Section 5.1 – How Do We Measure Distance Traveled?

Example 1. The following data is gathered as a small plane travels down the runway toward takeoff. How far did the plane travel in the 10 second period? (Give a range of values.)

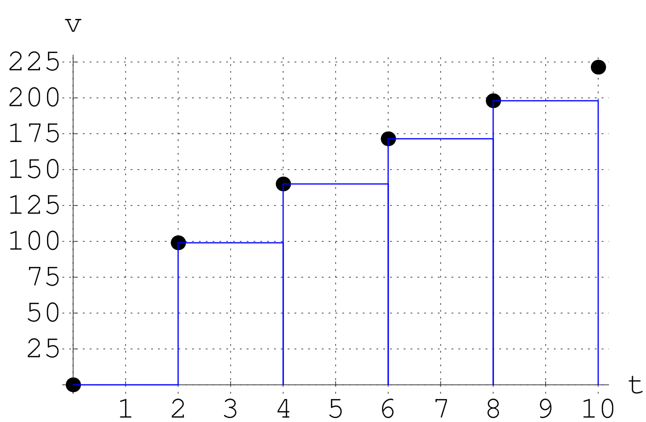
time (sec)	0	2	4	6	8	10
velocity (ft/sec)	0	99	140	171.5	198	221.4

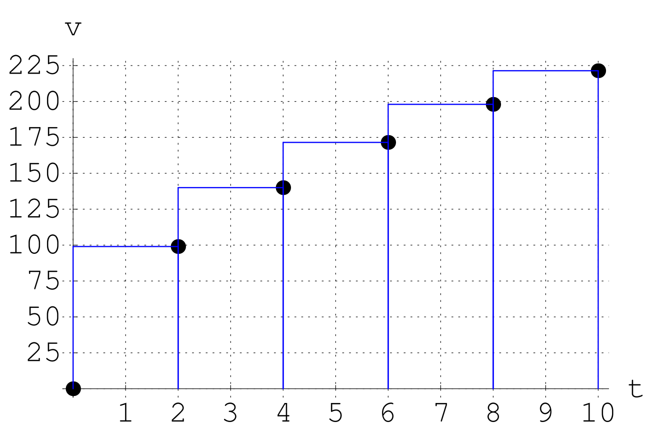
$$\begin{aligned}\texttt{Maximum Distance Traveled} &= (99)(2) + (140)(2) + (171.5)(2) + (198)(2) + (221.4)(2) \\ &= 2(99 + 140 + 171.5 + 198 + 221.4) \\ &= 1659.8 \texttt{ feet.}\end{aligned}$$

$$\begin{aligned}\texttt{Minimum Distance Traveled} &= 0(2) + (99)(2) + (140)(2) + (171.5)(2) + (198)(2) \\ &= 2(0 + 99 + 140 + 171.5 + 198) \\ &= 1217 \texttt{ feet.}\end{aligned}$$

Therefore, the plane travels between 1217 and 1659.8 feet.

Graphically, each term in the above two sums can be represented as the area of a rectangle. Specifically, the minimum distance traveled by the plane is the sum of the areas of the rectangles in the first diagram below, while the maximum distance traveled by the plane is the sum of the areas of the rectangles in the second diagram below.





Example 2. Suppose that the same small plane as in Example 1 is traveling toward takeoff, but that now, we are given the velocity of the plane every second (as shown in the table below). Give a new range of values representing the distance that the plane could have traveled, and illustrate your estimates with a new rectangles diagram.

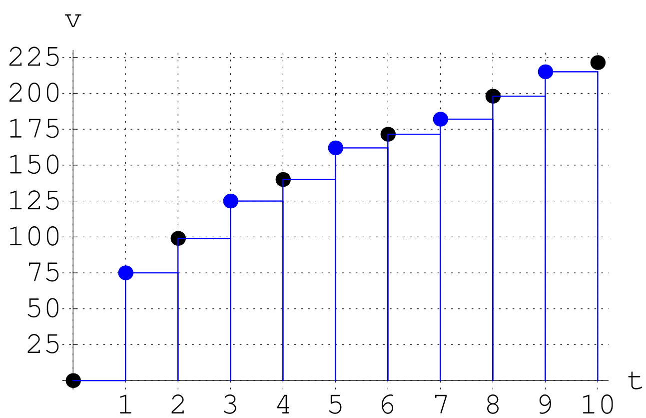
time (sec)	0	1	2	3	4	5	6	7	8	9	10
velocity (ft/sec)	0	75	99	125	140	162	171.5	182	198	215	221.4

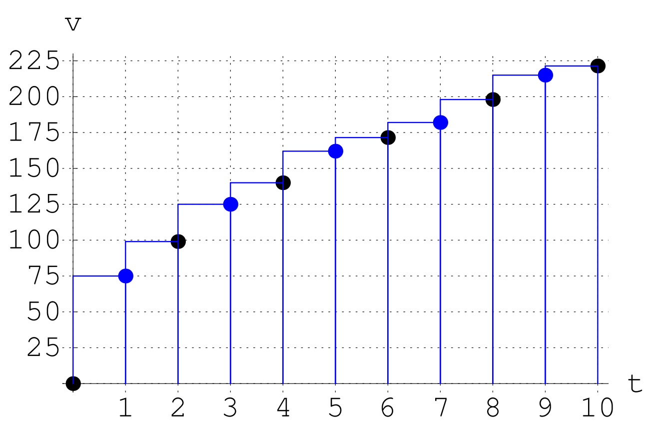
$$\begin{aligned}\texttt{Maximum Distance Traveled} &= 1(75 + 99 + 125 + 140 + 162 + 171.5 + 182 + 198 + 215 + 221.4) \\ &= 1588.9 \texttt{ feet.}\end{aligned}$$

$$\begin{aligned}\texttt{Minimum Distance Traveled} &= 1(0 + 75 + 99 + 125 + 140 + 162 + 171.5 + 182 + 198 + 215) \\ &= 1367.5 \texttt{ feet.}\end{aligned}$$

Therefore, the plane travels between 1367.5 and 1588.9 feet. Note that our range of possible distances traveled by the plane is smaller now that we are given more velocity data.

As before, each term in the above two sums can be represented as the area of a rectangle. Specifically, the minimum distance traveled by the plane is the sum of the areas of the rectangles in the first diagram below, while the maximum distance traveled by the plane is the sum of the areas of the rectangles in the second diagram below.





Section 5.1 – How Do We Measure Distance Traveled?

Officer "Tommy Boy" Hopkins and his evil brother Angry Mike are at it again. It seems that a hit and run accident took place the other night at midnight near Galloping Gulch, 40 miles from town. Farmer Bob claims that he witnessed the offending car traveling at approximately 60 miles per hour when his prize-winning Holstein, Bessie, was hit. Tommy Boy had figured his evil brother was up to no good that night and had planted a hidden camera in Angry Mike's car before Angry Mike left town at 11:00 p.m. Unfortunately, the resolution was so bad that he could only read the speedometer and the dashboard clock on the night in question. Tommy Boy states that not only was Angry Mike out driving that night, but at the time in question, was traveling 60 miles per hour and his car seemed to hit something. In his defense, Mike says that he had not yet arrived at Galloping Gulch but that he does remember hitting a nasty bump around midnight.
Your goal is to figure out which brother is correct. The following questions will help you achieve this goal. You may assume that Angry Mike's speed never decreased over the course of the entire trip.

Time	11:00 pm	11:10 pm	11:20 pm	11:30 pm	11:40 pm	11:50 pm	Midnight
Velocity, mph	0	30	40	45	50	53	60

Use only the data provided above to answer the following questions. Put all your work on a separate sheet of paper.

1. What is the maximum possible distance that Angry Mike could have traveled between 11:00 p.m. and midnight? Clearly show your calculations.

   For the first 10 minutes (i.e. $(1/6)$th of an hour), Tommy Boy's maximum speed was 30 miles per hour, so the maximum possible distance he could have traveled in the first 10 minutes is $(30\text{ mi/hr}) \cdot ((1/6)\text{ hr}) = 5$ miles. Similarly, for the second 10-minute interval, his maximum possible distance traveled is $40 \cdot (1/6) \approx 6.67$ miles. Continuing in this way, we see that the maximum possible distance he could have traveled during the entire hour is

$$30 \cdot \frac{1}{6} + 40 \cdot \frac{1}{6} + 45 \cdot \frac{1}{6} + 50 \cdot \frac{1}{6} + 53 \cdot \frac{1}{6} + 60 \cdot \frac{1}{6} = \frac{1}{6}(30 + 40 + 45 + 50 + 53 + 60) \approx 46.33 \text{ miles.}$$

2. What is the minimum possible distance that Angry Mike could have traveled between 11:00 p.m. and midnight? Clearly show your calculations.

   To find the minimum possible distance Angry Mike could have traveled during this time, we do a calculation similar to the one above, except that we assume Angry Mike travels at his slowest speed on each 10-minute subinterval. Therefore, the minimum possible distance Angry Mike could have traveled is

$$0 \cdot \frac{1}{6} + 30 \cdot \frac{1}{6} + 40 \cdot \frac{1}{6} + 45 \cdot \frac{1}{6} + 50 \cdot \frac{1}{6} + 53 \cdot \frac{1}{6} = \frac{1}{6}(0 + 30 + 40 + 45 + 50 + 53) \approx 36.33 \text{ miles.}$$

3. Now, assume that you are an investigating officer writing up a report of the incident. Your report should be a paragraph long and contain AT MINIMUM the following:
   - A description of the incident and the available evidence. Please do not copy the description above but rather rephrase in your own words.
   - A description of the calculations that you made in questions (1) and (2) above, including an explanation of how you know that they really do give the maximum and minimum distance traveled by Angry Mike's car over the time in question.

- A discussion of whether or not there is sufficient evidence to conclude that Angry Mike reached Galloping Gulch at or before midnight. Remember, use only the data in the above table to support any claims that you make.

A key observation in this report should be that, based on our calculations, it is possible that Angry Mike covered the 40 miles to the location of the cow during the hour (based on his maximum possible distance traveled), but we cannot be certain that he reached the cow (based on his minimum possible distance traveled).

Part 2: Now, consider the following more complete list of data, also compiled that night. Continue to assume that Angry Mike's speed never decreases over the entire one-hour period. Answer the questions again using the more complete data. As before, put all of your work on a separate sheet of paper.

Time	11:00 pm	11:05 pm	11:10 pm	11:15 pm	11:20 pm	11:25 pm	11:30 pm
Velocity, mph	0	25	30	38	40	43	45
Time	11:35 pm	11:40 pm	11:45 pm	11:50 pm	11:55 pm	Midnight	
Velocity, mph	49	50	51	53	58	60	

4. What is the maximum possible distance that Angry Mike could have traveled between 11:00 p.m. and midnight? Clearly show your calculations.

   The maximum possible distance that Angry Mike could have traveled is

   $$\frac{1}{12}(25+30+38+40+43+45+49+50+51+53+58+60) \quad = \quad 45.17 \text{ miles.}$$

5. What is the minimum possible distance that Angry Mike could have traveled between 11:00 p.m. and midnight? Clearly show your calculations.

   The minimum possible distance that Angry Mike could have traveled is

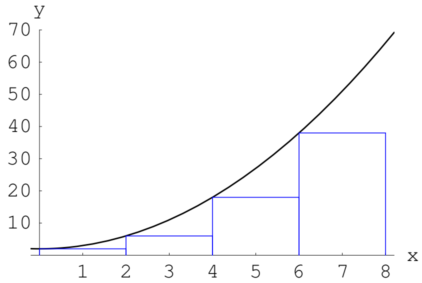
   $$\frac{1}{12}(0+25+30+38+40+43+45+49+50+51+53+58) \quad = \quad 40.17 \text{ miles.}$$

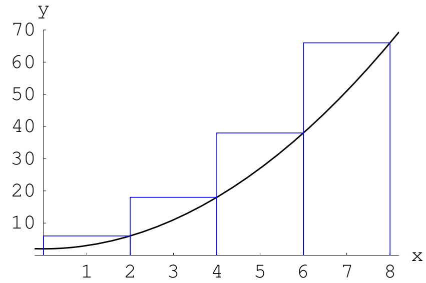
6. Write the same type of paragraph that you did for number 3, basing your conclusions on the more complete data. How do your conclusions change, if at all?

   A key observation in this report should be that, in the case of the new, more complete data, we can be sure that Angry Mike made it to the cow within the one-hour time interval because the minimum possible distance he could have traveled is more than 40 miles.

Section 5.2 – The Definite Integral

Example 1. Use a left sum and a right sum, with $n = 4$, to estimate the area under the curve $f(x) = x^2 + 2$ on the interval $0 \leq x \leq 8$.



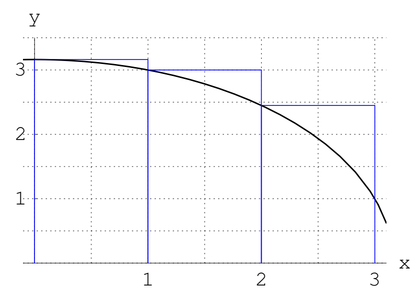


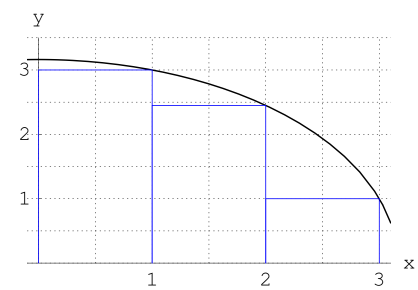
$$\begin{aligned} L_4 &= f(0)\cdot 2 + f(2)\cdot 2 + f(4)\cdot 2 + f(6)\cdot 2 \\ &= 2(f(0)+f(2)+f(4)+f(6)) \\ &= 2(2+6+18+38) \\ &= 128 \end{aligned}$$

$$\begin{aligned} R_4 &= f(2)\cdot 2 + f(4)\cdot 2 + f(6)\cdot 2 + f(8)\cdot 2 \\ &= 2(f(2)+f(4)+f(6)+f(8)) \\ &= 2(6+18+38+66) \\ &= 256 \end{aligned}$$

Therefore, the exact area under the curve is between 128 and 256.

Example 2. Use a left sum and a right sum, with $n = 3$, to estimate the area under the curve $g(x)$ (shown below) on the interval $0 \leq x \leq 3$.



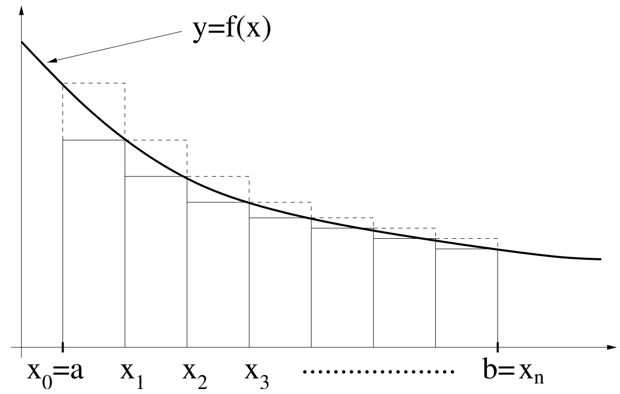


$$\begin{aligned} L_4 &= f(0)\cdot 1 + f(1)\cdot 1 + f(2)\cdot 1 \\ &= 3.2 + 3 + 2.5 \\ &= 8.7 \end{aligned}$$

$$\begin{aligned} R_4 &= f(1)\cdot 1 + f(2)\cdot 1 + f(3)\cdot 1 \\ &= 3 + 2.5 + 1 \\ &= 6.5 \end{aligned}$$

Therefore, the area under the curve is between 6.5 and 8.7.

Goal. Describe, in general, a way to find the *exact* area under a curve.



n = number of rectangles

Δx = width of one rectangle

Left Sum $= f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_{n-1})\Delta x = \sum_{i=0}^{n-1} f(x_i)\Delta x$

Right Sum $= f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + \cdots + f(x_n)\Delta x = \sum_{i=1}^{n} f(x_i)\Delta x$

Definition. The *definite integral* of $f(x)$ from a to b is the limit of the left and right hand sums as the number of rectangles approaches infinity. We write

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}[\text{Right-hand sums}] = \lim_{n\to\infty}[\text{Left-hand sums}]$$

Example 3. Use the results of Examples 1 and 2 to give your best estimate of $\int_0^8 (x^2+2)\,dx$ and $\int_0^3 g(x)\,dx$. Then, explain what these integrals represents geometrically.

To estimate each of these integrals, we can take the average value of the left sums and the right sums that we calculated in Examples 1 and 2. Therefore,

$$\int_0^8 (x^2+2)\,dx \approx \frac{128+256}{2} = 192,$$

which means that 192 is our estimate of the exact area under the curve $y = x^2+2$ between $x=0$ and $x=8$. Similarly,

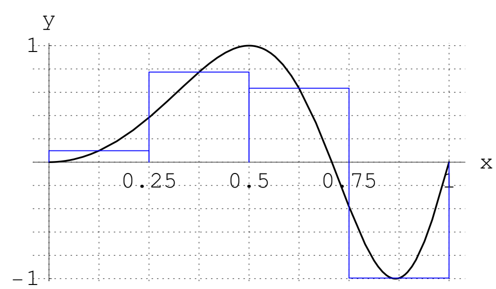
$$\int_0^3 g(x)\,dx \approx \frac{8.7+6.5}{2} = 7.6,$$

which means that 7.6 is our estimate of the area under the curve $y=g(x)$ between $x=0$ and $x=3$.

Example 4. Use a "middlesum" with $n=4$ to estimate the value of

$$\int_0^1 h(x)\,dx,$$

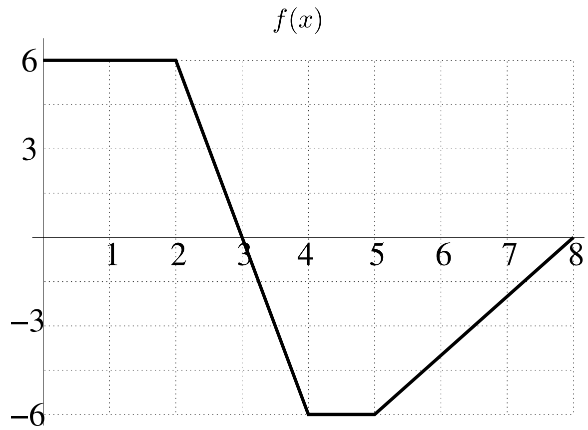
where the graph of h is given to the right.



$$\begin{aligned}\int_0^1 h(x)\,dx &\approx h\left(\frac{1}{8}\right)\cdot\frac{1}{4} + h\left(\frac{3}{8}\right)\cdot\frac{1}{4} + h\left(\frac{5}{8}\right)\cdot\frac{1}{4} + h\left(\frac{7}{8}\right)\cdot\frac{1}{4} \\ &= \frac{1}{4}\cdot(0.1+0.8+0.6+(-1)) \\ &= 0.125\end{aligned}$$

Note that the area of the 4th rectangle was subtracted because $h(7/8)$ is negative. In general, area gets added when f is above the x-axis and subtracted when f is below the x-axis.

Example 5. Let f be the graph of the function shown to the right. Calculate each of the integrals that follow *exactly*.



$$\int_0^1 f(x)\,dx = 6$$

$$\int_5^8 f(x)\,dx = -9$$

$$\int_0^2 f(x)\,dx = 12$$

$$\int_0^5 f(x)\,dx = 6$$

$$\int_0^3 f(x)\,dx = 15$$

$$\int_2^4 f(x)\,dx = 0$$

$$\int_4^5 f(x)\,dx = -6$$

Example 6. Let f be the function defined on $0 \le x \le 12$, some of whose values are shown in the table below. Estimate the value of $\int_0^{12} f(x)\,dx$.

x	0	3	6	9	12
$f(x)$	20	10	5	2	1

We have

$$L_4 = 20(3) + 10(3) + 5(3) + 2(3) = 111,$$

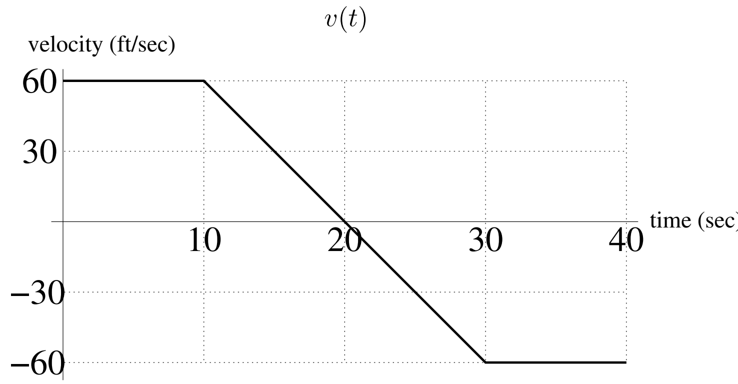
and

$$R_4 = 10(3) + 5(3) + 2(3) + 1(3) = 54,$$

so our estimate of the value of the integral is $(111 + 54)/2 = 82.5$.

Section 5.3 – The Fundamental Theorem and Interpretations

Example. Let $s(t)$ be the position, in feet, of a car along a straight east/west highway at time t seconds. Positive values of s indicate eastward displacement of the car from home, and negative values indicate westward displacement. Let $v(t)$ represent the velocity of this same car, in feet per second, at time t seconds (see graph to the right).



1. Use the velocity graph above to help you fill in the chart below.

t	0	10	20	30	40
$s(t)$	0	600	900	600	0

2. Fill in the chart below.

Integral of Velocity	Change in Position
$\int_0^{10} v(t)\,dt = 600$ feet	$s(10) - s(0) = 600 - 0 = 600$ feet
$\int_0^{20} v(t)\,dt = 900$ feet	$s(20) - s(0) = 900 - 0 = 900$ feet
$\int_{20}^{40} v(t)\,dt = -900$ feet	$s(40) - s(20) = 0 - 900 = -900$ feet
$\int_0^{40} v(t)\,dt = 0$ feet	$s(40) - s(0) = 0 - 0 = 0$ feet

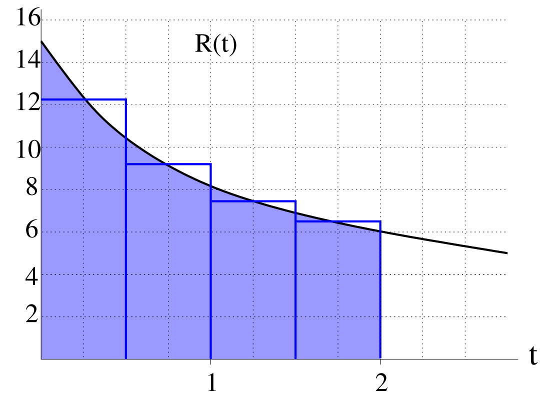
Total Change Principle. Let $F(t)$ be some quantity with a continuous rate of change $F'(t)$. Then

$$\int_a^b F'(t)\,dt = F(b) - F(a).$$

In other words, the integral of a rate of change gives total change.

Exercises.

1. Water is leaking out of a tank at a rate of $R(t)$ gallons per hour, where t is measured in hours (see graph to the right).



   (a) Write a definite integral that expresses the total amount of water that leaks out in the first two hours.

   $$\text{Total Change in Volume} = \int_0^2 R(t)\,dt \text{ gallons}$$

   (b) On the graph to the right, shade the region whose area represents the total amount of water that leaks out in the first two hours.

   (c) Give your best estimate of the amount of water that leaks out of the tank in the first two hours.

   Using a middlesum with $n = 4$, we obtain

   $$\begin{aligned}\int_0^2 R(t)\,dt &\approx 12.4(0.5) + 9.2(0.5) + 7.5(0.5) + 6.5(0.5)\\ &= 17.8,\end{aligned}$$

   so we estimate that about 17.8 gallons leak out of the tank in the first two hours. Graphically, this number is the sum of the areas of the four rectangles we drew in on the above diagram.

2. (Adapted from *Hughes-Hallett, et. al.*) A news broadcast in early 1993 said that the average American's annual income was changing at a rate of $r(t)$ dollars per month (see the table below), where t is the number of months after January 1, 1993. Estimate the amount that the average American's income changed in 1993.

t (months)	0	2	4	6	8	10	12
$r(t)$ (dollars per month)	40.00	40.16	40.32	40.48	40.64	40.81	40.97

   Since $r(t)$ represents the rate of change of income at time t, the total change in income in 1993 is given by $\int_0^{12} r(t)\,dt$. Using a leftsum, the approximate value of this integral is

   $$40(2) + 40.16(2) + 40.32(2) + 40.48(2) + 40.64(2) + 40.81(2) = 484.82,$$

   so the average American's income changed by about $\$484.82$ in 1993.

3. A can of soda is put into a refrigerator to cool. The rate at which the temperature of the soda is changing is given by

   $$f(t) = -25\mathrm{e}^{-2t} \text{ degrees Fahrenheit per hour,}$$

   where t represents the time (in hours) after the soda was placed in the refrigerator.

   (a) How fast is the can of soda cooling after 1 hour has passed? Include the appropriate units with your answer.

   Since $f(1) = -25\mathrm{e}^{-2}$, which is approximately equal to -3.38, we conclude that the can of soda is cooling at a rate of about $3.38°$F per hour after 1 hour has passed.

(b) If the temperature of the can of soda is 60°F when it is placed in the refrigerator, estimate the temperature of the can of soda after 3 hours have passed.

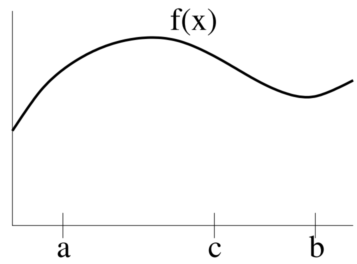
Using a calculator to approximate the value of the integral below, we have

$$\begin{aligned}
\text{Temp. at } t=3 &= (\text{Temp. at } t=0) + (\text{Change in Temp. between } t=0 \text{ and } t=3) \\
&= 60 + \int_0^3 (-25e^{-2t})\, dt \\
&= 60 + (-12.47) \\
&= 47.53.
\end{aligned}$$

Therefore, the temperature of the can of soda is about 47.53°F after 3 hours have passed.

Section 5.4 – Theorems About Definite Integrals

Properties of the Integral



1. $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \underline{\int_a^b f(x)\,dx}$

2. $\int_a^a f(x)\,dx = \underline{\quad 0 \quad}$

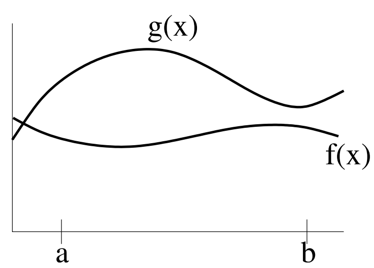
3. $\int_b^a f(x)\,dx = \underline{-\int_a^b f(x)\,dx}$

4. $\int_a^b [f(x) \pm g(x)]\,dx = \underline{\int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx}$

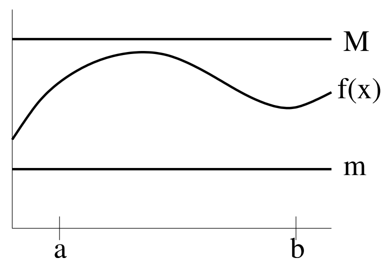
5. $\int_a^b cf(x)\,dx = \underline{c\int_a^b f(x)\,dx}$ (if c is constant)

Comparison Properties

6. If $f(x) \leq g(x)$ for all $a \leq x \leq b$, then $\underline{\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx}$.



7. If $m \leq f(x) \leq M$ for all $a \leq x \leq b$, then $\underline{m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)}$.



Exercises.

1. Assume that $\int_a^b f(x)\,dx = 9$, $\int_b^c f(x)\,dx = -7$, $\int_a^b (f(x))^2\,dx = 36$, and $\int_a^b g(x)\,dx = -2$. Use this information to calculate each of the following.

   (a) $\int_a^c f(x)\,dx$

   $$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx = 9 + (-7) = 2.$$

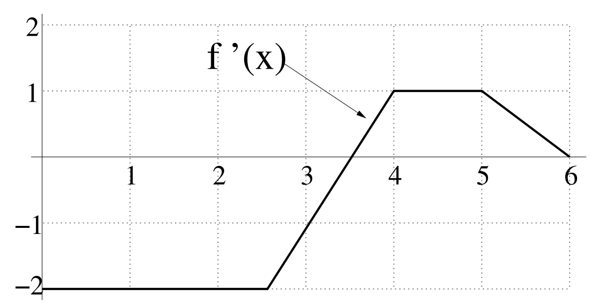
   (b) $\int_a^b (f(x))^2\,dx - \left(\int_a^b f(x)\,dx\right)^2$

   $$\int_a^b (f(x))^2\,dx - \left(\int_a^b f(x)\,dx\right)^2 = 36 - 9^2 = -45.$$

   (c) $\int_a^b (2f(x) - g(x))\,dx$

   $$\int_a^b (2f(x) - g(x))\,dx = 2\int_a^b f(x)\,dx - \int_a^b g(x)\,dx = 2(9) - (-2) = 20.$$

2. (a) Given to the right is the graph of $f'(x)$, the DERIVATIVE of a function. Given that $f(0) = 4$, fill in the chart below.



x	0	1	2	3	4	5	6
$f(x)$	4	2	0	-1.75	-1.75	-0.75	-0.25

By the Total Change Principle, we have $\int_a^b f'(x)\,dx = f(b) - f(a)$, which can be rewritten as $f(b) = f(a) + \int_a^b f'(x)\,dx$. Therefore,

$$\begin{aligned}
f(1) &= f(0) + \int_0^1 f'(x)\,dx = 4 + (-2) = 2 \\
f(2) &= f(1) + \int_1^2 f'(x)\,dx = 2 + (-2) = 0 \\
f(3) &= f(2) + \int_2^3 f'(x)\,dx = 0 + (-1.75) = -1.75 \\
f(4) &= f(3) + \int_3^4 f'(x)\,dx = -1.75 + 0 = -1.75 \\
f(5) &= f(4) + \int_4^5 f'(x)\,dx = -1.75 + 1 = -0.75 \\
f(6) &= f(5) + \int_5^6 f'(x)\,dx = -0.75 + 0.5 = -0.25
\end{aligned}$$

(b) Where does f attain its minimum value? What is this minimum value?

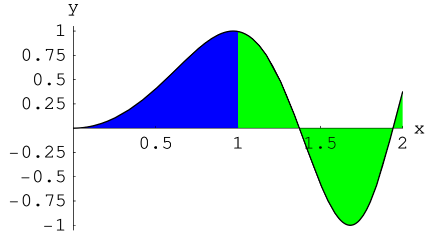
We can see from the graph of f' that $f'(x) \le 0$ for $0 \le x \le 3.5$ and $f'(x) \ge 0$ for $3.5 \le x \le 6$. Therefore, f is decreasing everywhere to the left of 3.5 and increasing everywhere to the right of 3.5, which means that f attains its minimum value at $x = 3.5$. We have

$$f(3.5) = f(3) + \int_3^{3.5} f'(x)\,dx = -1.75 - 0.25 = -2,$$

so -2 is the minimum value of f.

3. Let $f'(x) = \sin(5x^2/3)$ (see graph to the right).

$f'(x) = \sin(5x^2/3)$



(a) Which is larger, $f(1) - f(0)$ or $f(2) - f(1)$?

Note that the shaded area on the interval $0 \le x \le 1$ lies entirely above the x-axis (see diagram to the right); therefore, $\int_0^1 f'(x)\,dx > 0$. On the other hand, on the interval $1 \le x \le 2$, there is more shaded area below the x-axis than above it, so $\int_1^2 f'(x)\,dx < 0$. Therefore, by the Total Change Principle, we have

$$f(2) - f(1) = \int_1^2 f'(x)\,dx < 0 < \int_0^1 f'(x)\,dx = f(1) - f(0),$$

so $f(1) - f(0)$ is larger than $f(2) - f(1)$.

(b) Which is larger, $f(1.5) - f(1)$ or $f(2) - f(1.5)$?

Using similar reasoning as in part (a) above, we have

$$f(2) - f(1.5) = \int_{1.5}^2 f'(x)\,dx < 0 < \int_1^{1.5} f'(x)\,dx = f(1.5) - f(1),$$

so $f(1.5) - f(1)$ is larger than $f(2) - f(1.5)$.

(c) Rank the following quantities in order from smallest to largest:

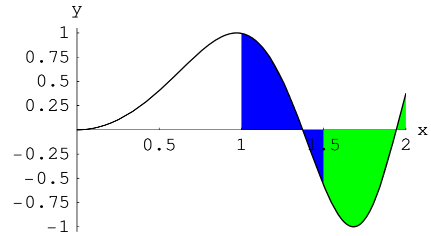
$$f(0),\ f(0.5),\ f(1),\ f(1.5),\ f(2)$$

Since $f'(x) > 0$ for $0 \le x \le 1$, we know that f is increasing on the interval $0 \le x \le 1$. Therefore,

$$f(0) < f(0.5) < f(1).$$

On the interval $1 \le x \le 1.5$, the function f increases more than it decreases because, in examining the shaded area on the interval $1 \le x \le 1.5$, there is more area above the x-axis than below it; therefore,

$$f(1) < f(1.5).$$



By similar logic, the function f decreases on the interval $1.5 \le x \le 2$, so $f(2) < f(1.5)$. However, by examining <u>all</u> of the shaded area on the interval $1 \le x \le 2$, there is more area below the x-axis than above it, so f decreases more than it increases on this interval; therefore, $f(2) < f(1)$. By similar logic, $f(1) < f(1.5)$, so by combining all of the above information, we arrive at the following conclusion:

$$f(0) < f(0.5) < f(2) < f(1) < f(1.5)$$