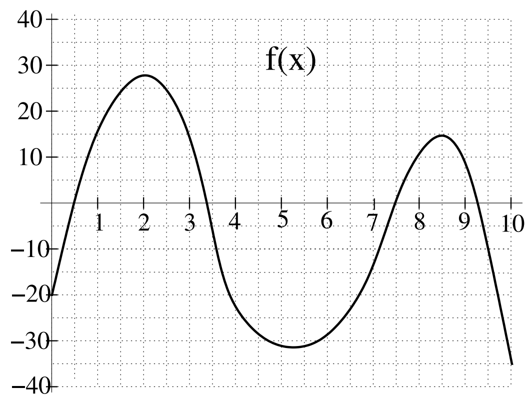
Section 4.1 – Using the First and Second Derivatives

Definitions. Let f be a function.

1. A *critical point* of f is a point p in the domain of f such that either $f'(p) = 0$ or $f'(p)$ is undefined.
2. We say that f has a *local minimum* at p if $f(p)$ is less than or equal to the values of f for points near p.
3. We say that f has a *local maximum* at p if $f(p)$ is greater than or equal to the values of f for points near p.
4. An *inflection point* of f is a point at which the function f changes concavity.

Example. Given to the right is the graph of a function f.



(a) Estimate the critical point(s) of f.

$$x = 2, \quad x = 5.3, \quad x = 8.5$$

(b) Estimate the inflection point(s) of f.

$$x = 3.4, \quad x = 7.2$$

(c) Does f have any local maximum or local minimum values? If so, list them, making it clear which are which.

$f(2) = 27$ is a local maximum.
$f(5.3) = -31$ is a local minimum.
$f(8.5) = 15$ is a local maximum.

First Derivative Test. Suppose that p is a critical point of a continuous function f.

1. If f' changes from negative to positive at p, then f has a local minimum at $x = p$.
2. If f' changes from positive to negative at p, then f has a local maximum at $x = p$.

Second Derivative Test.

1. If $f'(p) = 0$ and $f''(p) > 0$, then f has a local minimum at $x = p$.
2. If $f'(p) = 0$ and $f''(p) < 0$, then f has a local maximum at $x = p$.

EXERCISES. Please do the following on a separate sheet of paper.

1. Let $f(x) = x^{2/3}(4-x)^{1/3}$.

   (a) Given that $f'(x) = \dfrac{8-3x}{3x^{1/3}(4-x)^{2/3}}$, find the intervals on which f is increasing/decreasing.

   First, we note that $f'(x) = 0$ when $8 - 3x = 0$, so solving for x indicates that $x = 8/3$ is one critical point. In addition, we can see that $x = 0$ and $x = 4$ are critical points since $f'(x)$ is undefined for these two values of x. We therefore obtain the sign chart shown to the right:

Interval	Sign of $f'(x)$
$x < 0$	$-$
$0 < x < 8/3$	$+$
$8/3 < x < 4$	$-$
$x > 4$	$-$

   We therefore conclude from the sign chart that f is increasing on $0 < x < 8/3$ and decreasing for $x < 0$, $8/3 <\leq 4$, and $x > 4$.

   (b) Given that $f''(x) = \dfrac{-32}{9x^{4/3}(4-x)^{5/3}}$, find the intervals on which f is concave up/concave down.

   We can see from the formula for $f''(x)$ that $f''(x)$ never equals zero; however, $f''(x)$ is undefined for $x = 0$ and $x = 4$. We therefore obtain the sign chart shown to the right:

Interval	Sign of $f''(x)$
$x < 0$	$-$
$0 < x < 4$	$-$
$x > 4$	$+$

   We therefore conclude that f is concave up for $x > 4$ and concave down for $x < 0$ and $0 < x < 4$.

   (c) Find all local maxima, local minima, and inflection points of f.

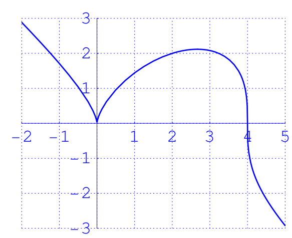
   From the sign chart for f', we can see that f has a local minimum at $x = 0$ and a local maximum at $x = 8/3$, and we can see from the sign chart for f'' that f has an inflection point at $x = 4$. We summarize this information below:

   $$f(0) = 0^{2/3}(4-0)^{1/3} = 0 \text{ is a local minimum.}$$

   $$f(8/3) = \left(\frac{8}{3}\right)^{2/3}\left(4-\frac{8}{3}\right)^{1/3} = \frac{4}{3}\sqrt[3]{4} \text{ is a local maximum.}$$

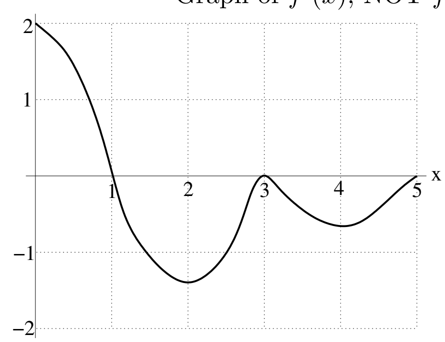
   $$(4, f(4)) = (4, 0) \text{ is an inflection point}$$

   Graph of f



2. Given to the right is the graph of the DERIVATIVE of a function. Use this graph to help you answer the following questions about the ORIGINAL FUNCTION f.

Graph of $f'(x)$, NOT $f(x)$



   (a) What are the critical points of f?

   $x = 1$, $x = 3$, $x = 5$

   (b) Where is f increasing? decreasing?

   increasing on $0 \leq x \leq 1$
   decreasing on $1 \leq x \leq 5$

   (c) Does f have any local maxima? If so, where?

   Yes, f has a local maximum at $x = 1$.

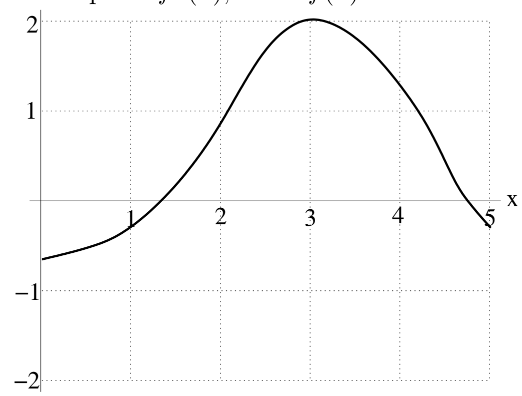
   (d) Does f have any local minima? If so, where?

   No, f has no local minima.

   (e) Where is f concave up? concave down?

   concave up on $2 < x < 3$ and $4 < x < 5$
   concave down on $0 < x < 2$ and $3 < x < 4$

3. Given to the right is the graph of the SECOND DERIVATIVE of a function. Use this graph to help you answer the following questions about the ORIGINAL FUNCTION f.



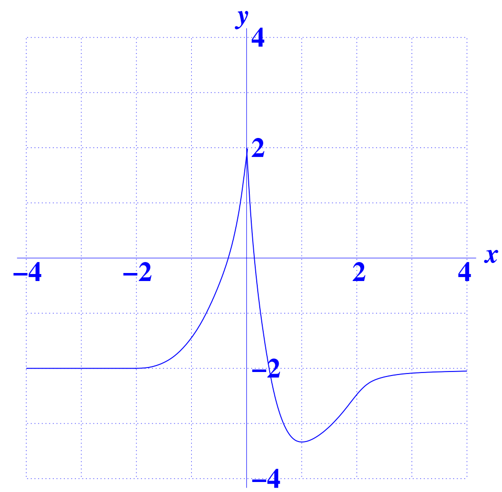
   (a) Where is f concave up? concave down?

   `concave up on` $1.3 < x < 4.7$
   `concave down on` $0 < x < 1.3$ `and` $4.7 < x < 5$

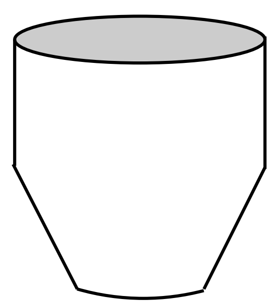
   (b) Does f have any inflection points? If so, where?

   `Yes, at` $x = 1.3$ `and` $x = 4.7$

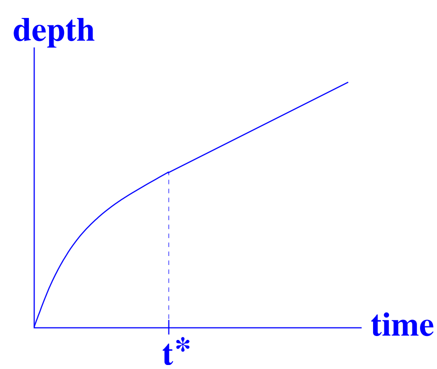
4. Sketch the graph of ONE FUNCTION f that has ALL of the following properties.
   - f is continuous everywhere.
   - $f(0) = 2$.
   - $f'(x) = 0$ for $-4 \le x \le -2$.
     $f'(x) < 0$ for $0 < x < 1$.
     $f'(x) > 0$ for $-2 < x < 0$ and for $1 < x < 4$.
   - $f''(x) > 0$ for $-2 < x < 0$ and for $0 < x < 2$.
     $f''(x) < 0$ for $2 < x < 4$
   - $\lim_{x\to\infty} f(x) = -2$.



5. If water is flowing at a constant rate (i.e. constant volume per unit time) into the urn pictured to the right, sketch a graph of the depth of the water in the urn against time. Mark on the graph the time at which the water reaches the corner of the urn.



`Because the urn increases in width from floor level to the corner of the urn, the graph of depth versus time should be increasing at a decreasing rate (and therefore concave down) until` t^*`, the time when the water level reaches the corner. After this time, the width of the urn becomes constant, so the water level should increase at a constant rate, meaning that depth is a linear function of time to the right of` t^* `(see graph to the right).`

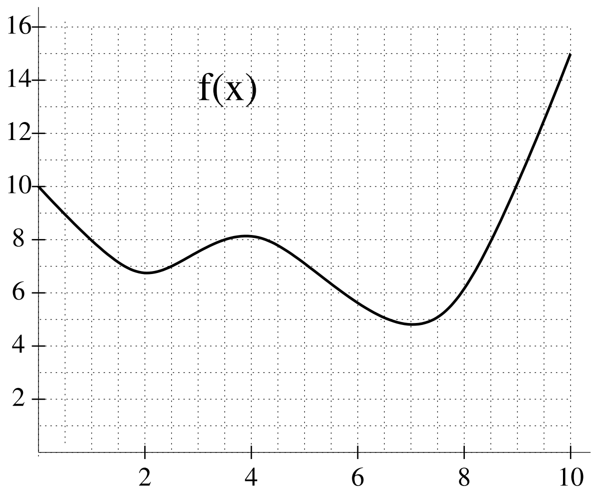


Section 4.2 – Optimization

Some Definitions. Let f be a function.

1. f has a *global maximum* at $x = p$ if $f(p)$ is greater than or equal to all output values of f.
2. f has a *global minimum* at $x = p$ if $f(p)$ is less than or equal to all output values of f.
3. *Optimization* refers to the process of finding the global maximum or global minimum of a function.

Example. For the function f given below, locate all local and global maxima and minima on the interval $[0, 10]$.



$f(2) = 6.8$ and $f(7) = 4.9$ are local minima.

$f(3.9) = 8.1$ is a local maximum.

$f(7) = 4.9$ is a global minimum.

$f(10) = 15$ is a global maximum.

General Rule. To find the global maximum and the global minimum of a continuous function on a closed interval (i.e., an interval that contains its endpoints), compare the output values of the function at the following locations:

1. critical points
2. endpoints

Exercises

1. Find the global maximum and global minimum value of $f(x) = x + \frac{3}{x}$ on the interval $[1, 4]$.

We have

$$f'(x) = 1 - \frac{3}{x^2} = \frac{x^2 - 3}{x^2},$$

so our critical points occur when $x^2 - 3 = 0$, or when $x = \pm\sqrt{3}$. Since the endpoints of our interval are $x = 1$ and $x = 4$, the only relevant critical point is $x = \sqrt{3}$. Calculating the value of f at these three points, we have

$$\begin{aligned} f(1) &= 1 + \frac{3}{1} = 4 \\ f(4) &= 4 + \frac{3}{4} = 4.75 \\ f(\sqrt{3}) &= \sqrt{3} + \frac{3}{\sqrt{3}} = 2\sqrt{3} \approx 3.46 \end{aligned}$$

Therefore, $f(\sqrt{3}) = 2\sqrt{3}$ is the global minimum and $f(4) = 4.75$ is the global maximum.

2. (Taken from *Hughes-Hallett, et. al.*) When you cough, your windpipe contracts. The speed, v, at which the air comes out depends on the radius, r, of your windpipe. If R is the normal (rest) radius of your windpipe, then for $0 \leq r \leq R$, the speed is given by $v = a(R-r)r^2$, where a is a positive constant. What value of r maximizes the speed?

Since $v = a(R-r)r^2 = aRr^2 - ar^3$, we have

$$\frac{dv}{dr} = 2arR - 3ar^2 = ar(2R - 3r),$$

so $dv/dr = 0$ when $r = (2/3)R$ and $r = 0$ meaning that $r = (2/3)R$ is the only critical point of our speed function that is not also an endpoint. Since $r = 0$ and $r = R$ are the endpoints of our interval of consideration, we can calculate and compare the values of v at the relevant three points.

$$\begin{aligned} v|_{r=0} &= a(R-0)(0)^2 = 0 \\ v|_{r=R} &= a(R-R)R^2 = 0 \\ v|_{r=(2/3)R} &= a\left(R - \frac{2R}{3}\right)\left(\frac{2R}{3}\right)^2 \\ &= a \cdot \frac{R}{3} \cdot \frac{4R^2}{9} \\ &= \frac{4aR^3}{27} \end{aligned}$$

Therefore, the maximum coughing speed occurs when $r = (2/3)R$, that is, when the radius of the windpipe is two-thirds of its normal (rest) radius.

3. (Taken from *Hughes-Hallett, et. al.*) The potential energy, U, of a particle moving along the x-axis is given by

$$U = b\left(\frac{a^2}{x^2} - \frac{a}{x}\right),$$

where a and b are positive constants and $x > 0$. What value of x minimizes the potential energy?

First, we have

$$U'(x) = b \cdot \frac{d}{dx}\left(\frac{a^2}{x^2} - \frac{a}{x}\right) \quad = \quad b \cdot \frac{d}{dx}\left(\frac{a^2 - ax}{x^2}\right) = ab\left(\frac{x - 2a}{x^3}\right),$$

so $U'(x) = 0$ when $x - 2a = 0$, or when $x = 2a$. Therefore, $x = 2a$ is the only critical point. Also, since we can see from our formula for $U'(x)$ that $U'(x) < 0$ for $0 < x < 2a$ and $U'(x) > 0$ for $x > 2a$, we see that U is decreasing everywhere to the left of $x = 2a$ and increasing everywhere to the right of $x = 2a$. It follows that U has a global minimum at $x = 2a$, meaning that $2a$ is the value of x that minimizes the potential energy.

4. Let $f(x) = xe^{-x^2}$.

   (a) Locate all local maximum and all local minimum values of f.

We have

$$f'(x) = xe^{-x^2} \cdot (-2x) + e^{-x^2} \cdot 1 = e^{-x^2}(1 - 2x^2),$$

so $f'(x) = 0$ when $1 - 2x^2 = 0$, that is, when $x = \pm\sqrt{1/2}$. From the sign chart to the right, we see that f has a local minimum at $x = -1/\sqrt{2}$ and a local maximum at $x = 1/\sqrt{2}$.

Interval	Sign of $f'(x)$
$x < -\sqrt{1/2}$	$-$
$-\sqrt{1/2} < x < \sqrt{1/2}$	$+$
$x > \sqrt{1/2}$	$-$

(b) Find the global maximum and the global minimum values of f on the interval $[0, 2]$.

To determine the global maximum and minimum values of f on $[0, 2]$, we begin by comparing the value of f at the endpoints of our interval and the one critical point that lies within the interval.

$$\begin{aligned} f(0) &= 0\mathrm{e}^{-0^2} = 0 \\ f(2) &= 2\mathrm{e}^{-2^2} = 2\mathrm{e}^{-4} \approx 0.037 \\ f\left(\frac{1}{\sqrt{2}}\right) &= \frac{1}{\sqrt{2\mathrm{e}}} \approx 0.43 \end{aligned}$$

Therefore, $f(0) = 0$ is the global minimum value of f and $f(1/\sqrt{2}) = 1/\sqrt{2\mathrm{e}}$ is the global maximum value of f on $[0, 2]$.

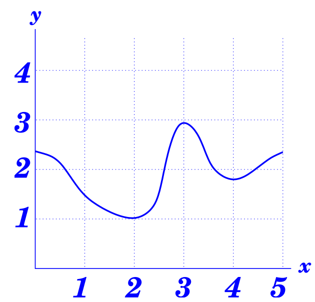
5. Give an example of a function that does not have a global maximum or a global minimum value.

Since the function f defined by $f(x) = x^3$ can get arbitrarily large and arbitrarily small on the interval $(-\infty, \infty)$, we conclude that f has no global maximum and no global minimum value on this interval.

Section 4.2 – Optimization

1. Sketch a continuous, differentiable graph with the following properties:

   - local minima at 2 and 4
   - global minimum at 2
   - local and global maximum at 3
   - no other extrema



2. A warehouse orders and stores boxes. The cost of storing boxes is proportional to q, the quantity ordered. The cost of ordering boxes is proportional to $1/q$, because the warehouse gets a price cut for larger orders. The total cost of operating the warehouse is the sum of ordering costs and storage costs. What value of q gives the minimum cost?

Let $C = f(q)$ represent the total cost of operating the warehouse as a function of the quantity ordered, q. Then, from the given information, we have

$$C = f(q) = k_1 q + k_2\left(\frac{1}{q}\right) = k_1 q + \frac{k_2}{q},$$

where k_1 and k_2 are positive constants. Rewriting, we see that $f(q) = k_1 q + k_2 q^{-1}$, so $f'(q) = k_1 - k_2 q^{-2}$. To find the critical points of C, we set $f'(q)$ equal to zero and solve for q.

$$\begin{aligned} k_1 - k_2 q^{-2} &= 0 \\ k_1 &= k_2 q^{-2} \\ q &= \pm\sqrt{\frac{k_2}{k_1}} \end{aligned}$$

Therefore, $q = \sqrt{k_2/k_1}$ is the only relevant critical point. Since $f'(q) < 0$ for all $0 < q < \sqrt{k_2/k_1}$ and $f'(q) > 0$ for all $q > \sqrt{k_2/k_1}$, we see that C is decreasing for $0 < q < \sqrt{k_2/k_1}$ and increasing for $q > \sqrt{k_2/k_1}$. It follows that $q = \sqrt{k_2/k_1}$ gives the minimum total cost of operating the warehouse.

3. Find the best possible bounds for $f(t) = t + \sin t$ for t between 0 and 2π.

We begin by noting that $f'(t) = 1 + \cos t$, and we can find the critical points of f by setting $f'(t)$ equal to zero and solving for t.

$$\begin{aligned} 1 + \cos t &= 0 \\ \cos t &= -1 \\ t &= \pi \end{aligned}$$

Therefore, $t = \pi$ is the only critical point between 0 and 2π. To find the best possible bounds, we compare the values of f at our two endpoints and the critical points (see table to the right). Since 0 is the smallest output value and 2π is the largest output value, we conclude that

t	$f(t)$
0	0
π	π
2π	2π

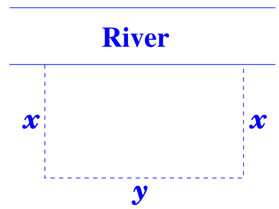
$$0 \leq t + \sin t \leq 2\pi$$

for all t between 0 and 2π, and that these are the best possible bounds.

Section 4.3 – Optimization and Modeling

1. A farmer wants to fence a rectangular grazing area along a straight river (no fence is needed along the river). There are 1700 total feet of fencing available. What dimensions (length and width) will maximize the grazing area?

   Let x and y represent the dimensions of the rectangular pen, in feet, and let A represent its area.

   Given: $2x + y = 1700$

   Find: x and y that maximize A

   Since the pen is rectangular, we know that $A = xy$, and from our given information, we see that $y = 1700 - 2x$. Therefore,

   $$A = x(1700 - 2x) = 1700x - 2x^2.$$

   Next, we find the critical points of A; we have $A' = 1700 - 4x$, so

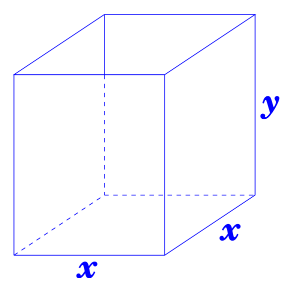
   $$\begin{aligned} 1700 - 4x &= 0 \\ 4x &= 1700 \\ x &= 425 \end{aligned}$$

   Therefore, $x = 425$ is the only critical point. Also, since 1700 feet is the maximum amount of fencing available, the largest value that x can have is $1700/2 = 850$. Therefore, the endpoints of our domain are $x = 0$ and $x = 850$. The table to the right confirms that the maximum value of A occurs when $x = 425$. Therefore, our final answer is $x = 425$ feet and $y = 1700 - 2(425) = 850$ feet.

x	A
0	0
425	361,250
850	0

2. A box with an open top of fixed volume V with a square base is to be constructed. Find the dimensions of the box that minimize the amount of material used in its construction.

   Let x represent the length and width of the box, let y represent the height of the box, and let V represent the volume of the box (see diagram below).

   Given: $V = x^2 y$

   Find: x and y that minimize S, the surface area of the box.

   We begin by finding a formula for S, the surface area of the box. Since

   $$\begin{aligned} S &= \text{(Sum of the areas of the 4 sides)} + \text{(Area of the bottom)} \\ &= 4xy + x^2 \\ &= 4x(Vx^{-2}) + x^2 \\ &= 4Vx^{-1} + x^2, \end{aligned}$$

   we have $S' = -4Vx^{-2} + 2x = (2x^3 - 4V)/x^2$, so the critical points of S occur when

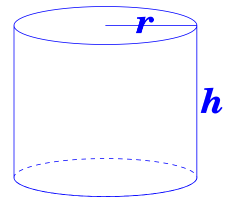
   $$\begin{aligned} 2x^3 - 4V &= 0 \\ x^3 &= 2V \\ x &= \sqrt[3]{2V}. \end{aligned}$$

Interval	Sign of S'
$0 < x < \sqrt[3]{2V}$	$-$
$x > \sqrt[3]{2V}$	$+$

Therefore, since the above sign chart confirms that S is decreasing for all x values to the left of $\sqrt[3]{2V}$ and increasing for all x values to the right of $\sqrt[3]{2V}$, we conclude that S has a global minimum at $x = \sqrt[3]{2V}$. In other words, the dimensions of the box that will minimize the amount of material used in the construction are $x = \sqrt[3]{2V}$ and $y = Vx^{-2} = V(2V)^{-2/3} = \sqrt[3]{V/4}$.

3. A metal can manufacturer needs to build cylindrical cans with volume 300 cubic centimeters. The material for the side of a can costs 0.03 cents per cm^2, and the material for the bottom and top of the can costs 0.06 cents per cm^2. What is the cost of the least expensive can that can be built?

Let r represent the radius of the can, let h represent the height of the can, and let C represent the cost of building the can, in cents.



Given: $\pi r^2 h = 300$

Find: Minimum Value of C

We begin by finding a formula for the cost of building the can. We have

$$\begin{aligned} C &= \text{(Cost of the top and bottom)} + \text{(Cost of the outside)} \\ &= \text{(Area of Top and Bottom)(Cost Per Unit Area)} + \text{(Area of Outside)(Cost Per Unit Area)} \\ &= (2\pi r^2)(0.06) + (2\pi r h)(0.03) \\ &= 0.12\pi r^2 + 0.06\pi r\left(\frac{300}{\pi r^2}\right) \\ &= 0.12\pi r^2 + 18r^{-1}, \end{aligned}$$

so $C' = 0.24\pi r - 18r^{-2} = (0.24\pi r^3 - 18)/r^2$, which means that $C' = 0$ when $0.24\pi r^3 = 18$, or when $r = \sqrt[3]{75/\pi}$. Since the sign chart to the right confirms that C has a global minimum at $r = \sqrt[3]{75/\pi}$, we conclude that the cost of the least expensive can is given by

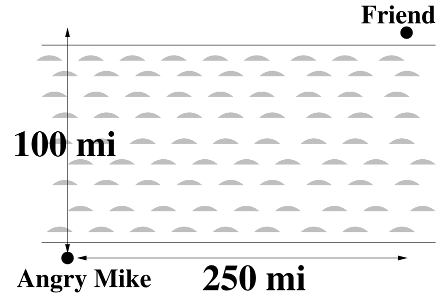
Interval	Sign of C'
$0 < r < \sqrt[3]{75/\pi}$	$-$
$r > \sqrt[3]{75/\pi}$	$+$

$$\begin{aligned} C &= 0.12\pi(\sqrt[3]{75/\pi})^2 + 18(\sqrt[3]{75/\pi})^{-1} \\ &= \left(\frac{75}{\pi}\right)^{-1/3}\left(0.12\pi \cdot \frac{75}{\pi} + 18\right) \\ &= 27\sqrt[3]{\frac{\pi}{75}}, \end{aligned}$$

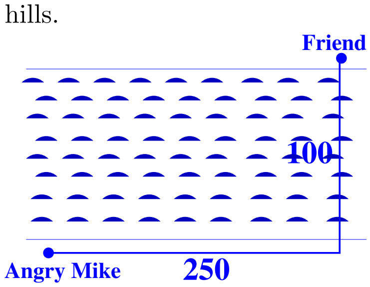
or about 9.38 cents.

Section 4.3 – Optimization and Modeling with Angry Mike

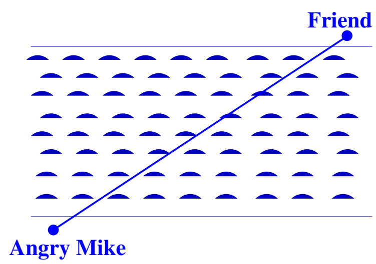
Angry Mike is on the run. He finally couldn't outsmart his brother sheriff Tommy Boy Hopkins any longer and there is a warrant out for his arrest. In his desperation Angry Mike has hijacked a dirt bike and is making a run for the state line. A friend is waiting for him 100 miles north and 250 miles east with a car. Unfortunately, a range of hills stretches all the way to the north and east from Angry Mike's starting position. He knows that he can go 70 mph as long as he stays out of the hills, but that he can go no faster than 40 mph in the hills.



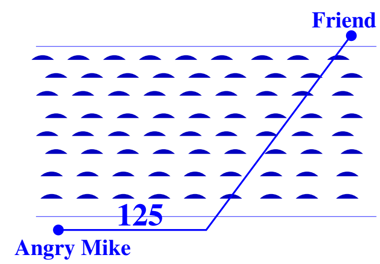
1. Draw three different possible routes (straight lines, or line segments) Angry Mike can take. One of them should maximize the part of the trip outside the hills, one of them should maximize the part of the trip in the hills.



Route 1



Route 2



Route 3

2. Compute the total driving time for those three routes.

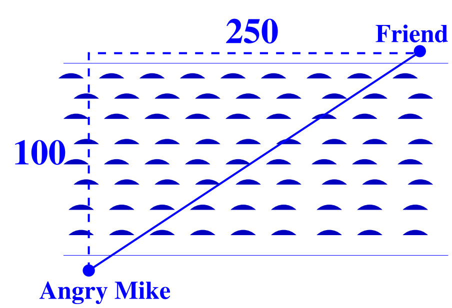
First, we note that for an object moving at a constant speed, the travel time equals the distance traveled divided by the speed. We use this principle for each of the three routes below.

Route 1: Since Angry Mike drives 250 miles at a speed of 70 miles per hour and 100 miles at a speed of 40 miles per hour, his travel time is

$$\frac{250}{70} + \frac{100}{40} \approx 6.07 \text{ hours.}$$

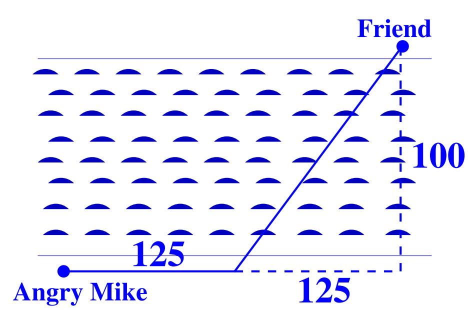
Route 2: Along this route, Angry Mike does all of his traveling through the hills, so his speed is 40 mph for the entire distance he travels. Referring to the diagram to the right, we use Pythagorean's Theorem to conclude that he travels a distance of $\sqrt{100^2 + 250^2} = 50\sqrt{29}$ miles. Therefore, his travel time is

$$\frac{50\sqrt{29}}{40} \approx 6.73 \text{ hours.}$$



Route 3: Referring to the diagram to the right, we see that Angry Mike travels the first 125 miles at a speed of 70 mph. Then, he travels a distance of $\sqrt{125^2 + 100^2} = 25\sqrt{41}$ miles through the hills at a speed of 40 mph. Therefore, his total travel time is

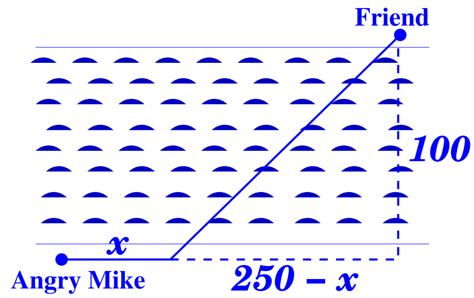
$$\frac{125}{70} + \frac{25\sqrt{41}}{40} \approx 5.79 \text{ hours.}$$



3. There are many more possible routes Mike could take. Out of all of them, which one should Angry Mike take to get to his waiting friend as fast as possible?

In general, suppose that Angry Mike first travels a distance of x miles outside of the hills (for $0 \leq x \leq 250$), and then cuts across the hills for the remaining distance, which, by Pythagorean's Theorem, would equal

$$\sqrt{100^2 + (250 - x)^2} \text{ miles.}$$



Since his speed is 70 mph for the first leg of the trip and 40 mph for the second leg of the trip, his total travel time is given by the function

$$t(x) = \frac{x}{70} + \frac{\sqrt{100^2 + (250 - x)^2}}{40} \text{ hours.}$$

To find the fastest travel time, we need to find the global minimum value of t on the interval $0 \leq x \leq 250$. We begin by solving the equation $t'(x) = 0$ for x to find the critical points.

$$\begin{aligned} 0 &= \frac{1}{70} + \frac{1}{40} \cdot \frac{1}{2}(100^2 + (250 - x)^2)^{-1/2} \cdot 2(250 - x) \cdot (-1) \\ \frac{250 - x}{40\sqrt{100^2 + (250 - x)^2}} &= \frac{1}{70} \\ 7(250 - x) &= 4\sqrt{100^2 + (250 - x)^2} \\ 49(250 - x)^2 &= 16 \cdot 100^2 + 16(250 - x)^2 \\ 33(250 - x)^2 &= 16 \cdot 100^2 \\ x &= 250 \pm \frac{400}{\sqrt{33}} \end{aligned}$$

Therefore, only one of the critical points above lies in the interval $0 \leq x \leq 250$, and its approximate value is 180.37. We now compare the value of the function t at this critical point with the approximate values of t at our two interval endpoints (see table to the right). Note that Angry Mike's fastest route is to drive the first 180.37 miles outside of the hills, and then to cut diagonally across the hills straight toward his destination.

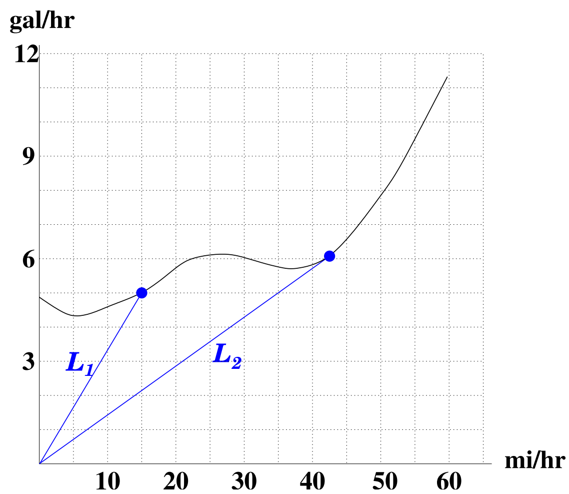
x	$t(x)$
0	6.73
180.37	5.62
250	6.07

4. How long is the part of the trip that Angry Mike has to ride through the hills?

Using the diagram from question (3) above as a guide, we see that Angry Mike's distance traveled through the hills is $\sqrt{100^2 + (250 - x)^2}$ miles. Therefore, since $x = 180.37$ for his most time efficient route, we conclude that the length of his journey through the hills is

$$\sqrt{100^2 + (250 - 180.37)^2} \approx 121.9 \text{ miles.}$$

Meanwhile, Tommy Boy is in hot pursuit and facing an entirely different problem. He is driving a county-issued cross country vehicle. The hills are absolutely no problem, but he can only use one tank of gas plus one reserve tank to follow Angry Mike. The tank of his vehicle holds 25 gallons; the reserve tank holds 16 gallons. Since Tommy Boy knows his brother and his friends, he quickly figured out where Angry Mike is going, and so he is taking the most direct route through the hills which is about 270 miles long. The fuel consumption (in gallons per hour) of Tommy's vehicle as a function of speed (in mph) is given in the graph to the right. Tommy Boy knows that he can maximize the fuel efficiency of his usual patrol car (in miles per gallon) by going 50 mph. He figures that the cross country vehicle is close enough to a normal car to have the same properties.



1. Is he right?

   To figure out how to analyze this question, let us first take a "nice" point on the provided graph, like $(15, 5)$, and try to interpret it in terms of fuel efficiency. The presence of this point on the graph indicates that when this vehicle is driven at a speed of 15 miles per hour, it uses fuel at a rate of 5 gallons per hour. Therefore, the fuel used per mile for this vehicle when driven at a speed of 15 miles per hour is

   $$\frac{5 \text{ gal/hr}}{15 \text{mi/hr}} = \frac{5 \text{ gal}}{\text{hr}} \times \frac{1 \text{ hr}}{15 \text{ mi}} = \frac{1}{3} \text{ gal/mi}.$$

   Note that the number calculated above is the slope of the line segment L_1 from the origin to $(5, 15)$ that we drew on the above diagram. Also, note that if we take the reciprocal of the preceding calculation, we get a fuel efficiency figure of 3 miles per gallon at a driving speed of 15 mph.

   From the above demonstration, we deduce that the fuel efficiency, in miles per gallon, can be determined by taking the reciprocal of the slope of a line segment drawn from the origin to the relevant point in the diagram above. To maximize fuel efficiency, we are therefore looking for the point on the curve above for which this line segment has the smallest possible slope. Using a straightedge to help us graphically estimate, we see that the segment labeled L_2 in the diagram above has the smallest possible slope, and that it intersects the graph at the approximate point $(42, 6)$. Therefore, fuel efficiency for this vehicle is maximized at a driving speed of about 42 miles per hour, so Tommy Boy was incorrect in his assumption.

2. Will he actually make it to Angry Mike's destination at a driving speed of 50 mph?

   First, note that Tommy Boy's travel time to the destination at a speed of 50 mph is given by

   $$\frac{270 \text{ mi}}{50 \text{ mi/hr}} = 5.4 \text{ hours}.$$

   The above graph indicates that, at a speed of 50 mph, Tommy Boy's vehicle uses about 7.9 gallons of fuel per hour. Therefore, the total amount of fuel he needs for the trip is $(5.4) \cdot (7.9) = 42.66$ gallons. Since his total gas supply is only 41 gallons, it appears that he will not make it to the destination at this driving speed.

3. How fast should he go to maximize fuel efficiency?

   According to our calculations in problem (1) above, he should drive at a speed of about 42 miles per hour to maximize fuel efficiency.

4. How much gas would he have to spare if he drove with optimal speed?

As in problem (2) above, we first calculate Tommy Boy's traveling time to the destination; in this case, it is

$$\frac{270 \text{ mi}}{42 \text{ mi/hr}} \approx 6.43 \text{ hours.}$$

The graph above indicates that his vehicle will use about 6.1 gallons of fuel per hour at the optimal speed, so the total amount of fuel used is $(6.43) \cdot (6.1) \approx 39.2$ gallons. He will therefore have about 1.8 gallons of gas to spare.

Section 4.4 – Families of Functions and Modeling

1. (Taken from *Hughes-Hallett, et. al.*) The number, N, of people who have heard a rumor spread by mass media at time, t, is given by $N(t) = a(1 - e^{-kt})$. There are 200,000 people in the population who hear the rumor eventually. If 10% of them heard it the first day, find a and k, assuming that t is measured in days.

   To begin, we are given the following information

   $$\begin{aligned} N(1) &= 0.1 \cdot 200{,}000 = 20{,}000 \\ \lim_{t\to\infty} N(t) &= 200{,}000 \end{aligned}$$

   Because $\lim_{t\to\infty} N(t) = 200{,}000$, we have

   $$\begin{aligned} \lim_{t\to\infty} a(1 - e^{-kt}) &= 200{,}000 \\ a(1 - 0) &= 200{,}000 \end{aligned}$$

   Therefore, $a = 200{,}000$, which means that $N(t) = 200{,}000(1 - e^{-kt})$. But we also know that $N(1) = 20{,}000$, so we have

   $$\begin{aligned} 200{,}000(1 - e^{-k(1)}) &= 20{,}000 \\ 1 - e^{-k} &= 0.1 \\ e^{-k} &= 0.9 \\ k &= -\ln(0.9) \end{aligned}$$

   Therefore, our answers are $a = 200{,}000$ and $k = -\ln(0.9)$.

2. Let $f(x) = x^4 - ax^2$.

   (a) Find all possible critical points of f in terms of a.

   We have $f'(x) = 4x^3 - 2ax = 2x(2x^2 - a)$, so we can see that $f'(x) = 0$ when $x = 0$ or when $x = \pm\sqrt{a/2}$. Therefore, our critical points are $x = 0$, $x = \sqrt{a/2}$, and $x = -\sqrt{a/2}$.

   (b) If $a < 0$, how many critical points does f have?

   If $a < 0$, then $a/2 < 0$, which means that $\sqrt{a/2}$ and $-\sqrt{a/2}$ are not real numbers. Therefore, using our answer to part (a), we see that $x = 0$ is the only critical point of f, i.e., f has exactly one critical point.

   (c) If $a > 0$, find the x and y coordinates of all critical points of f.

   If $a > 0$, then, by part (a), $x = 0$, $x = \sqrt{a/2}$, and $x = -\sqrt{a/2}$ are all critical points of f. We have

   $$\begin{aligned} f(0) &= 0^4 - a(0)^2 = 0 \\ f\left(\pm\sqrt{\frac{a}{2}}\right) &= \left(\sqrt{\frac{a}{2}}\right)^4 - a\left(\sqrt{\frac{a}{2}}\right)^2 = \frac{a^2}{4} - \frac{a^2}{2} = -\frac{a^2}{4}, \end{aligned}$$

Interval	Sign of $f'(x)$
$x < -\sqrt{a/2}$	$-$
$-\sqrt{a/2} < x < 0$	$+$
$0 < x < \sqrt{a/2}$	$-$
$x > \sqrt{a/2}$	$+$

   so the critical points are

   $$(0,0), \qquad \left(\sqrt{\frac{a}{2}},\ -\frac{a^2}{4}\right), \qquad \text{and} \qquad \left(-\sqrt{\frac{a}{2}},\ -\frac{a^2}{4}\right),$$

   which can be classified as local maxima or minima by referring to the above sign chart.

(d) Find a value of a such that the two local minima of f occur at $x = \pm 2$.

The sign chart from part (c) reveals that $(\pm\sqrt{a/2},\ -a^2/4)$ are the coordinates of the two local minima of f. We have

$$\begin{aligned}\sqrt{a/2} &= 2\\ a/2 &= 4\\ a &= 8\end{aligned}$$

Therefore, the local minima of f occur at $x = \pm 2$ when $a = 8$.

3. Let $f(x) = axe^{-bx}$. ASSUME THAT a AND b ARE BOTH POSITIVE.

(a) Find all inflection points of f in terms of a and b.

Interval	Sign of $f''(x)$
$x < 2/b$	$-$
$x > 2/b$	$+$

Since $f'(x) = axe^{-bx} \cdot (-b) + ae^{-bx} = ae^{-bx}(1 - bx)$, we see that

$$f''(x) = ae^{-bx} \cdot (-b) + ae^{-bx} \cdot (-b) \cdot (1 - bx) = abe^{-bx}(bx - 2).$$

Therefore, $f''(x) = 0$ if and only if $x = 2/b$, and the sign chart above confirms that an inflection point does indeed occur at this point. Since $f(2/b) = (2a/b)e^{-2}$, we see that the one and only inflection point of f is

$$\left(\frac{2}{b},\ \frac{2a}{b}e^{-2}\right).$$

(b) Find a and b so that the inflection point of f occurs at $(1, 2)$.

In order for the inflection point that we found in part (a) above to occur at $(1, 2)$, we must have $2/b = 1$ and $(2a/b)e^{-2} = 2$. Since $2/b = 1$, we have $b = 2$, which leads to

$$\begin{aligned}\frac{2a}{2}e^{-2} &= 2\\ a &= 2e^2\end{aligned}$$

Therefore, we conclude that $a = 2e^2$ and $b = 2$.

Section 4.7 – L'Hopital's Rule, Growth, and Dominance

1. Find each of the following limits **exactly**.

(a) $\lim_{x\to 0} \frac{\sin x}{x}$

Since this limit has the form "0/0", we can use L'Hopital's Rule as follows:

$$\begin{aligned} \lim_{x\to 0} \frac{\sin x}{x} &= \lim_{x\to 0} \frac{\cos x}{1} \\ &= \cos 0 \\ &= 1 \end{aligned}$$

(b) $\lim_{x\to\infty} \frac{\ln x}{x}$

Since $\ln x$ and x both approach infinity as x approaches infinity, this limit has the form "∞/∞", so we can use L'Hopital's Rule as follows:

$$\begin{aligned} \lim_{x\to\infty} \frac{\ln x}{x} &= \lim_{x\to\infty} \frac{1/x}{1} \\ &= \lim_{x\to\infty} \frac{1}{x} \\ &= 0 \end{aligned}$$

(c) $\lim_{x\to 0} \frac{x}{e^x}$

Since this limit does not fit any of the Indeterminate Forms, we may not apply L'Hopital's Rule. Instead, we have

$$\lim_{x\to 0} \frac{x}{e^x} = \frac{0}{e^0} = 0.$$

(d) $\lim_{x\to\infty} \frac{\ln(\ln x)}{\sqrt{x}}$

Since this limit has the form "∞/∞", we begin by applying L'Hopital's Rule. We have

$$\begin{aligned} \lim_{x\to\infty} \frac{\ln(\ln x)}{\sqrt{x}} &= \lim_{x\to\infty} \frac{\frac{1}{\ln x}\cdot\frac{1}{x}}{\frac{1}{2x^{1/2}}} \\ &= \lim_{x\to\infty} \frac{2}{x^{1/2}\ln x} \\ &= 0 \end{aligned}$$

(e) $\lim_{x\to 1^+} (x-1)\tan\left(\frac{\pi}{2}x\right)$

Since this limit has the form "$0\cdot(-\infty)$", we may use L'Hopital's rule <u>after</u> rewriting the involved function as a ratio. We have

$$\begin{aligned} \lim_{x\to 1^+} (x-1)\tan\left(\frac{\pi}{2}x\right) &= \lim_{x\to 1^+} \frac{x-1}{\cot((\pi/2)x)} \\ &= \lim_{x\to 1^+} \frac{1}{-\csc^2((\pi/2)x)\cdot(\pi/2)} \\ &= \lim_{x\to 1^+} \frac{-\sin^2((\pi/2)x)}{\pi/2} \\ &= \lim_{x\to 1^+} \frac{-2\sin^2((\pi/2)x)}{\pi} \\ &= \frac{-2\sin^2(\pi/2)}{\pi} \\ &= -\frac{2}{\pi}. \end{aligned}$$

Definition. We say that a function g *dominates* a function f as $x \to \infty$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \underline{\ 0\ }$.

2. For each of the following, determine which function dominates as $x \to \infty$.

(a) $1000x^2$ and x^3

Using L'Hopital's Rule twice, we have

$$\lim_{x\to\infty} \frac{1000x^2}{x^3} = \lim_{x\to\infty} \frac{2000x}{3x^2} = \lim_{x\to\infty} \frac{2000}{6x} = 0,$$

so we conclude that x^3 dominates $1000x^2$.

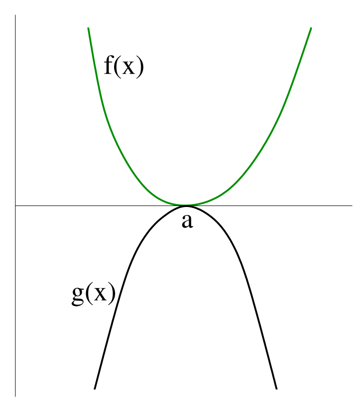
(b) $e^{0.1x}$ and x^3

Using L'Hopital's Rule three times, we have

$$\lim_{x\to\infty} \frac{x^3}{e^{0.1x}} = \lim_{x\to\infty} \frac{3x^2}{0.1e^{0.1x}} = \lim_{x\to\infty} \frac{6x}{(0.1)^2 e^{0.1x}} = \lim_{x\to\infty} \frac{6}{(0.1)^3 e^{0.1x}} = 0,$$

so we conclude that $e^{0.1x}$ dominates x^3.

3. Use the graph to the right to determine the sign of the limit $\lim_{x\to a} \frac{f(x)}{g(x)}$. Briefly explain how you determined your answer. (Note: You may assume that the limit exists and that all derivatives of f and g exist.)



First, note that, from the graph, it appears that $f(a) = g(a) = 0$ and that $f'(a) = g'(a) = 0$, so we conclude that the limits

$$\lim_{x\to a} \frac{f(x)}{g(x)} \quad \text{and} \quad \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

both have the form $0/0$. Therefore, using L'Hopital's Rule, we have

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} = \lim_{x\to a} \frac{f''(x)}{g''(x)} = \frac{f''(a)}{g''(a)}$$

But the above graph also reveals that f is concave up near a and that g is concave down near a, so $f''(a) > 0$ and $g''(a) < 0$. Therefore, it follows from the previous set of equations that

$$\lim_{x\to a} \frac{f(x)}{g(x)} < 0.$$