UTD Putnam Seminar: Induction

The Context. The Putnam exam is a spicy test. Solving a problem is like eating spicy food: it's enough that you gave it your best try, and it is no issue if you don't finish.

The Rules. There are too many problems to consider: just pick a few problems you like and play around with them.

You are not allowed to try a problem that you already know how to solve. Otherwise, work on the problems you want to work on.

Stuck on a problem? Try one or more: Work in groups. Try small cases. Plug in smaller numbers. Do examples. Look for patterns. Draw pictures. Use lots of paper. Talk it over. Choose effective notation. Look for symmetry. Divide into cases. Work backwards. Argue by contradiction. Consider extreme cases. Eat a sandwich. Drink water. Modify the problem. Generalize. Don't give up after five minutes. Don't be afraid of a little algebra. Sleep on it if need be. Ask.

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1. Prove that the number $\underbrace{111\ldots11}_{1997}\underbrace{22\ldots22}_{1998}5$ (which has 1997 1s and 1998 2s) is a perfect square.

   [1998 Junior Balkan Math Olympiad #1]

2. Find the sum of $1\cdot 1! + 2\cdot 2! + 3\cdot 3! + \cdots + (n-1)(n-1)! + n\cdot n!$, where $n! = n(n-1)(n-2)\cdots 2\cdot 1$.

   [1969 Canada #6]

3. Prove that $t^{n-1} + t^{1-n} < t^n + t^{-n}$ when $t \neq 1$, $t > 0$ and n is a positive integer.

   [1982 VTRMC #4]

4. Let n be a given positive integer. Show that we can choose numbers $c_k \in \{-1, 1\}$ $(i \le k \le n)$ such that
$$0 \le \sum_{k=1}^{n} c_k \cdot k^2 \le 4.$$

   [2009 Baltic Way #3]

5. Prove that a list can be made of all the subsets of a finite set in such a way that (i) the empty set is first in the list, (ii) each subset occurs exactly once, (iii) each subset in the list is obtained either by adding one element to the preceding subset or by deleting one element of the preceding subset.

   [1968 Putnam A3]

6. Let k be a fixed positive integer. The n-th derivative of $\dfrac{1}{x^k - 1}$ has the form $\dfrac{P_n(x)}{(x^k-1)^{n+1}}$ where $P_n(x)$ is a polynomial. Find $P_n(1)$.

   [2002 Putnam A1]

7. Let $f_0 = f_1 = 1$ and $f_{i+2} = f_{i+1} + f_i$ for all $n \ge 0$. Find all real solutions to the equation
$$x^{2010} = f_{2009}\cdot x + f_{2008}$$

   [2009 Baltic Way #5]

8. You have coins $C_1, C_2, \ldots, C_n$. For each k, C_k is biased so that, when tossed, it has probability $1/(2k+1)$ of falling heads. If the n coins are tossed, what is the probability that the number of heads is odd? Express the answer as a rational function of n.

   [2001 Putnam A2]

9. Prove that after deleting the perfect squares from the list of positive integers the number we find in the nth position is equal to $n + \{\sqrt{n}\}$, where $\{\sqrt{n}\}$ denotes the integer closest to $\sqrt{n}$.

   [1966 Putnam A4]

10. A triangle is called a "parabolic triangle" if its vertices lie on a parabola $y = x^2$. Prove that for every nonnegative integer n, there is an odd number m and a parabolic triangle with vertices at three distinct points with integer coordinates with area $(2^n m)^2$.

    [2010 USAJMO #4]

11. Prove that for every natural number n, and for every real number $x \neq \frac{k\pi}{2^t}$ ($t = 0, 1, \ldots, n$; k any integer)

    $$\frac{1}{\sin 2x} + \frac{1}{\sin 4x} + \cdots + \frac{1}{\sin 2^n x} = \cot x - \cot 2^n x$$

    [1966 IMO #4]

12. Santa Claus has at least n gifts for n children. For $i \in \{1, 2, \ldots, n\}$, the i-th child considers $x_i > 0$ of these items to be desirable. Assume that

    $$\frac{1}{x_1} + \cdots + \frac{1}{x_n} \leq 1.$$

    Prove that Santa Claus can give each child a gift that this child likes.

    [2013 Baltic Way #6]

13. Point O lies on line g; $\overrightarrow{OP_1}$, $\overrightarrow{OP_2}$, …, $\overrightarrow{OP_n}$ are unit vectors such that points $P_1, P_2, \ldots, P_n$ all lie in a plane containing g and on one side of g. Prove that if n is odd,

    $$\left|\overrightarrow{OP_1} + \overrightarrow{OP_2} + \ldots + \overrightarrow{OP_n}\right| \geq 1$$

    Here $\left|\overrightarrow{OM}\right|$ denotes the length of vector $\overrightarrow{OM}$.

    [1973 IMO #1]

14. Let D_n be the determinant of order n of which the element in the ith row and the jth column is the absolute value of the difference of i and j. Show that D_n is equal to

    $$(-1)^{n-1}(n-1)2^{n-2}.$$

    [1969 Putnam A2]

15. For all real numbers x, y define $x \star y = \frac{x+y}{1+xy}$. Evaluate the expression

    $$(\cdots(((2 \star 3) \star 4) \star 5) \star \cdots) \star 1995.$$

    [1995 Balkan MO #1]

16. Let $f_0(x) = e^x$ and $f_{n+1}(x) = x f_n'(x)$ for $n = 0, 1, 2, \ldots$. Show that

    $$\sum_{n=0}^{\infty} \frac{f_n(1)}{n!} = e^e$$

    [1975 Putnam B5]

17. Prove that for any sequence of positive real numbers b_1, b_2, …, b_k,

    $$\frac{1}{b_1+1} + \frac{1}{b_2+1} + \cdots + \frac{1}{b_k+1} \geq \frac{1}{b_1 b_2 \cdots b_k + 1}.$$

18. For each integer $a_0 > 1$, define the sequence $a_0, a_1, a_2, \ldots$ by:

    $$a_{n+1} = \begin{cases} \sqrt{a_n} & \text{if } \sqrt{a_n} \text{ is an integer,} \\ a_n + 3 & \text{otherwise,} \end{cases} \quad \text{for each } n > 0.$$

    Determine all values of a_0 for which there is a number A such that $a_n = A$ for infinitely many values of n.

    [2017 IMO #1]