# Section 4.1 – Using the First and Second Derivatives

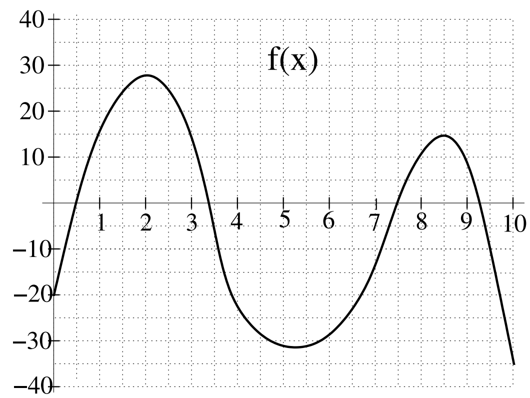
**Definitions.** Let $f$ be a function.

1. A *critical point* of $f$ is a point $p$ in the domain of $f$ such that either $f'(p) = 0$ or $f'(p)$ is undefined.
2. We say that $f$ has a *local minimum* at $p$ if $f(p)$ is less than or equal to the values of $f$ for points near $p$.
3. We say that $f$ has a *local maximum* at $p$ if $f(p)$ is greater than or equal to the values of $f$ for points near $p$.
4. An *inflection point* of $f$ is a point at which the function $f$ changes concavity.

**Example.** Given to the right is the graph of a function $f$.

(a) Estimate the critical point(s) of $f$.

(b) Estimate the inflection point(s) of $f$.




(c) Does $f$ have any local maximum or local minimum values? If so, list them, making it clear which are which.

**First Derivative Test.** Suppose that $p$ is a critical point of a continuous function $f$.

1. If $f'$ changes from negative to positive at $p$, then $f$ has a ____________________ at $x = p$.
2. If $f'$ changes from positive to negative at $p$, then $f$ has a ____________________ at $x = p$.

**Second Derivative Test.**

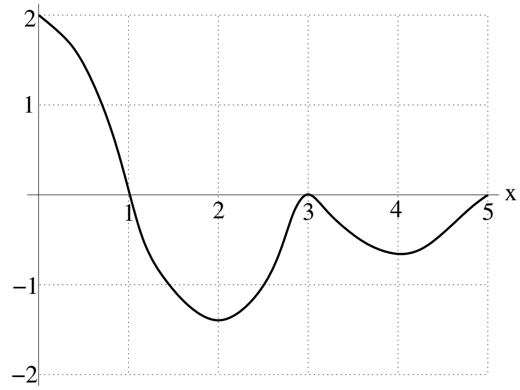
1. If $f'(p) = 0$ and $f''(p) > 0$, then $f$ has a ____________________ at $x = p$.
2. If $f'(p) = 0$ and $f''(p) < 0$, then $f$ has a ____________________ at $x = p$.

**EXERCISES.** Please do the following on a separate sheet of paper.

1. Let $f(x) = x^{2/3}(4-x)^{1/3}$.
   (a) Given that $f'(x) = \dfrac{8-3x}{3x^{1/3}(4-x)^{2/3}}$, find the intervals on which $f$ is increasing/decreasing.
   (b) Given that $f''(x) = \dfrac{-32}{9x^{4/3}(4-x)^{5/3}}$, find the intervals on which $f$ is concave up/concave down.
   (c) Find all local maxima, local minima, and inflection points of $f$.

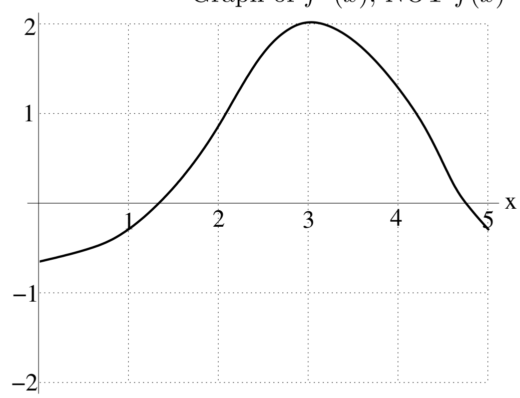
2. Given to the right is the graph of the DERIVATIVE of a function. Use this graph to help you answer the following questions about the ORIGINAL FUNCTION $f$.
   (a) What are the critical points of $f$?
   (b) Where is $f$ increasing? decreasing?
   (c) Does $f$ have any local maxima? If so, where?
   (d) Does $f$ have any local minima? If so, where?
   (e) Where is $f$ concave up? concave down?

Graph of $f'(x)$, NOT $f(x)$



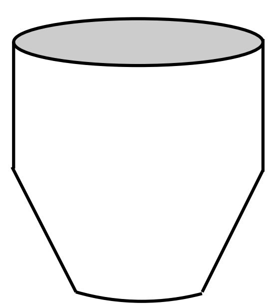

3. Given to the right is the graph of the SECOND DERIVATIVE of a function. Use this graph to help you answer the following questions about the ORIGINAL FUNCTION $f$.
   (a) Where is $f$ concave up? concave down?
   (b) Does $f$ have any inflection points? If so, where?

Graph of $f''(x)$, NOT $f(x)$




4. Sketch the graph of ONE FUNCTION $f$ that has ALL of the following properties.
   - $f$ is continuous everywhere.
   - $f(0) = 2$.
   - $f'(x) = 0$ for $-4 \le x \le -2$.
     $f'(x) < 0$ for $0 < x < 1$.
     $f'(x) > 0$ for $-2 < x < 0$ and for $1 < x < 4$.
   - $f''(x) > 0$ for $-2 < x < 0$ and for $0 < x < 2$.
     $f''(x) < 0$ for $2 < x < 4$
   - $\lim_{x\to\infty} f(x) = -2$.

5. If water is flowing at a constant rate (i.e. constant volume per unit time) into the urn pictured to the right, sketch a graph of the depth of the water in the urn against time. Mark on the graph the time at which the water reaches the corner of the urn.

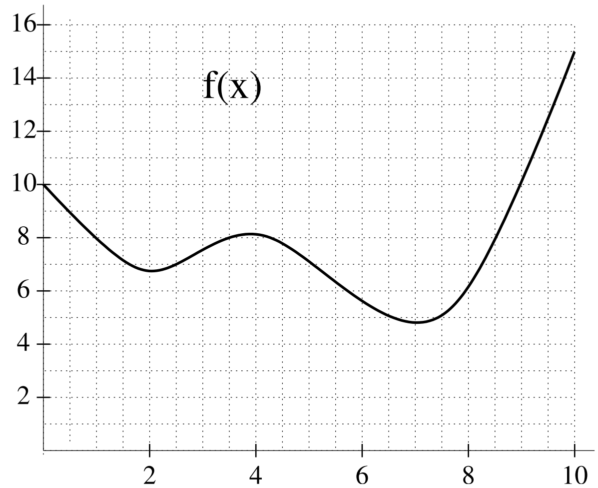


# Section 4.2 – Optimization

**Some Definitions.** Let $f$ be a function.

1. $f$ has a *global maximum* at $x = p$ if $f(p)$ is greater than or equal to all output values of $f$.
2. $f$ has a *global minimum* at $x = p$ if $f(p)$ is less than or equal to all output values of $f$.
3. *Optimization* refers to the process of finding the global maximum or global minimum of a function.

**Example.** For the function $f$ given below, locate all local and global maxima and minima on the interval $[0, 10]$.




**General Rule.** To find the global maximum and the global minimum of a continuous function on a closed interval (i.e., an interval that contains its endpoints), compare the output values of the function at the following locations:

1.

2.

## Exercises

1. Find the global maximum and global minimum value of $f(x) = x + \frac{3}{x}$ on the interval $[1, 4]$.

2. (Taken from *Hughes-Hallett, et. al.*) When you cough, your windpipe contracts. The speed, $v$, at which the air comes out depends on the radius, $r$, of your windpipe. If $R$ is the normal (rest) radius of your windpipe, then for $0 \leq r \leq R$, the speed is given by $v = a(R - r)r^2$, where $a$ is a positive constant. What value of $r$ maximizes the speed?

3. (Taken from *Hughes-Hallett, et. al.*) The potential energy, $U$, of a particle moving along the $x$-axis is given by

$$U = b\left(\frac{a^2}{x^2} - \frac{a}{x}\right),$$

where $a$ and $b$ are positive constants and $x > 0$. What value of $x$ minimizes the potential energy?

4. Let $f(x) = xe^{-x^2}$.

   (a) Locate all local maximum and all local minimum values of $f$.

   (b) Find the global maximum and the global minimum values of $f$ on the interval $[0, 2]$.

5. Give an example of a function that does not have a global maximum or a global minimum value.

# Section 4.2 – Optimization

1. Sketch a continuous, differentiable graph with the following properties:
   - local minima at 2 and 4
   - global minimum at 2
   - local and global maximum at 3
   - no other extrema

2. A warehouse orders and stores boxes. The cost of storing boxes is proportional to $q$, the quantity ordered. The cost of ordering boxes is proportional to $1/q$, because the warehouse gets a price cut for larger orders. The total cost of operating the warehouse is the sum of ordering costs and storage costs. What value of $q$ gives the minimum cost?

3. Find the best possible bounds for $f(t) = t + \sin t$ for $t$ between 0 and $2\pi$.

# Section 4.3 – Optimization and Modeling

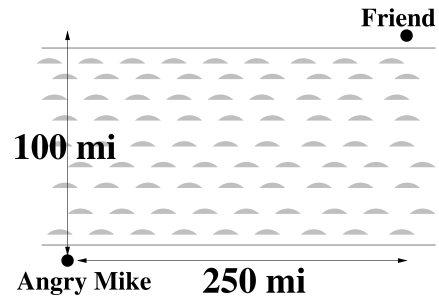
1. A farmer wants to fence a rectangular grazing area along a straight river (no fence is needed along the river). There are 1700 total feet of fencing available. What dimensions (length and width) will maximize the grazing area?

2. A box with an open top of fixed volume $V$ with a square base is to be constructed. Find the dimensions of the box that minimize the amount of material used in its construction.

3. A metal can manufacturer needs to build cylindrical cans with volume 300 cubic centimeters. The material for the side of a can costs 0.03 cents per $\text{cm}^2$, and the material for the bottom and top of the can costs 0.06 cents per $\text{cm}^2$. What is the cost of the least expensive can that can be built?

# Section 4.3 – Optimization and Modeling with Angry Mike

Angry Mike is on the run. He finally couldn't outsmart his brother sheriff Tommy Boy Hopkins any longer and there is a warrant out for his arrest. In his desperation Angry Mike has hijacked a dirt bike and is making a run for the state line. A friend is waiting for him 100 miles north and 250 miles east with a car. Unfortunately, a range of hills stretches all the way to the north and east from Angry Mike's starting position. He knows that he can go 70 mph as long as he stays out of the hills, but that he can go no faster than 40 mph in the hills.



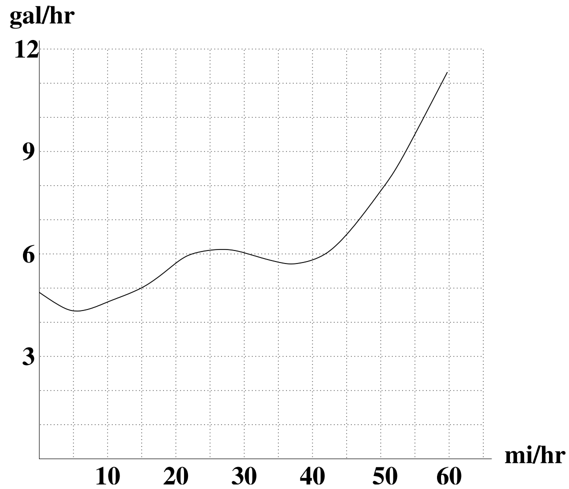

1. Draw three different possible routes (straight lines, or line segments) Angry Mike can take. One of them should maximize the part of the trip outside the hills, one of them should maximize the part of the trip in the hills.

2. Compute the total driving time for those three routes.

3. There are many more possible routes Mike could take. Out of all of them, which one should Angry Mike take to get to his waiting friend as fast as possible?

4. How long is the part of the trip that Angry Mike has to ride through the hills?

Meanwhile, Tommy Boy is in hot pursuit and facing an entirely different problem. He is driving a county-issued cross country vehicle. The hills are absolutely no problem, but he can only use one tank of gas plus one reserve tank to follow Angry Mike. The tank of his vehicle holds 25 gallons; the reserve tank holds 16 gallons. Since Tommy Boy knows his brother and his friends, he quickly figured out where Angry Mike is going, and so he is taking the most direct route through the hills which is about 270 miles long. The fuel consumption (in gallons per hour) of Tommy's vehicle as a function of speed (in mph) is given in the graph to the right. Tommy Boy knows that he can maximize the fuel efficiency of his usual patrol car (in miles per gallon) by going 50 mph. He figures that the cross country vehicle is close enough to a normal car to have the same properties.




1. Is he right?

2. Will he actually make it to Angry Mike's destination at a driving speed of 50 mph?

3. How fast should he go to maximize fuel efficiency?

4. How much gas would he have to spare if he drove with optimal speed?

# Section 4.4 – Families of Functions and Modeling

1. (Taken from *Hughes-Hallett, et. al.*) The number, $N$, of people who have heard a rumor spread by mass media at time, $t$, is given by $N(t) = a(1 - \mathrm{e}^{-kt})$. There are 200,000 people in the population who hear the rumor eventually. If 10% of them heard it the first day, find $a$ and $k$, assuming that $t$ is measured in days.

2. Let $f(x) = x^4 - ax^2$.

   (a) Find all possible critical points of $f$ in terms of $a$.

   (b) If $a < 0$, how many critical points does $f$ have?

   (c) If $a > 0$, find the $x$ and $y$ coordinates of all critical points of $f$.

   (d) Find a value of $a$ such that the two local minima of $f$ occur at $x = \pm 2$.

3. Let $f(x) = ax\mathrm{e}^{-bx}$. ASSUME THAT $a$ AND $b$ ARE BOTH POSITIVE.

   (a) Find all inflection points of $f$ in terms of $a$ and $b$.

   (b) Find $a$ and $b$ so that the inflection point of $f$ occurs at $(1, 2)$.

# Section 4.7 – L'Hopital's Rule, Growth, and Dominance

1. Find each of the following limits **exactly**.

   (a) $\lim_{x \to 0} \frac{\sin x}{x}$

   (b) $\lim_{x \to \infty} \frac{\ln x}{x}$

   (c) $\lim_{x \to 0} \frac{x}{e^x}$

   (d) $\lim_{x \to \infty} \frac{\ln(\ln x)}{\sqrt{x}}$

   (e) $\lim_{x \to 1^+} (x-1) \tan\left(\frac{\pi}{2} x\right)$

**Definition.** We say that a function $g$ *dominates* a function $f$ as $x \to \infty$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} =$ ______.

2. For each of the following, determine which function dominates as $x \to \infty$.

   (a) $1000x^2$ and $x^3$

   (b) $e^{0.1x}$ and $x^3$

3. Use the graph to the right to determine the sign of the limit $\lim_{x\to a} \frac{f(x)}{g(x)}$. Briefly explain how you determined your answer. (Note: You may assume that the limit exists and that all derivatives of $f$ and $g$ exist.)