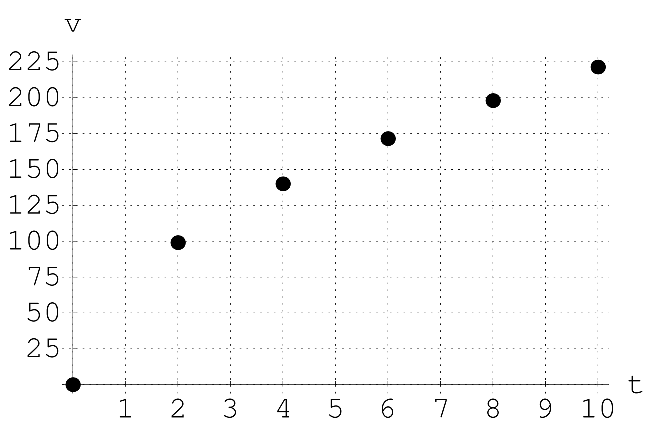
# Section 5.1 – How Do We Measure Distance Traveled?

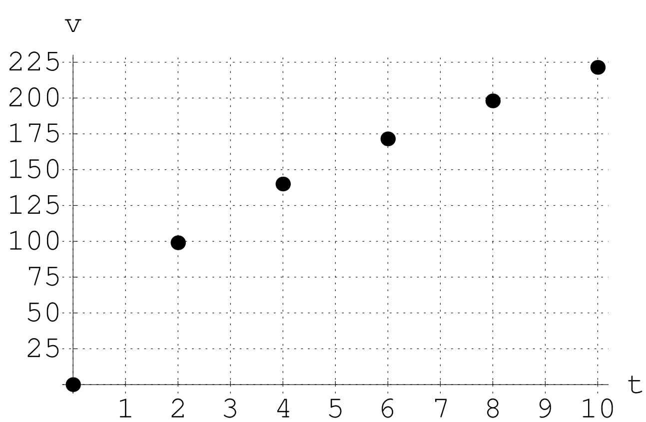
**Example 1.** The following data is gathered as a small plane travels down the runway toward takeoff. How far did the plane travel in the 10 second period? (Give a range of values.)

| time (sec) | 0 | 2 | 4 | 6 | 8 | 10 |
|---|---|---|---|---|---|---|
| velocity (ft/sec) | 0 | 99 | 140 | 171.5 | 198 | 221.4 |



**Example 2.** Suppose that the same small plane as in Example 1 is traveling toward takeoff, but that now, we are given the velocity of the plane every second (as shown in the table below). Give a new range of values representing the distance that the plane could have traveled, and illustrate your estimates with a new rectangles diagram.

| time (sec) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| velocity (ft/sec) | 0 | 75 | 99 | 125 | 140 | 162 | 171.5 | 182 | 198 | 215 | 221.4 |



# Section 5.1 – How Do We Measure Distance Traveled?

Officer "Tommy Boy" Hopkins and his evil brother Angry Mike are at it again. It seems that a hit and run accident took place the other night at midnight near Galloping Gulch, 40 miles from town. Farmer Bob claims that he witnessed the offending car traveling at approximately 60 miles per hour when his prize-winning Holstein, Bessie, was hit. Tommy Boy had figured his evil brother was up to no good that night and had planted a hidden camera in Angry Mike's car before Angry Mike left town at 11:00 p.m. Unfortunately, the resolution was so bad that he could only read the speedometer and the dashboard clock on the night in question. Tommy Boy states that not only was Angry Mike out driving that night, but at the time in question, was traveling 60 miles per hour and his car seemed to hit something. In his defense, Mike says that he had not yet arrived at Galloping Gulch but that he does remember hitting a nasty bump around midnight.
Your goal is to figure out which brother is correct. The following questions will help you achieve this goal. You may assume that Angry Mike's speed never decreased over the course of the entire trip.

| Time | 11:00 pm | 11:10 pm | 11:20 pm | 11:30 pm | 11:40 pm | 11:50 pm | Midnight |
|---|---|---|---|---|---|---|---|
| Velocity, mph | 0 | 30 | 40 | 45 | 50 | 53 | 60 |

Use only the data provided above to answer the following questions. Put all your work on a separate sheet of paper.

1. What is the maximum possible distance that Angry Mike could have traveled between 11:00 p.m. and midnight? Clearly show your calculations.

2. What is the minimum possible distance that Angry Mike could have traveled between 11:00 p.m. and midnight? Clearly show your calculations.

3. Now, assume that you are an investigating officer writing up a report of the incident. Your report should be a paragraph long and contain AT MINIMUM the following:
   - A description of the incident and the available evidence. Please do not copy the description above but rather rephrase in your own words.
   - A description of the calculations that you made in questions (1) and (2) above, including an explanation of how you know that they really do give the maximum and minimum distance traveled by Angry Mike's car over the time in question.
   - A discussion of whether or not there is sufficient evidence to conclude that Angry Mike reached Galloping Gulch at or before midnight. Remember, use only the data in the above table to support any claims that you make.

Part 2: Now, consider the following more complete list of data, also compiled that night. Continue to assume that Angry Mike's speed never decreases over the entire one-hour period. Answer the questions again using the more complete data. As before, put all of your work on a separate sheet of paper.

| Time | 11:00 pm | 11:05 pm | 11:10 pm | 11:15 pm | 11:20 pm | 11:25 pm | 11:30 pm |
|---|---|---|---|---|---|---|---|
| Velocity, mph | 0 | 25 | 30 | 38 | 40 | 43 | 45 |
| Time | 11:35 pm | 11:40 pm | 11:45 pm | 11:50 pm | 11:55 pm | Midnight | |
| Velocity, mph | 49 | 50 | 51 | 53 | 58 | 60 | |

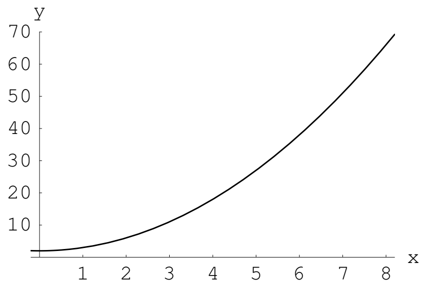
4. What is the maximum possible distance that Angry Mike could have traveled between 11:00 p.m. and midnight? Clearly show your calculations.

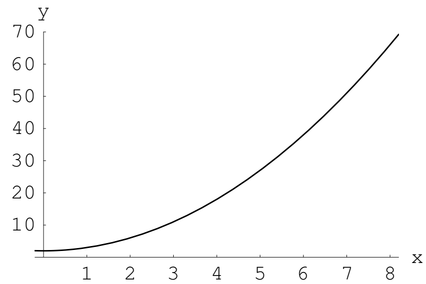
5. What is the minimum possible distance that Angry Mike could have traveled between 11:00 p.m. and midnight? Clearly show your calculations.

6. Write the same type of paragraph that you did for number 3, basing your conclusions on the more complete data. How do your conclusions change, if at all?

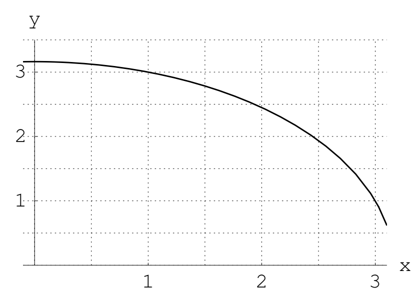
# Section 5.2 – The Definite Integral

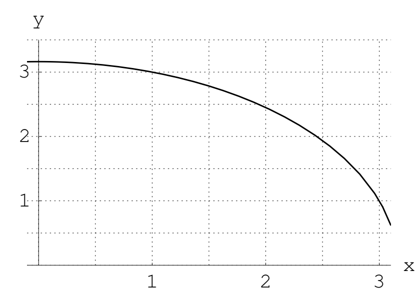
**Example 1.** Use a left sum and a right sum, with $n = 4$, to estimate the area under the curve $f(x) = x^2 + 2$ on the interval $0 \le x \le 8$.



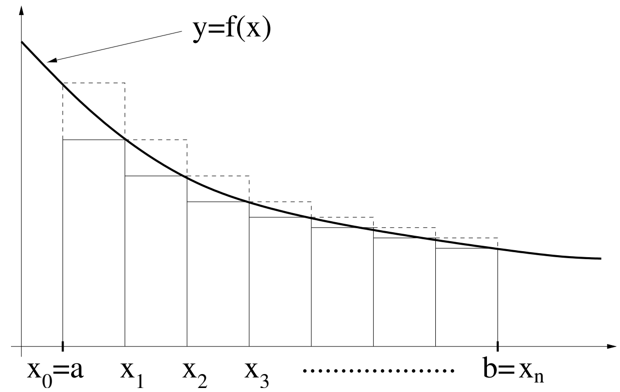




**Example 2.** Use a left sum and a right sum, with $n = 3$, to estimate the area under the curve $g(x)$ (shown below) on the interval $0 \le x \le 3$.



**Goal.** Describe, in general, a way to find the *exact* area under a curve.




$n$ = number of rectangles

$\Delta x$ = width of one rectangle

Left Sum =

Right Sum =

**Definition.** The *definite integral* of $f(x)$ from $a$ to $b$ is the limit of the left and right hand sums as the number of rectangles approaches infinity. We write

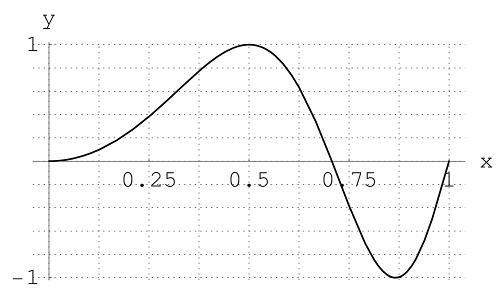
$$\int_a^b f(x)\,dx = \lim_{n\to\infty}[\text{Right-hand sums}] = \lim_{n\to\infty}[\text{Left-hand sums}]$$

**Example 3.** Use the results of Examples 1 and 2 to give your best estimate of $\int_0^8 (x^2+2)\,dx$ and $\int_0^3 g(x)\,dx$. Then, explain what these integrals represents geometrically.

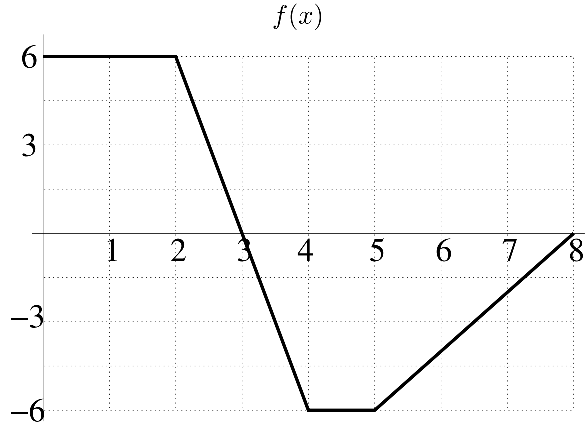
**Example 4.** Use a "middlesum" with $n = 4$ to estimate the value of

$$\int_0^1 h(x)\,dx,$$

where the graph of $h$ is given to the right.



**Example 5.** Let $f$ be the graph of the function shown to the right. Calculate each of the integrals that follow *exactly.*




$$\int_0^1 f(x)\,dx \quad =$$

$$\int_5^8 f(x)\,dx \quad =$$

$$\int_0^2 f(x)\,dx \quad =$$

$$\int_0^5 f(x)\,dx \quad =$$

$$\int_0^3 f(x)\,dx \quad =$$

$$\int_2^4 f(x)\,dx \quad =$$

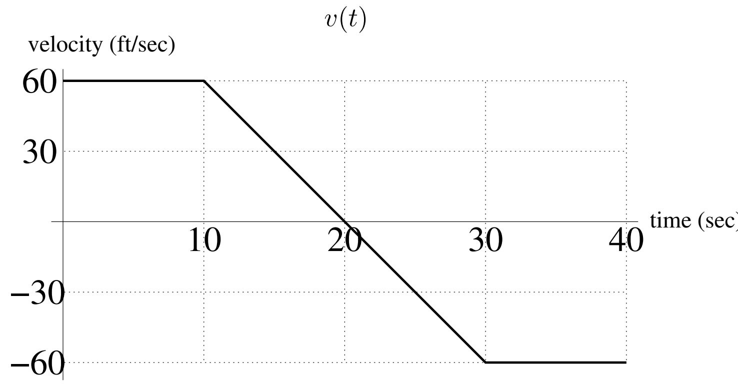
$$\int_4^5 f(x)\,dx \quad =$$

**Example 6.** Let $f$ be the function defined on $0 \le x \le 12$, some of whose values are shown in the table below. Estimate the value of $\int_0^{12} f(x)\,dx$.

| $x$ | 0 | 3 | 6 | 9 | 12 |
|---|---|---|---|---|---|
| $f(x)$ | 20 | 10 | 5 | 2 | 1 |

# Section 5.3 – The Fundamental Theorem and Interpretations

**Example.** Let $s(t)$ be the position, in feet, of a car along a straight east/west highway at time $t$ seconds. Positive values of $s$ indicate eastward displacement of the car from home, and negative values indicate westward displacement. Let $v(t)$ represent the velocity of this same car, in feet per second, at time $t$ seconds (see graph to the right).




1. Use the velocity graph above to help you fill in the chart below.

| $t$ | 0 | 10 | 20 | 30 | 40 |
|---|---|---|---|---|---|
| $s(t)$ | 0 | | | | |

2. Fill in the chart below.

| Integral of Velocity | Change in Position |
|---|---|
| $\int_0^{10} v(t)\,dt =$ | $s(10) - s(0) =$ |
| $\int_0^{20} v(t)\,dt =$ | $s(20) - s(0) =$ |
| $\int_{20}^{40} v(t)\,dt =$ | $s(40) - s(20) =$ |
| $\int_0^{40} v(t)\,dt =$ | $s(40) - s(0) =$ |

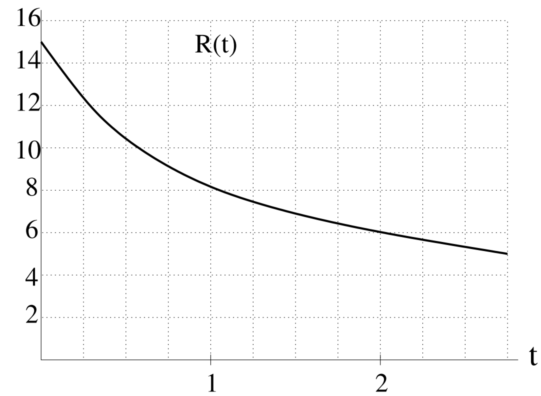
**Total Change Principle.** Let $F(t)$ be some quantity with a continuous rate of change $F'(t)$. Then

$$\int_a^b F'(t)\,dt = F(b) - F(a).$$

In other words, the integral of a rate of change gives total change.

## Exercises.

1. Water is leaking out of a tank at a rate of $R(t)$ gallons per hour, where $t$ is measured in hours (see graph to the right).




   (a) Write a definite integral that expresses the total amount of water that leaks out in the first two hours.

   (b) On the graph to the right, shade the region whose area represents the total amount of water that leaks out in the first two hours.

   (c) Give your best estimate of the amount of water that leaks out of the tank in the first two hours.

2. (Adapted from *Hughes-Hallett, et. al.*) A news broadcast in early 1993 said that the average American's annual income was changing at a rate of $r(t)$ dollars per month (see the table below), where $t$ is the number of months after January 1, 1993. Estimate the amount that the average American's income changed in 1993.

| $t$ (months) | 0 | 2 | 4 | 6 | 8 | 10 | 12 |
|---|---|---|---|---|---|---|---|
| $r(t)$ (dollars per month) | 40.00 | 40.16 | 40.32 | 40.48 | 40.64 | 40.81 | 40.97 |

3. A can of soda is put into a refrigerator to cool. The rate at which the temperature of the soda is changing is given by

$$f(t) = -25e^{-2t} \text{ degrees Fahrenheit per hour,}$$

   where $t$ represents the time (in hours) after the soda was placed in the refrigerator.

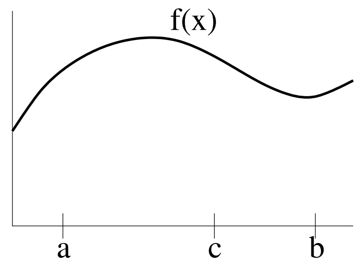
   (a) How fast is the can of soda cooling after 1 hour has passed? Include the appropriate units with your answer.

   (b) If the temperature of the can of soda is 60°F when it is placed in the refrigerator, estimate the temperature of the can of soda after 3 hours have passed.

# Section 5.4 – Theorems About Definite Integrals

## Properties of the Integral

1. $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx =$ ____________________




2. $\int_a^a f(x)\,dx =$ ____________________

3. $\int_b^a f(x)\,dx =$ ____________________

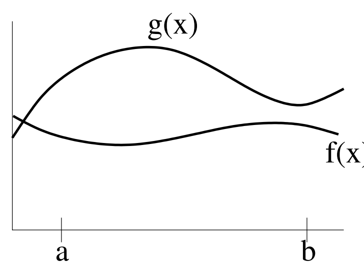
4. $\int_a^b [f(x) \pm g(x)]\,dx =$ ______________________________________________________

5. $\int_a^b cf(x)\,dx =$ ________________________________________ (if $c$ is constant)

## Comparison Properties

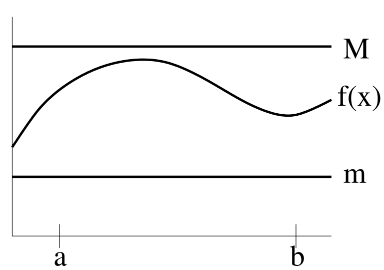
6. If $f(x) \leq g(x)$ for all $a \leq x \leq b$, then

   ____________________________________________________.




7. If $m \leq f(x) \leq M$ for all $a \leq x \leq b$, then

   ____________________________________________________.



## Exercises.

1. Assume that $\int_a^b f(x)\,dx = 9$, $\int_b^c f(x)\,dx = -7$, $\int_a^b (f(x))^2\,dx = 36$, and $\int_a^b g(x)\,dx = -2$. Use this information to calculate each of the following.

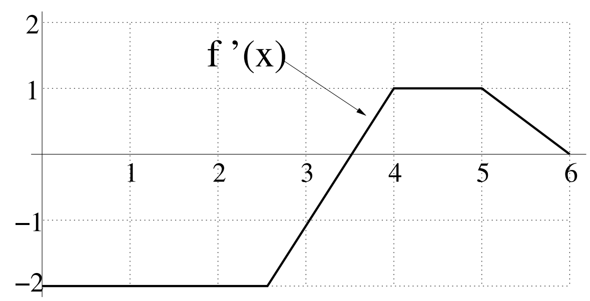
   (a) $\int_a^c f(x)\,dx$

   (b) $\int_a^b (f(x))^2\,dx - \left(\int_a^b f(x)\,dx\right)^2$

   (c) $\int_a^b (2f(x) - g(x))\,dx$

2. (a) Given to the right is the graph of $f'(x)$, the DERIVATIVE of a function. Given that $f(0) = 4$, fill in the chart below.

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 4 | | | | | | |




   (b) Where does $f$ attain its minimum value? What is this minimum value?

3. Let $f'(x) = \sin(5x^2/3)$ (see graph to the right).

   (a) Which is larger, $f(1) - f(0)$ or $f(2) - f(1)$?

   (b) Which is larger, $f(1.5) - f(1)$ or $f(2) - f(1.5)$?

   (c) Rank the following quantities in order from smallest to largest:

$$f(0),\ f(0.5),\ f(1),\ f(1.5),\ f(2)$$

$f'(x) = \sin(5x^2/3)$