac{s}{\eta}} \sqrt{\frac{\mu}{\nu}} \right) \end{array} \right] \quad (B9) \]

where \( \nu = \sqrt{-\sqrt{\nu} \beta + 1} / (d+2) \)

The values of \( c_1 \) and \( c_2 \) that satisfy the given IWBC (Eq. B3) and ORBC (Eq. B4) conditions can be found as, respectively,

\[ c_1 = 0 \quad (B10) \]

and

\[ c_2 = \frac{q_w B}{\alpha \eta_0 r_w^{\frac{d-\beta}{2}}} \frac{\sqrt{\nu}}{r_w^{\frac{d-\beta}{2}}} K_\kappa \left( \frac{r_w}{r_w^{\frac{d-\beta}{2}}} \sqrt{\frac{s}{\eta}} \sqrt{\frac{\mu}{\nu}} \right) \quad (B11) \]

where

\[ \nu = -\sqrt{-\sqrt{\nu} \beta + 1} / (d+2) \quad (B12) \]

It should be noted that the determination of \( c_2 \) is quite tedious and lengthy and hence its determination is not given here but can be found in Ozkan (2020). Using the values of \( c_1 \) and \( c_2 \) in Eq. (B9) and knowing that \( K_\kappa(z) = K_\kappa(z) \) gives the solution of the IVBP described by Eq. (B1)-B4 in Laplace space

\[ \Delta \tilde{p}(r, s) = \frac{q_w B \sqrt{\nu}}{\alpha \eta_0 r_w^{\frac{d-\beta}{2}}} \frac{r^{d-\frac{\beta}{\theta+2}}}{\sqrt{\nu}} K_\kappa \left( \frac{r_w}{r_w^{\frac{d-\beta}{2}}} \sqrt{\frac{s}{\eta}} \sqrt{\frac{\mu}{\nu}} \right) \quad (B13) \]

Evaluating Eq. (B14) at \( r = r_w \) gives the pressure change at the sandface as

\[ \Delta p_{12}(s) = \Delta \tilde{p}(r = r_w, s) = \frac{q_w B \sqrt{\nu}}{\alpha \eta_0 R_w^{\frac{d-\beta}{2}}} \frac{1}{\sqrt{\nu}} K_\kappa \left( \frac{r_w}{r_w^{\frac{d-\beta}{2}}} \sqrt{\frac{s}{\eta}} \sqrt{\frac{\mu}{\nu}} \right) \quad (B14) \]
Using the dimensionless pressure ($p_d$) definition given by Eq. (33), we can express Eq. (B14) in terms of $p_d$ as

$$
p_d(t_d, s) = \frac{1}{\alpha_p} r_D^{\frac{1}{n+1/2}} \frac{\sqrt{\eta}}{r_c} K_i \left( \sqrt{\alpha} \left( \frac{r_w}{r_c} \right)^{1/2} \right)
$$

and we can express Eq. (B15) as

$$
p_d(t_d, s) = \frac{1}{\alpha_p} r_D^{\frac{1}{n+1/2}} \frac{\sqrt{\eta}}{r_c} K_i \left( \sqrt{\alpha} \right)
$$

where

$$u = \frac{4r_w^2 \eta}{\eta d_c^2}
$$

It is worth noting that inverting the solutions given by Eqs. B16 and B.17 by the Stehfest (1970) algorithm does not generate the solutions of the dimensionless $p_D$ and $p_{wD}$ vs. dimensionless time $t_D$ given by Eq. (35) rather than they are $p_D$ and $p_{wD}$ vs. real-time $t$.

**Early-Time Asymptotic Approximation**

At small real-time $t$ values, the Laplace variable $s$ should be large. When $v$ is fixed, and $x$ is large, $K_v(x)$ can be approximated as (Abramowitz and Stegun, 1972)

$$K_v(x) \approx \sqrt{\frac{x}{2\pi}} e^{-x}
$$

Using Eq. (B19) in B14 and then simplifying the resulting equation gives

$$\Delta p(r, s) = \frac{q \sigma}{\alpha_p \lambda_h \sqrt{\eta}} \left( \frac{r}{r_c} \right)^{1/2} \sqrt{\frac{2}{1-\alpha} e^{-r}}
$$

The analytic inverse of Eq. (B20) cannot be found in tables of Laplace transform, though it can be numerically inverted by using the Stehfest algorithm (1970). If we evaluate Eq. (B20) at the sandface, i.e., $r = r_w$, then Eq. (B20) simplifies to

$$\Delta p_w(s) = \Delta p(r = r_w, s) = \frac{q \sigma}{\alpha_p \lambda_h \sqrt{\eta}} \left( \frac{r}{r_c} \right)^{1/2} \sqrt{\frac{2}{1-\alpha} e^{-\frac{r}{r_w}}}
$$

As $\alpha$ always less than 2, the analytic inverse of Eq. (B21) to real-time is given by

$$\Delta p_w(t) = \Delta p(r = r_w, t) = \frac{q \sigma}{\alpha_p \lambda_h \sqrt{\eta}} \left( \frac{r}{r_c} \right)^{1/2} \sqrt{\frac{2}{1-\alpha} e^{-\frac{r}{r_w}}} e^{-\frac{t}{t_0}}
$$

which shows that the early-time solutions should yield a straight line with a slope equal to $1-\alpha/2$ on a log-log plot of $\Delta p_w(t)$ vs. $t$. The Bourdet derivative of the early-time approximation given by Eq. (B22a) is given by

$$\frac{d\Delta p_w(t)}{dt} = \frac{q \sigma}{\alpha_p \lambda_h \sqrt{\eta}} \frac{1}{\Gamma(2-\frac{\alpha}{2})} e^{-\frac{r}{r_w}}
$$

The early-time approximation for the dimensionless pressure, $p_{wD}$, is obtained using the approximation given by Eq. (B19) in Eq. (B17), simplifying it, and then inverting it to real-time gives

$$p_{wD}(t_D, t_0) = \frac{1}{\Gamma(2-\frac{\alpha}{2})} \frac{\eta d_c^2}{\alpha_p} e^{-\frac{r_w^2}{r_w^2}}
$$

where $\alpha_p$

$$\alpha_p = \left( \frac{1}{\eta d_c^2} \right)^{1/2} \left( \frac{r_w}{r_c} \right)
$$

and $t_0$ is given by

$$t_0 = \frac{\eta d_c^2}{r_w^2}
$$

It is also clear from Eq. (B23a) that $p_{wD}$ is dimensionless if $d = 2$. Otherwise, it is not. The Bourdet derivative of the early-time approximation given by Eq. (B23a) is given by

$$\frac{d p_{wD}(t_D, t_0)}{dt_D} = \frac{2 - \alpha}{2} \frac{1}{\Gamma(2-\frac{\alpha}{2})} \frac{\eta d_c^2}{\alpha_p} e^{-\frac{r_w^2}{r_w^2}}
$$

(B23d)
Late-Time Asymptotic Approximation

At large real-time $t$ values, the Laplace variable $s$ should be small. Before we derive the approximate late-time approximation, we first check the sign of the order of the modified Bessel function appearing in the rigorous solution given by Eq. (B14). Recall Eq. (B13), that is, $\nu = (\beta - 1)/ (\theta + 2)$. Note that $\beta = d - \theta - 1, \theta \geq 0$, and $1 \leq d \leq 3$. $\beta$ could take negative and positive values depending on the values of $\theta$ and $d$.

The case where $\nu \leq 0$. If $\beta \leq 1$, or $d \leq \theta + 1$ then $\nu \leq 0$, we know $K_{\nu}(x) = K_{-\nu}(x)$. So, we can express Eq. (B14) as

$$\Delta \rho (r, s) = \frac{q_r B \sqrt{\eta r}}{\alpha \rho_i h} \left[ \frac{r \nu}{\nu + 1} \right]^{1/2} K_{-\nu} \left( \frac{\nu + 1}{\nu + 2} \right)$$

(B24)

Then, we use the approximation of the modified Bessel’s functions of the second kind and order $-\nu > 0$ and $\nu + 1 > 0$ (note that $\nu + 1 = \beta \geq 0$) for small arguments

$$K_{-\nu}(x) \approx \frac{\Gamma(-\nu)}{2} \left( \frac{x}{2} \right)^{-\nu} \frac{\Gamma(-\nu) / \Gamma(\nu)}{2}$$

(B25)

in Eq. (B24) to obtain (we skip the tedious algebra)

$$\Delta \rho (r, s) = \frac{q_r B}{\alpha \rho_i h d^{\nu + 1} \eta} \frac{\Gamma(-\nu)}{\nu + 1} \frac{1}{\nu + 1} \left[ \frac{\nu}{\nu + 1} \right]^{1/2}$$

(B26)

For $2 - a (1 + \nu) > 0$, the analytic inverse of Eq. (B26) is given by

$$\Delta \rho (r, t) = \frac{q_r B}{\alpha \rho_i h d^{\nu + 1} \eta} \frac{\Gamma(-\nu)}{\nu + 1} \frac{1}{\nu + 1} \left[ \frac{\nu}{\nu + 1} \right]^{1/2}$$

(B27a)

or

$$\Delta \rho (r, t) = \frac{q_r B}{\alpha \rho_i h d^{\nu + 1} \eta} \frac{\Gamma(-\nu)}{\nu + 1} \frac{1}{\nu + 1} \left[ \frac{\nu}{\nu + 1} \right]^{1/2}$$

(B27b)

The Bourdet derivative of Eq. (B27a) or (B27b) is given by

$$\frac{d \Delta \rho (r, t)}{dt} = \frac{q_r B}{\alpha \rho_i h d^{\nu + 1} \eta} \frac{\Gamma(-\nu)}{\nu + 1} \frac{1}{\nu + 1} \left[ \frac{\nu}{\nu + 1} \right]^{1/2}$$

(B27c)

It is interesting to note that the late-time approximation is independent of the radial distance and the wellbore radius. Raghavan (2011) has obtained the same late-time approximate equation but starting for a line-sink solution. Eq. (B27a) and 27c show that log-log plots $\Delta \rho$ and $d \Delta \rho / dt$ at the wellbore of an active well and an interference well located $r$ distance away from the active well yield a straight line with a slope equal to $1 - a (1 + \nu) = 1 - \frac{2d}{\nu + 1}$.

We can express the late-time approximate solution (Eq. (B27a) or 27 b) in terms of dimensionless pressure (Eq. (32)) and dimensionless time (Eq. (34)) (we skip tedious algebra to obtain the following equation):

$$p_{\nu}(r_0, t_0) = \frac{\Gamma(-\nu)}{\nu + 1} \frac{1}{\nu + 1} \left[ \frac{\nu}{\nu + 1} \right]^{1/2}$$

(B28a)

or replacing $a_r$ in Eq. (B28a) by the right-hand side of Eq. (B23b) gives

$$p_{\nu}(r_0, t_0) = \frac{\Gamma(-\nu)}{\nu + 1} \frac{1}{\nu + 1} \left[ \frac{\nu}{\nu + 1} \right]^{1/2}$$

(B28b)

The Bourdet derivative of Eq. (B28b) is given by

$$\frac{d p_{\nu}(r_0, t_0)}{dt} = \frac{\Gamma(-\nu)}{\nu + 1} \frac{1}{\nu + 1} \left[ \frac{\nu}{\nu + 1} \right]^{1/2}$$

(B29)

The case where $\nu \geq 0$. If $\beta \geq 1$, or $d \geq \theta + 1$ then $\nu \geq 0$, then using the approximation given by

$$K_{\nu}(x) \approx \frac{\Gamma(-\nu)}{2} \left( \frac{x}{2} \right)^{-\nu}$$

(B30)

we can approximate Eq. (B14) as

$$\Delta \rho (r, s) = \frac{q_r B}{\alpha \rho_i h} \frac{r^{-\nu}}{\nu} \frac{\Gamma(-\nu)}{\nu + 1} \frac{\Gamma(-\nu)}{\nu + 1} \left[ \frac{\nu}{\nu + 1} \right]^{1/2}$$

(B31)

The analytical inverse of Eq. (B31) is obtained as

$$p_{\nu}(r_0, t_0) = \frac{\Gamma(-\nu)}{\nu + 1} \frac{1}{\nu + 1} \left[ \frac{\nu}{\nu + 1} \right]^{1/2}$$

(B32a)

or
\[ \Delta p(r, t) = \frac{q_w B}{a d_h} r^{-d} \Gamma(\nu) \left( \frac{1}{\rho} \right) \Gamma(\nu + 1) \Gamma(2 - \alpha) \]  

The Bourdet derivative of Eq. (B32a) or B32b is given by

\[ \frac{d \Delta p(r, t)}{d t} = \frac{q_w B}{a d_h} r^{-d} \Gamma(\nu) (1 - \alpha) \Gamma(\nu + 1) \Gamma(2 - \alpha) \left( \frac{1}{\rho} \right) \Gamma(\nu + 1) \Gamma(2 - \alpha) \]  

Eq. (B32a) and 32c show that log-log plots \( \Delta p \) and \( d \Delta p/dt \) at the wellbore of an active well and an interference well located \( r \) distance away from the active well yield a straight line with a slope equal to \( 1 - \alpha = 1 - \frac{a}{2} \).

We can express the late-time approximate solution (Eq. (B32a) or 32 b) in terms of the dimensionless pressure (Eq. (32)) and dimensionless time (Eq. (34)):

\[ p(t, \tau) = \frac{\rho}{\rho_0} \frac{\Delta p(r, t)}{\Delta p_{w}} \]

and the Bourdet derivative of Eq. (B33a) is given by

\[ \frac{dp(t, \tau)}{d\tau} = \frac{\rho}{\rho_0} \frac{\Delta p(r, t)}{\Delta p_{w}} \]  


Here, we provide the solutions for the CBHP case for the Raghavan anomalous diffusion model. The PDE, the IC, and the ORBC are the same as those given by Eq. (B1), B2, and B4, respectively, for the CR case. The only difference to obtain the solution for the CBHP case is the IWBC, which is given by.

IWBC: \( \Delta p(r = r_w, t) = p_i - p_{wef} = \Delta p_{wef} \)

Following a similar solution procedure that uses the Laplace transformation, we obtain the pressure change in the reservoir for the CBHP case as

\[ \Delta p(r, s) = \frac{\Delta p_{wef} F_{p}(s, \tau)}{s} \frac{K_{\nu} \left( \frac{\sqrt{2} r_w}{\sqrt{\nu}} \sqrt{\frac{\mu}{\theta}} \right)}{K_{\nu} \left( \frac{r_w}{\sqrt{\nu}} \right)} \]  

The solution for the flow rate can be obtained from the flux condition given by

\[ q(r, t) B = \frac{\alpha d_h}{\rho} \rho^{-d} \frac{d \Delta p}{d r} \]  

Taking the Laplace transform of Eq. (C3) with respect to \( t \), we obtain

\[ \bar{q}(r, s) = -\frac{\alpha d_h}{\rho} \rho^{-d} \frac{d \Delta p(s, \tau)}{d r} \]  

where \( \bar{q} \) represents the Laplace transform of \( q \). Differentiating \( dp \) given by Eq. (C3) with respect to \( r \) and then using in Eq. (C4) (skipping tedious lengthy algebra) gives

\[ \bar{q}(r, s) = -\frac{\alpha d_h}{\rho} \rho^{-d} \frac{d \Delta p(s, \tau)}{d r} \]  

Using the dimensionless definition given by Raghavan (2011),

\[ q_{d}(r, t) = \frac{\alpha d_h}{\rho} \rho^{-d} \frac{d \Delta p(s, \tau)}{d r} \]

which has the Laplace transform with respect to \( t \),

\[ \bar{q}_{d}(r, s) = -\frac{\alpha d_h}{\rho} \rho^{-d} \bar{q}(r, s) \]  

we can express Eq. (C5) in the dimensionless form as

\[ \bar{q}_{d}(r, s) = \frac{1}{\rho} \frac{d}{\rho d h} \left( \frac{1}{\rho} \right) \frac{K_{\nu+1} \left( \frac{\sqrt{2} r_w}{\sqrt{\nu}} \sqrt{\frac{\mu}{\theta}} \right)}{s^{\nu+1} K_{\nu} \left( \frac{r_w}{\sqrt{\nu}} \right)} \]
Early-Time Asymptotic Approximation

At small real-time $t$ values, the Laplace variable $s$ should be large. So, using the approximation given by Eq. (B19) in Eq. (C8) gives

$$
\tilde{q}_{\alpha}(r_0, s) = d_{\nu} \left( \frac{r_0^\alpha}{\eta d_e^2} \right)^{\nu - 1/2} r_0 \left[ \frac{1}{\gamma_s (r_0^\alpha)^{1/4}} \exp \left[ - \frac{1}{2} \sqrt{\nu} \left( \frac{r_0^\alpha - r_D^\alpha}{s^{\nu/2}} \right) \right] \right]
$$

(C9)

which we cannot invert it analytically, but it can be inverted numerically. However, at the sandface, $r_D = 1$, Eq. (C9) reduces to

$$
\tilde{q}_{\alpha}\left(\gamma_s, 1, s\right) = d_{\nu} \left( \frac{r_0^\alpha}{\eta d_e^2} \right)^{\nu - 1/2} \frac{1}{s^{\nu/2}}
$$

(C10)

which has an analytic inverse in real-time $t$, given by

$$
q_{\alpha}\left(t\right) = d_{\nu} \left( \frac{r_0^\alpha}{\eta d_e^2} \right)^{\nu - 1/2} \frac{1}{\Gamma(\nu/2)^{1/2}}
$$

(C11)

Eq. (C11) can be expressed in terms of dimensionless time $t_0$ given by Eq. (34) as

$$
\rho_{\alpha}\left(t_0\right) = d_{\nu} \left( \frac{r_0^\alpha}{\eta d_e^2} \right)^{\nu - 1/2} \frac{1}{\Gamma(\nu/2)^{1/2}}
$$

(C12)

Late-Time Asymptotic Approximation

Like in the CR case, we can derive late-time asymptotic approximate solutions for the CBHP case. Here, we present the approximate solutions only for the sandface rate. It follows from Eq. (x) that the solution for the sandface is given by

$$
\tilde{q}_{\alpha}\left(\gamma_s, 1, s\right) = d_{\nu} \left( \frac{r_0^\alpha}{\eta d_e^2} \right)^{\nu - 1/2} \frac{1}{\Gamma(\nu/2)^{1/2}}
$$

(C13)

Recall that $\nu = (\beta - 1)/(\theta + 2)$. Note that $\beta = d - \theta - 1, \theta \geq 0$, and $1 \leq d \leq 3$.

**The case where $\nu \leq 0$.** If $\beta \leq 1$, or $d \leq \theta + 1$ then $\nu \leq 0$, we know $K_\nu(x) = K_{-\nu}(x)$. So, by using the approximation given by Eq. (B25. We can approximate Eq. (C13) as

$$
\tilde{q}_{\alpha}\left(\gamma_s, 1, s\right) = d_{\nu} \left( \frac{r_0^\alpha}{\eta d_e^2} \right)^{\nu - 1/2} \frac{1}{\Gamma(\nu/2)^{1/2}}
$$

(C14)

which has an analytical inverse in real-time, given by

$$
q_{\alpha}\left(t\right) = d_{\nu} \left( \frac{r_0^\alpha}{\eta d_e^2} \right)^{\nu - 1/2} \frac{1}{\Gamma(\nu/2)^{1/2}}
$$

(C15)

which is the approximate solution derived by Raghavan (2011). We can express Eq. (C15) in terms of dimensionless time $t_0$ (Eq. (34)) as

$$
q_{\alpha}\left(t_0\right) = d_{\nu} \left( \frac{r_0^\alpha}{\eta d_e^2} \right)^{\nu - 1/2} \frac{1}{\Gamma(\nu/2)^{1/2}}
$$

(C16)

**The case where $\nu \geq 0$.** If $\beta \geq 1$, or $d \geq \theta + 1$ then $\nu \geq 0$, then using the approximation given by Eq. (B30) in Eq. (C13) gives

$$
\tilde{q}_{\alpha}\left(\gamma_s, 1, s\right) = d_{\nu} \left( \frac{r_0^\alpha}{\eta d_e^2} \right)^{\nu - 1/2} \frac{1}{\Gamma(\nu/2)^{1/2}}
$$

(C17)

which has an analytic inverse given by

$$
q_{\alpha}\left(t\right) = d_{\nu} \left( \frac{r_0^\alpha}{\eta d_e^2} \right)^{\nu - 1/2} \frac{1}{\Gamma(\nu/2)^{1/2}}
$$

(C18)
or can be expressed in terms of dimensionless time $t_{D}$ (Eq. (34)) as

\[
q_{wD}(t) = \frac{c_{wD}}{r_{wD}^{2}} \left( \frac{t_{D}}{t_{wD}} \right) \Gamma \left( \frac{1}{2}, \frac{t_{D}}{t_{wD}} \right).
\]

(C19)

References


