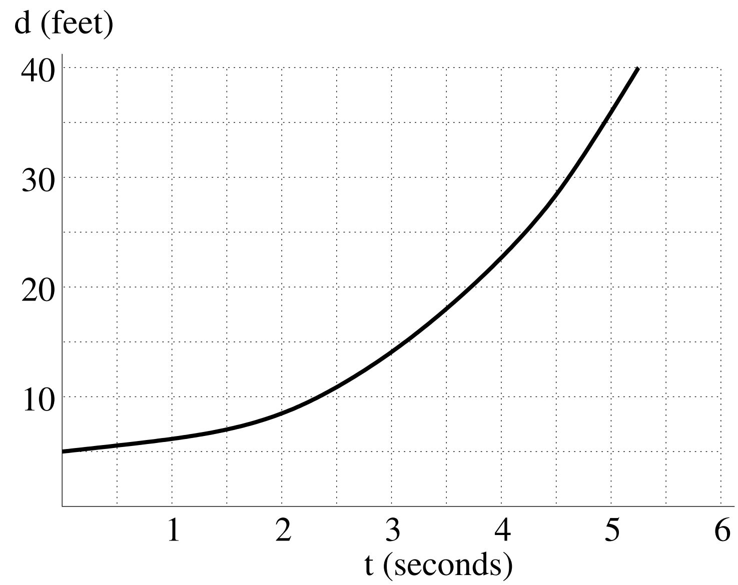
Section 2.1 – How Do We Measure Speed?

1. (a) Given to the right is the graph of the position of a runner as a function of time. Use the graph to complete each of the following.



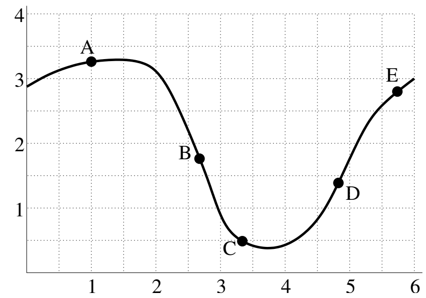
Time Interval	Average Velocity of Runner
$2 \leq t \leq 5$	
$2 \leq t \leq 3.5$	
$2 \leq t \leq 2.5$	

(b) Estimate the instantaneous velocity of the runner at $t = 2$ seconds.

2. For the function shown to the right, answer the following questions:

   (a) At what points is the slope of the curve positive?

   (b) At what points is the slope of the curve negative?



   (c) Rank the slopes at the 5 points in order from smallest to largest.

3. In a time of t seconds, a particle moves a distance of s meters from its starting point, where $s = f(t) = t^2 + 1$.

   (a) Find the average velocity between $t = 2$ and $t = 2 + h$ if $h = 0.1, h = 0.01$, and $h = 0.001$. (That is, compute the average velocity over 3 different time intervals).

   (b) Now, give your best estimate of the instantaneous velocity of the particle at $t = 2$.

4. The position of a car traveling along a straight east/west highway at various times is shown in the table below. Positive values of d indicate that the car is east of its starting point, while negative values of d indicate that the car is west of its starting point.

t (hours)	1	2	3	4	5
d (miles)	40	-10	20	90	-50

   Calculate the average velocity of the car on the following two time intervals: (a) between 1 and 2 hours, (b) between 2 and 4 hours. What does a positive velocity mean? What does a negative velocity mean?

Section 2.1 – How Do We Measure Speed - Angry Mike

Officer Tom "Tommy Boy" Hopkins and his long-time adversary and evil brother Mike, known throughout the county as "Angry Mike," are having another one of their disputes. Angry Mike decides to show off his love for lawlessness by speeding past the police station, but unbeknownst to him, Tommy Boy has set up an elaborate system of invisible laser beams and timing devices along the road passing the police station. As a result, Tommy Boy and his colleagues are able to construct a graph of Angry Mike's distance traveled, x, as a function of time, t (shown on the following page).

Alas, the case of Hopkins vs. Hopkins goes to trial, and you're playing the part of the prosecution's star witness.

1. Compute the average velocities (in miles per hour) on each of the following intervals.

   (a) 0 seconds $\leq t \leq$ 2 seconds

   (b) 2 seconds $\leq t \leq$ 4 seconds

   (c) 4 seconds $\leq t \leq$ 6 seconds

   (d) 6 seconds $\leq t \leq$ 8 seconds

   (e) 8 seconds $\leq t \leq$ 10 seconds

   (f) 10 seconds $\leq t \leq$ 12 seconds

2. In opening arguments, Angry Mike's defense attorney looks at the data on the graph and says that his client traveled less than 1190 feet in 12 seconds, giving a top possible velocity of under 68 miles per hour. He therefore claims that his client was traveling cautiously below the speed limit for the entire 12-second period. Is the attorney's claim accurate? Explain, based on your own calculations.

3. Estimate Angry Mike's maximum velocity during the 12-second interval. Explain how you got this number. How can you tell **from the graph** where Angry Mike is traveling the fastest?

4. The defense attorney, in response to your testimony, claims that since all your velocity calculations are averages over time intervals, rather than instantaneous velocities, that you can't prove that Angry Mike was speeding at any one given time. How would you respond to this statement? How would you justify your position to the jury?

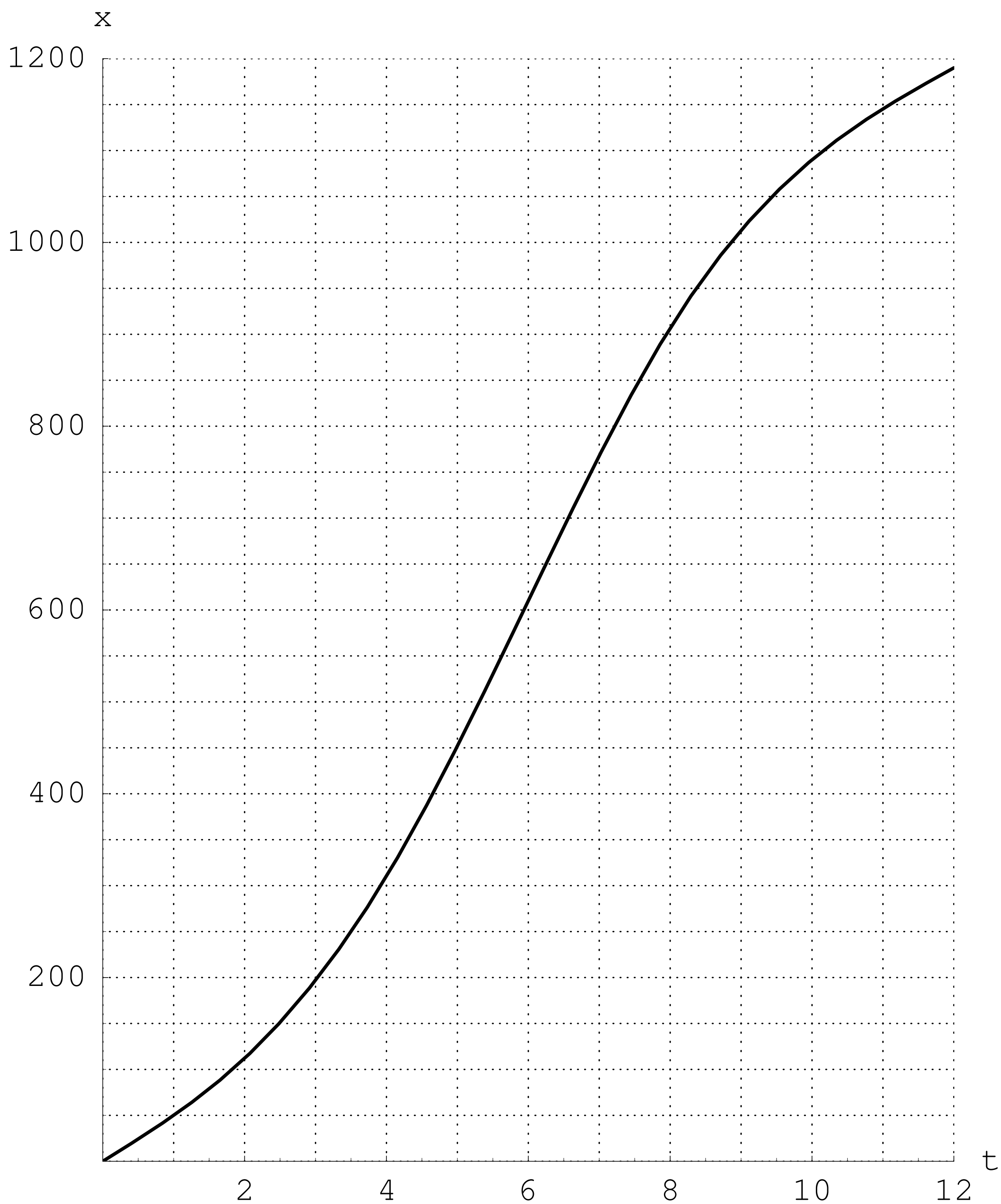
5. Finally, suppose (just this once!) that you're working for the defense attorney's team. As part of the closing argument, to show what a decent, law-abiding, courteous, and cautious citizen Angry Mike is, give your best estimate (as low as you can make it from the data) of his velocity (in miles per hour) at $t = 12$ seconds. Explain why you think your answer is a good estimate of his velocity right at $t = 12$ seconds.

Final Topics for Thought

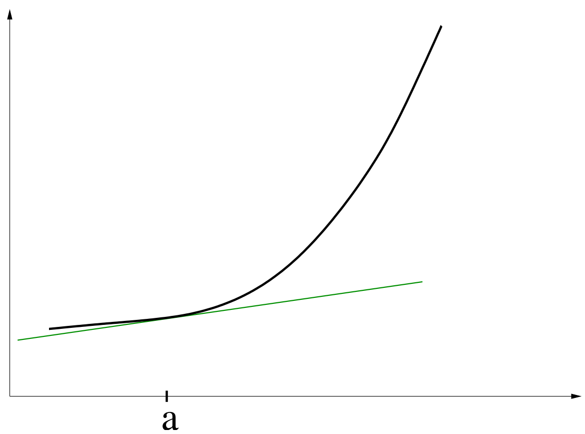
- What geometrical quantity on a graph of distance versus time of a moving object represents the velocity of the object?

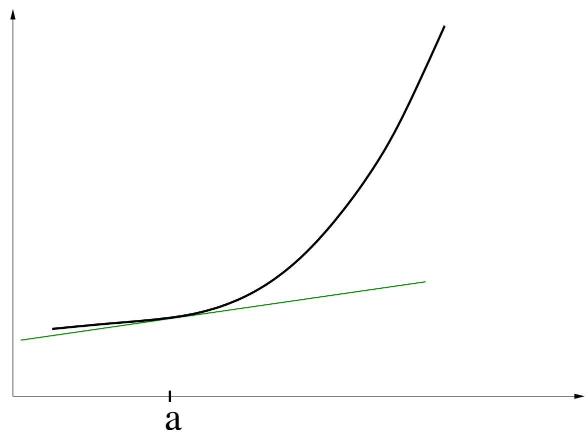
- What's the difference between average velocity and instantaneous velocity? How would you go about estimating instantaneous velocity?

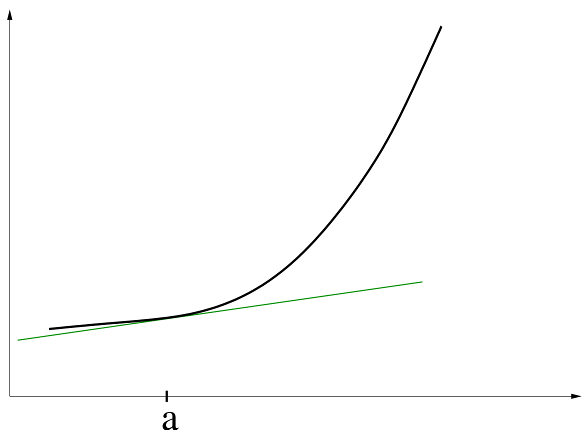
- What's the difference between velocity and speed?


x
1200
1000
800
600
400
200
2
4
6
8
10
12
t

Section 2.2 – The Derivative at a Point



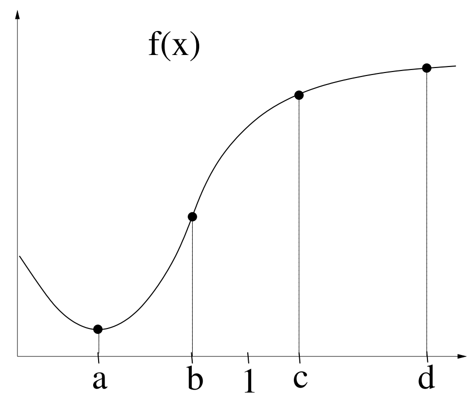




Definition. The *average rate of change* of f over the interval a to $a+h$ is given by

Definition. The instantaneous rate of change of f at a, called the *derivative of f at a*, is given by

1. Given to the right is the graph of a function f.



   (a) Rank the following quantities in order from smallest to largest: $f(0),\ f(a),\ f(b),\ f(1),\ f(c),\ f(d)$

   (b) Rank the following quantities in order from smallest to largest: $f'(0),\ f'(a),\ f'(b),\ f'(1),\ f'(c),\ f'(d)$

2. Let $f(x) = \ln x$. Estimate $f'(3)$ accurate to 2 decimal places.

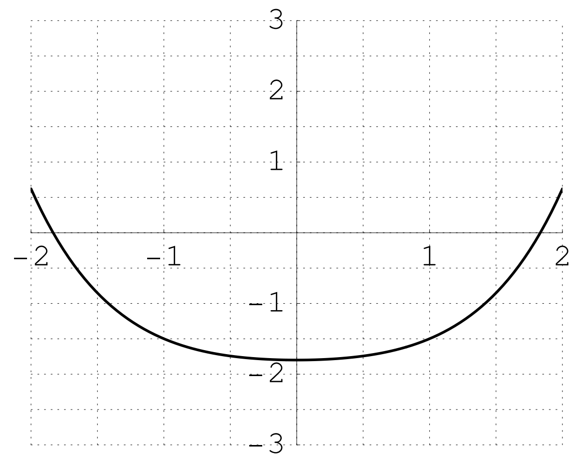
3. Let $f(x) = x^2 - 3x - 2$.

   (a) Use algebra to find $f'(x)$ at $x = 2$.

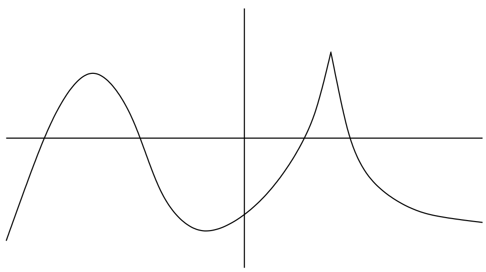
   (b) Find the equation of the tangent line to f at $x = 2$.

Section 2.3 – The Derivative Function

1. To the right you are given the graph of a function $f(x)$. With a straightedge, draw in tangent lines to $f(x)$ at every half unit and estimate their slope from the grid. Use your slopes to draw a graph of $f'(x)$ on the same set of axes.



2. Given the graph of $f(x)$ below, sketch a rough graph of $f'(x)$.



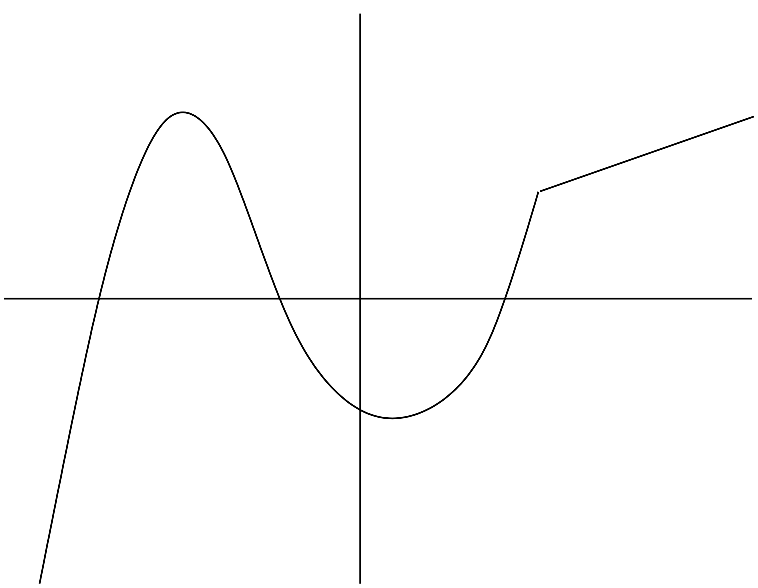
Notes: $f'(x) = 0$ (i.e. $f'(x)$ crosses the x-axis) when ____________________.

$f'(x) > 0$ (i.e. $f'(x)$ is above the x-axis) when ____________________.

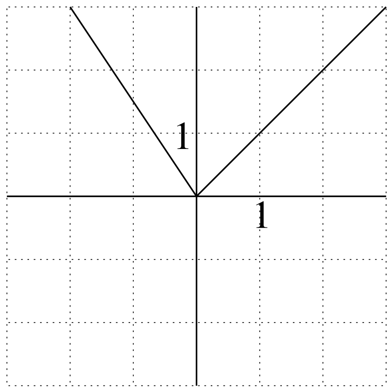
$f'(x) < 0$ (i.e. $f'(x)$ is below the x-axis) when ____________________.

If f has a cusp (sharp point) at x, then ____________________.

3. Given below is the graph of a function $y = f(x)$. Sketch a rough graph of $f'(x)$ and of $f''(x)$ on the same set of axes. Use different colors so you can tell the functions apart.



4. Given the graph of $f(x)$ below, sketch an accurate graph of $f'(x)$.



5. Use algebra (that is, use the **limit definition** of the derivative) to find a formula for $f'(x)$ for each of the following functions.

   (a) $f(x) = \frac{2}{x}$

   (b) $f(x) = \sqrt{x}$

Section 2.4 – Interpretations of the Derivative

1. Let $f(p)$ represent the daily demand for San Francisco '49ers T-shirts when the price for a shirt is p dollars. In other words, $f(p)$ gives the number of shirts purchased daily if the selling price is p dollars.

   (a) Is f increasing or decreasing?

   (b) What are the units of p, $f(p)$, and $f'(p)$?

   (c) Explain, in terms of shirts and dollars, the practical meaning of the following:

      i. $f(20) = 150$

      ii. $f'(20) = -5$

      iii. $f(30)$

   (d) Let d represent demand. Then $d = f(p)$, so the function f takes ______________ as an input and gives ______________ as an output. On the other hand, the inverse function f^{-1} takes ______________ as an input and gives ______________ as an output, so $f^{-1}(\_\_\_\_\_) = \_\_\_\_\_$.

   (e) Give practical interpretations of $f(25)$ and $f^{-1}(25)$.

2. (Taken from *Hughes-Hallett, et. al.*) If t is the number of years since 1993, the population, P, of China, in billions, can be approximated by the function

$$P = f(t) = 1.15(1.014)^t.$$

   (a) Calculate and interpret $f(6)$ in the context of this problem.

   (b) Use the table method to estimate $\frac{dP}{dt}$ at $t = 6$, and give an interpretation of this number in the context of this problem.

3. Between noon and 6 p.m., the temperature in a town rises continually, but rises at its quickest around 3 p.m., and slowest around noon and 6 p.m.

   (a) Sketch a possible graph of $H = f(t)$, where H is the temperature in the town (in degrees Fahrenheit) and t represents the time (in hours) after 12:00 noon.

   (b) Explain, in terms of degrees and hours, what each of the following represents:

   (i) $f'(2)$ (ii) $f'(3) = 7$ $f(4) = 40$ $f'(4) = 1$

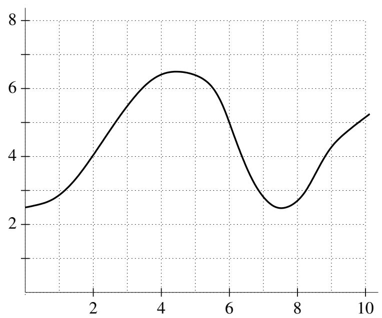
   (c) Use the statements given in parts (iii) and (iv) from above to estimate the temperature in the town at 5:30 p.m. Is the actual temperature higher or lower than the estimate?

Section 2.5 – The Second Derivative

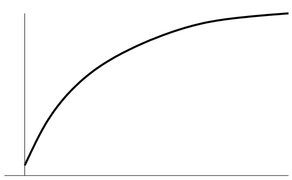
A function f is said to be...

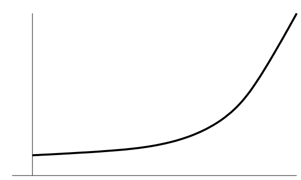
1. *increasing* on an interval if its graph rises from left to right on that interval.
2. *decreasing* on an interval if its graph falls from left to right on that interval.
3. *concave up* on an interval if its graph is shaped like part, or all, of a right-side up bowl on that interval.
4. *concave down* on an interval if its graph is shaped like part, or all, of an upside down bowl on that interval.

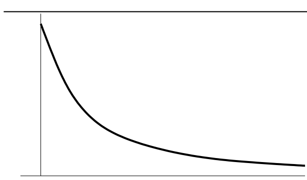
Example 1. Given below is the graph of a function f. Estimate the intervals on which f is increasing, decreasing, concave up, and concave down.

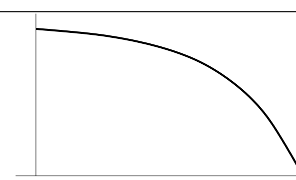


Example 2. Consider the four functions given below.







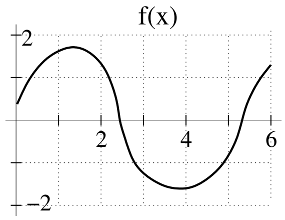


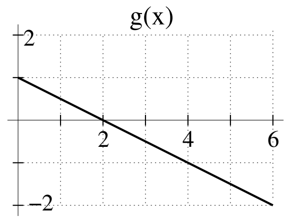
Summary of Key Facts. Assume that f is a function such that f' and f'' both exist.

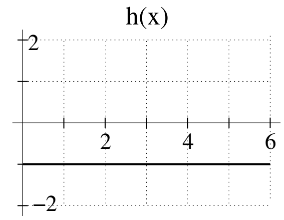
1. If f is increasing on an interval, then ______________ on that interval.
2. If f is decreasing on an interval, then ______________ on that interval.
3. If f is concave up on an interval, then ______________ on that interval.
4. If f is concave down on an interval, then ______________ on that interval.

Practice Problems

1. Given below are the graphs of three functions: f, g, and h.



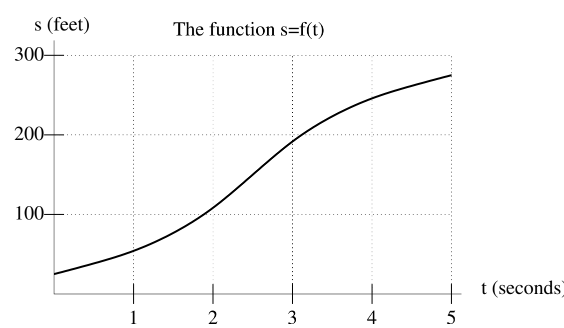




Use the graphs to decide whether each of the quantities that follow are positive, negative, or zero.

(a) $f'(1)$ (d) $f''(2)$ (g) $g'(3)$ (j) $h''(3)$
(b) $f''(1)$ (e) $f'(5)$ (h) $g''(3)$ (k) $f(1)$
(c) $f'(2)$ (f) $f''(5)$ (i) $h'(3)$ (l) $h(1)$

2. A South Dakota driver cruising along I-90 speeds up, sees a highway patrol car, and then begins to slow down. A graph of her displacement as a function of time is shown to the right. **Note.** Positive values of s indicate that the driver is east of her starting point, and negative values of s indicate that she is west of her starting point.



(a) On what approximate intervals is $f'(t)$ positive? negative? Interpret the meaning of these intervals in the context of this problem.

(b) On what approximate intervals is $f''(t)$ positive? negative? Interpret the meaning of these intervals in the context of this problem.

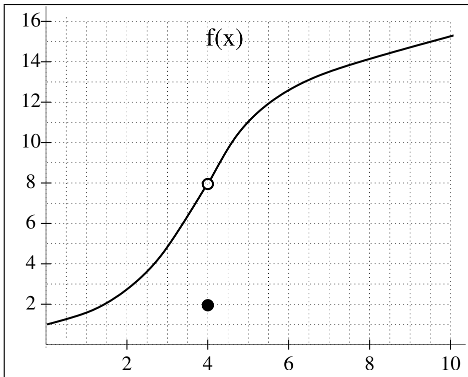
Section 2.5 – Derivatives

In each of the following situations, sketch the graph of a function $f(t)$ that has the indicated properties.

$f(t)$ is increasing $f(t) > 0$ $f'(t)$ is increasing	$f(t) > 0$ $f(t)$ is decreasing $f''(t) > 0$	$f(t)$ is increasing $f''(t) < 0$ $f(t) < 0$	$f''(t) < 0$ $f'(t) > 0$ $f(t) < 0$
$f(t) < 0$ $f'(t)$ is decreasing $f'(t) < 0$	$f'(t)$ is increasing $f(t) > 0$ $f'(t) < 0$	$f'(t) < 0$ $f''(t) < 0$ $f(t) > 0$	$f(t)$ is increasing $f(t) > 0$ $f''(t) > 0$
$f'(t)$ is increasing $f(t)$ is decreasing $f(t) < 0$	$f(t) > 0$ $f(t)$ is decreasing $f'(t)$ is decreasing	$f(t) < 0$ $f'(t)$ is increasing $f(t)$ is increasing	$f''(t) < 0$ $f(t) < 0$ $f(t)$ is decreasing
$f(t) > 0$ $f(t)$ is increasing $f''(t) < 0$	$f'(t) > 0$ $f(t) > 0$ $f''(t) > 0$	$f'(t) < 0$ $f''(t) > 0$ $f(t) > 0$	$f(t) < 0$ $f''(t) > 0$ $f'(t) < 0$

$f'(t)$ is decreasing $f(t) < 0$ $f'(t) > 0$	$f'(t)$ is increasing $f'(t) > 0$ $f(t) > 0$	$f(t) > 0$ $f''(t) < 0$ $f(t)$ is decreasing	$f(t)$ is decreasing $f'(t)$ is decreasing $f(t) < 0$
$f''(t) < 0$ $f(t) > 0$ $f'(t) > 0$	$f'(t) > 0$ $f(t) < 0$ $f'(t)$ is increasing	$f(t) < 0$ $f'(t) < 0$ $f''(t) < 0$	$f'(t)$ is increasing $f'(t) < 0$ $f(t) < 0$
$f(t) < 0$ $f'(t)$ is decreasing $f(t)$ is increasing	$f'(t)$ is decreasing $f(t) > 0$ $f(t)$ is increasing	$f(t)$ is decreasing $f''(t) > 0$ $f(t) < 0$	$f(t)$ is increasing $f(t) < 0$ $f''(t) > 0$
$f''(t) > 0$ $f'(t) > 0$ $f(t) < 0$	$f(t) > 0$ $f'(t)$ is decreasing $f'(t) > 0$	$f'(t) < 0$ $f'(t)$ is decreasing $f(t) > 0$	$f(t) > 0$ $f'(t)$ is increasing $f(t)$ is decreasing

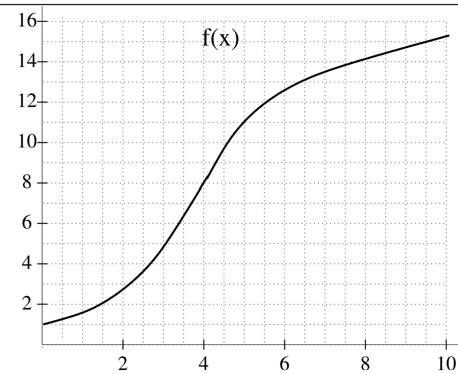
Section 2.6 – Differentiability



$\lim_{x\to 4} f(x)$

$f(4)$

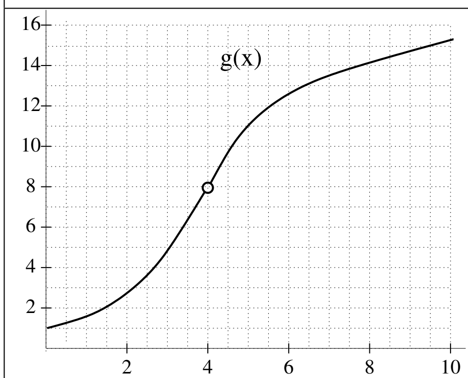
f ______________ continuous at $x = 4$.



$\lim_{x\to 4} f(x)$

$f(4)$

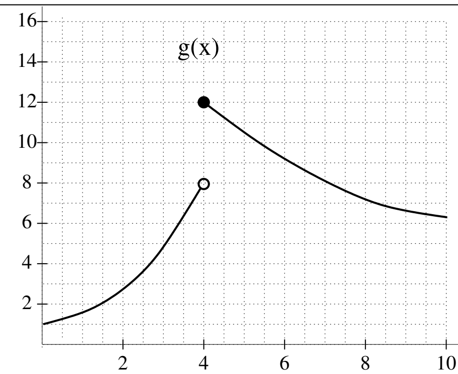
f ______________ continuous at $x = 4$.



$\lim_{x\to 4} g(x)$

$g(4)$

g ______________ continuous at $x = 4$.



$\lim_{x\to 4} g(x)$

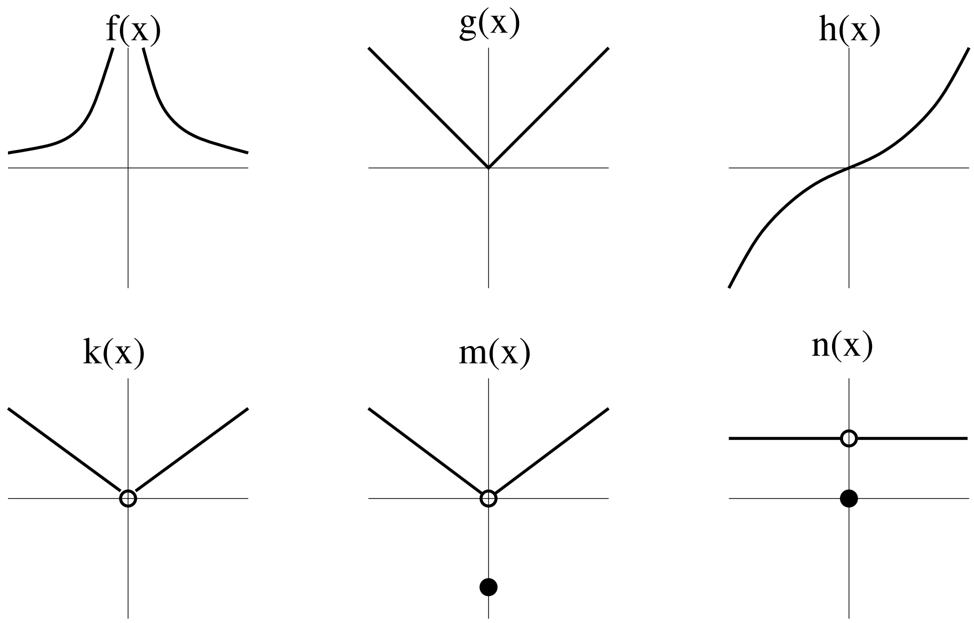
$g(4)$

g ______________ continuous at $x = 4$.

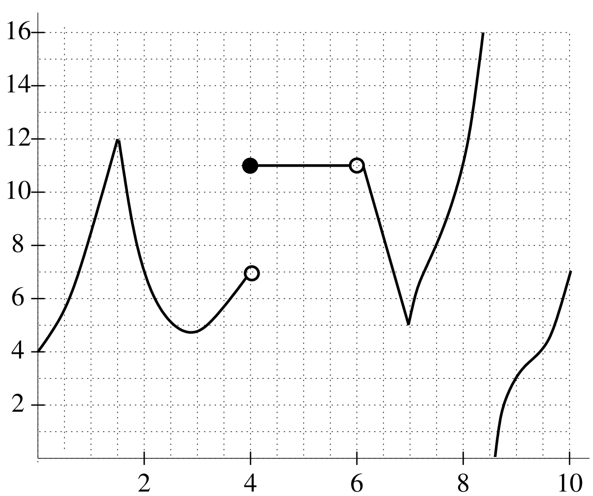
Theorem. The function f is continuous at $x = c$ if f is defined at $x = c$ and if

$$\lim_{x\to c} f(x) = \underline{\hspace{3cm}}.$$

1. Which of the following functions are continuous at $x = 0$?



2. Consider the function $f(x)$ given below.



(a) At what values of x is f not continuous?

(b) At what values of x is f not differentiable?

Section 2.6 – Differentiability

1. A magnetic field, B, is given as a function of the distance, r, from the center of a wire as follows:

$$B = \begin{cases} \frac{r}{r_0} B_0 & \text{for } r \leq r_0 \\ \frac{r_0}{r} B_0 & \text{for } r > r_0 \end{cases}$$

   (a) Is B continuous at r_0? Explain.

   (b) Is B differentiable at r_0? Explain.

2. Sketch the graph of $y = f(x)$ if f has the following properties:

   - $f(x)$ is continuous everywhere except at 3
   - $f(x)$ has a vertical tangent line at 2
   - $f(x)$ is not differentiable at 3
   - $f''(x) > 0$ wherever it is defined

3. Find the intersection point of the tangent line to $y = x^x$ at 1.1 and the x-axis.