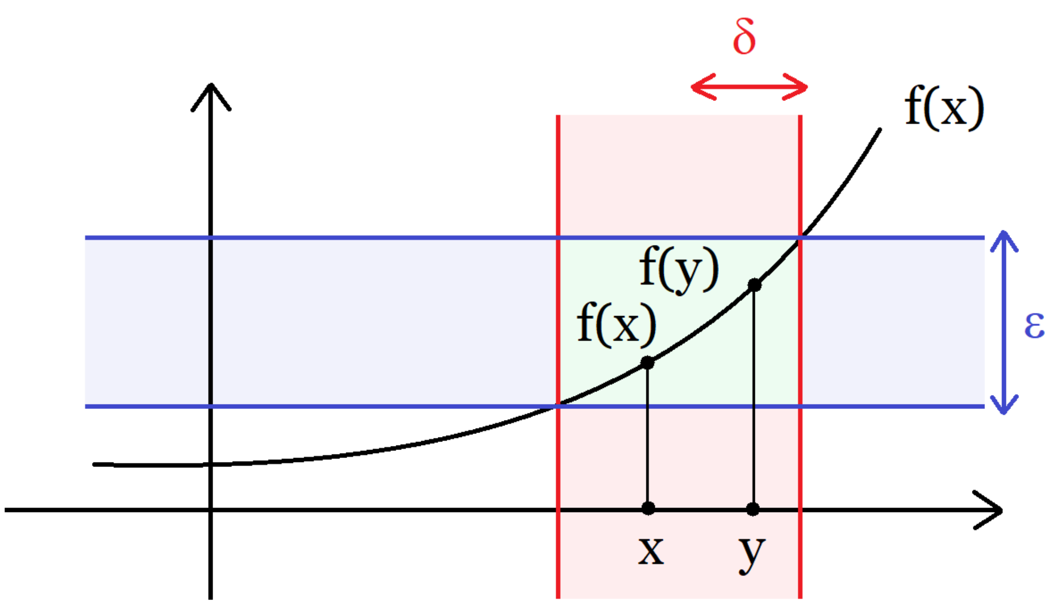
# LECTURE 28: UNIFORM CONTINUITY (II)

**Recall:**

$f$ is **uniformly continuous** on $S$ if: For all $\epsilon > 0$ there is $\delta > 0$ such that, for all $x, y \in S$

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$$




In other words, there is some $\delta$ independent of $x$ and $y$ that makes $f$ continuous.

**Today:** All about properties of uniform continuity.

*Date*: Wednesday, June 3, 2020.

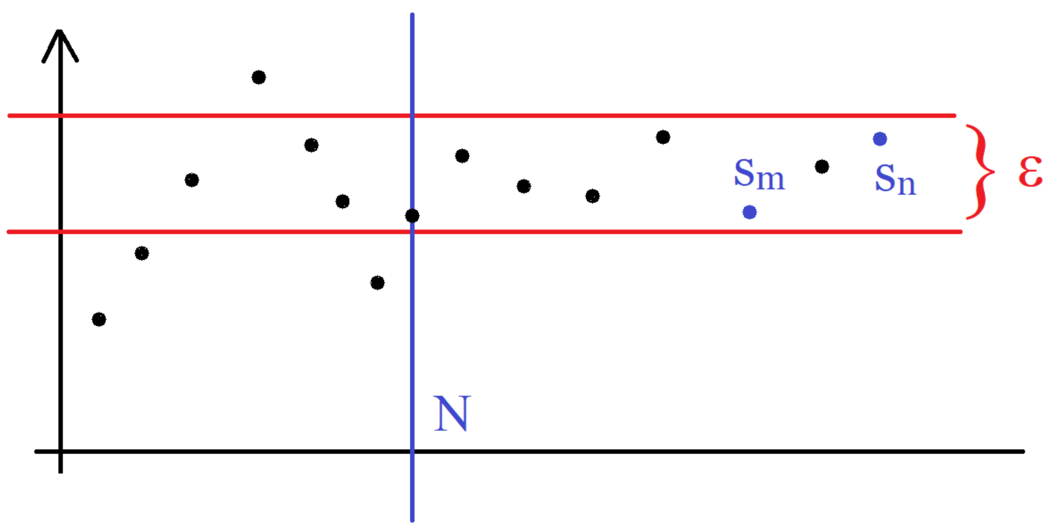
## 1. Uniform Continuity and Cauchy

**Video:** Uniform Continuity and Cauchy

Let's discuss a useful property that helps us understand how uniformly continuous behave. For this, let's recall the definition of Cauchy sequences from section 10:

**Recall (Section 10):**

$(s_n)$ is **Cauchy** if for all $\epsilon > 0$ there is $N$ such that if $m, n > N$, then $|s_m - s_n| < \epsilon$



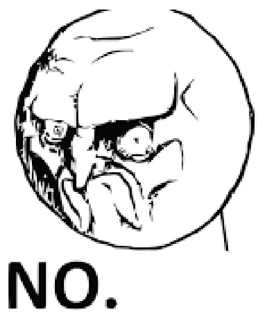

(The terms of the sequence $(s_n)$ are eventually as close to each other as we want. This is great way of talking about convergence without mentioning the limit)

If $f$ is continuous and $(s_n)$ converges (to $x_0$), then, by definition, $f(s_n)$ is converges as well (to $f(x_0)$)

But what if $(s_n)$ is just Cauchy?

**Question:** If $(s_n)$ is Cauchy and $f$ is continuous, is $f(s_n)$ Cauchy?

In general, the answer is:

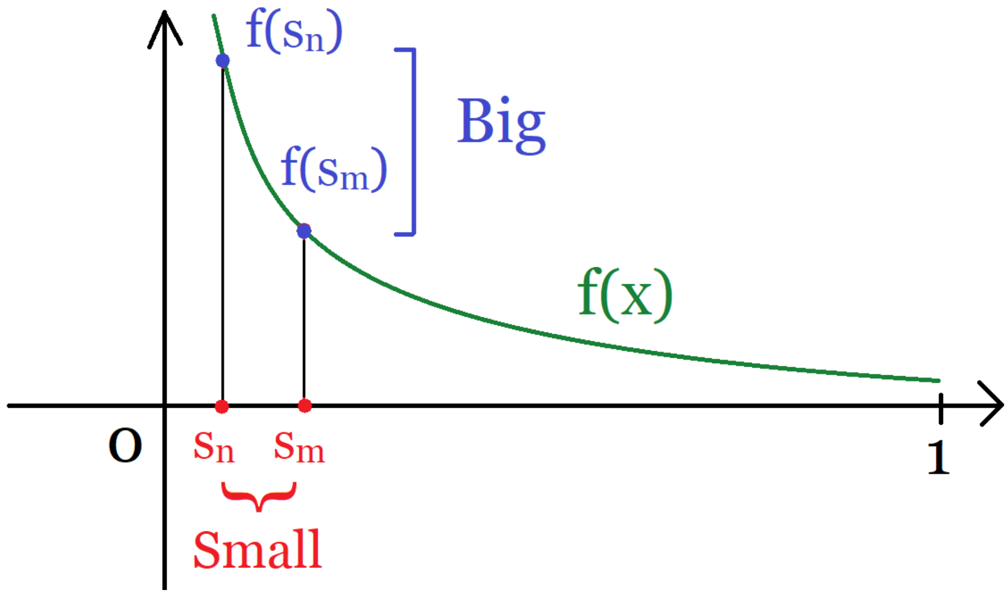



**Example:**

Let $f(x) = \frac{1}{x}$ on $(0,1)$

Then $s_n = \frac{1}{n} \in (0,1)$ is Cauchy (because it converges; here $n \geq 2$), but $f(s_n) = \frac{1}{s_n} = n \to \infty$ is not Cauchy (because it doesn't converge)

What is going on here? Even though the inputs $(s_n)$ are close together, the outputs $f(s_n)$ are very far apart. In some sense, $f$, even though continuous, "spreads out" points near 0.

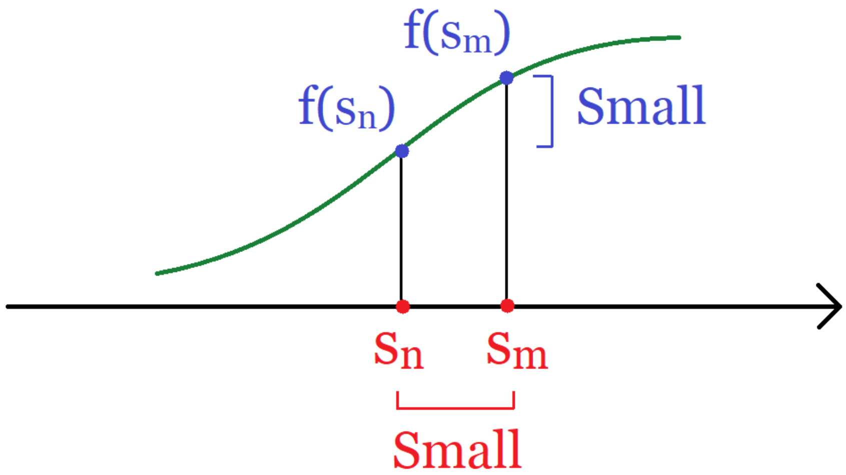



You may have guessed it, but the reason this fails is because $f$ is not *uniformly* continuous. In fact, if $f$ is uniformly continuous, then the answer to the question above is **YES**:

**Fact:**

If $f$ is uniformly continuous on a set $S$ and $(s_n)$ is a Cauchy sequence in $S$, then $f(s_n)$ is Cauchy as well

In other words, uniformly continuous functions take Cauchy sequences to Cauchy sequences.




**Application:**

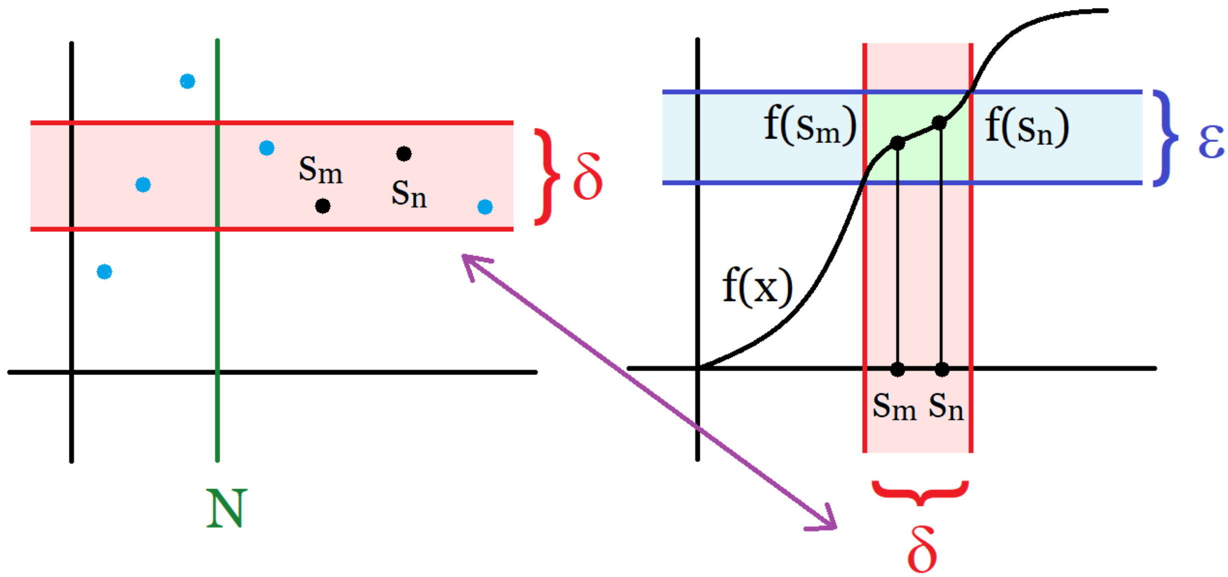
$f(x) = \frac{1}{x}$ is **not** uniformly continuous on $(0, 1)$ because $s_n = \frac{1}{n}$ is Cauchy in $(0, 1)$ but $f(s_n)$ is not Cauchy.

**Idea of Proof:** Since $(s_n)$ is Cauchy, the inputs $s_n$ and $s_m$ are close to each other. Since $f$ is uniformly continuous, close inputs give you close outputs. Hence $f(s_n)$ and $f(s_m)$ are close to each other, so $f(s_n)$ is Cauchy.

**Proof:** Suppose $(s_n)$ is Cauchy and let $\epsilon > 0$ be given. Since $f$ is uniformly continuous on $S$, there is $\delta > 0$ such that if $x, y \in S$ and $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$

But since $(s_n)$ is Cauchy (with $\delta$ instead of $\epsilon$), there is $N$ such that if $m, n > N$, then $|s_n - s_m| < \delta$, and therefore we get (using $x = s_n$ and $y = s_m$ ) $|f(s_n) - f(s_m)| < \epsilon$ ✓

Hence $f(s_n)$ is Cauchy $\square$




**Note:** This proof works *precisely* because $f$ is uniformly continuous. Since $(s_n)$ is Cauchy, we can make $s_n$ to be $\delta-$close, and therefore $f(s_n)$ are $\epsilon-$close. It's very important that $\delta$ is independent of $x$, since we don't know where the $s_n$ are; they could be near 0 or near 1 or near some other number, we don't know!

## 2. Continuous Extensions

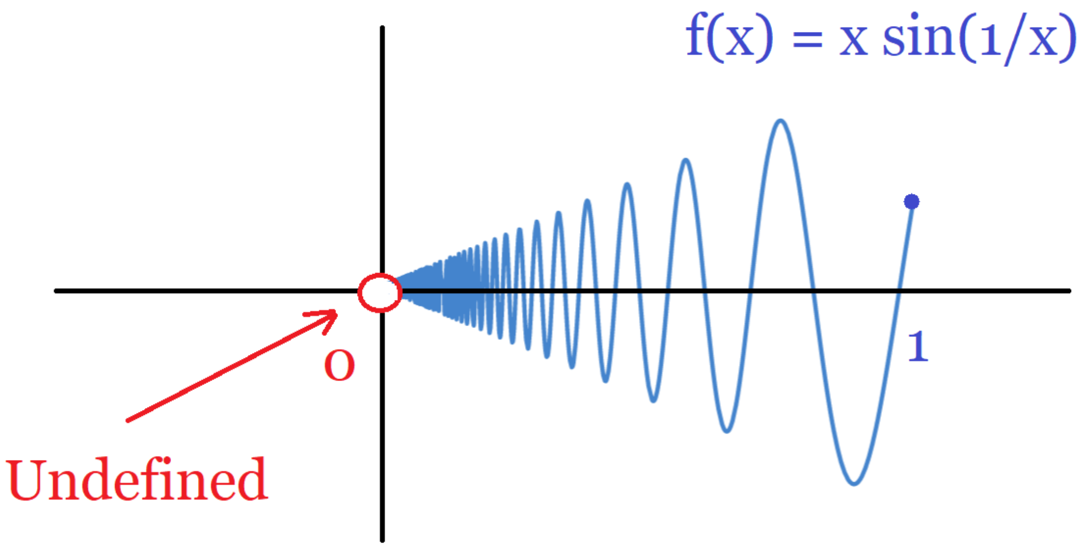
**Video:** Continuous Extensions

This is, in my opinion, the most important property because it relates uniform continuity, a really abstract concept, with continuous extensions, something much more concrete. Let's motivate this with a couple of examples:

**Example 1:**

Let $f(x) = x \sin\left(\frac{1}{x}\right)$ on $(0, 1]$.

(notice that $f$ is undefined at 0)

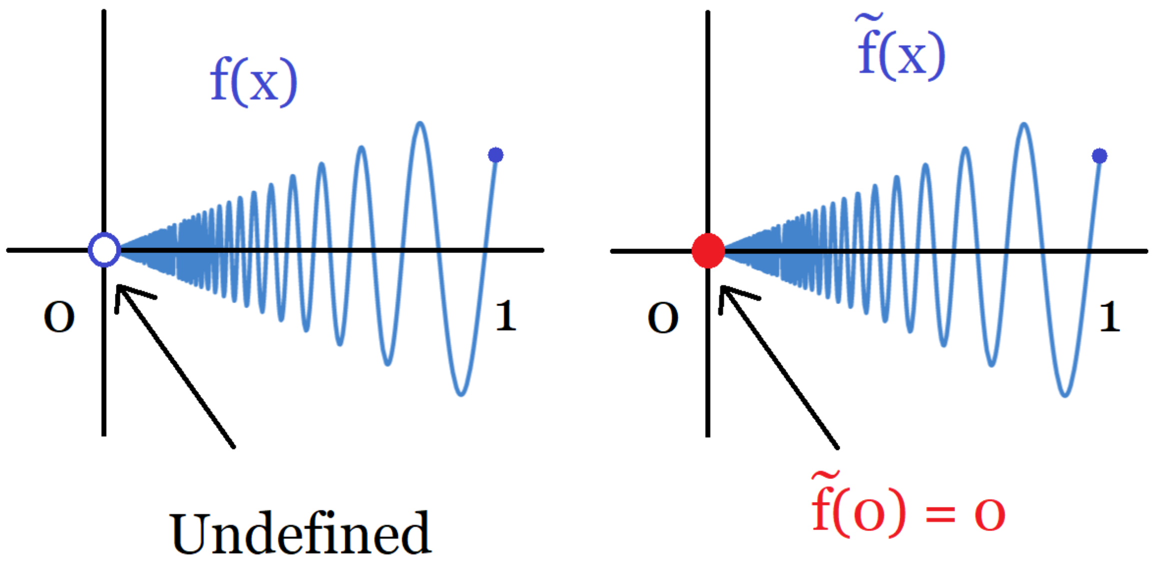



**Problem:** Is there some way of defining $f$ at 0 such that $f$ becomes continuous at 0?

YES, just let $f(0) = 0$

In other words, if you let

$$\tilde{f}(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & \text{if } x \in (0, 1] \\ 0 & \text{if } x = 0 \end{cases}$$




Then:

(1) $\tilde{f}$ is continuous on $[0,1]$ and

(2) For $x \in (0,1]$, $\tilde{f}(x) = f(x)$

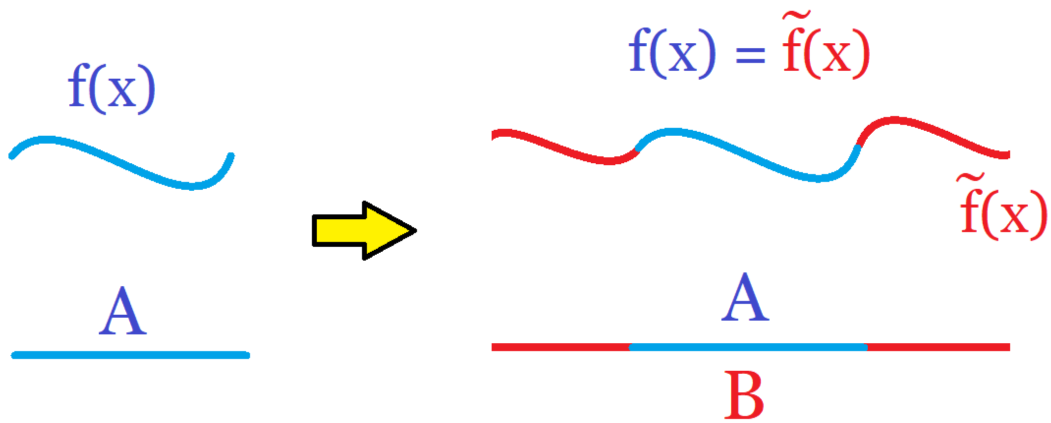
We call $\tilde{f}$ a **continuous extension** of $f$:

**Definition:**

Suppose $A \subseteq B$ (think $A = (0,1]$ and $B = [0,1]$)and $f : A \to \mathbb{R}$ is continuous. Then $\tilde{f} : B \to \mathbb{R}$ is a **continuous extension** of $f$ if

(1) $\tilde{f}$ is continuous on $B$ and

(2) For all $x \in A$ we have $\tilde{f}(x) = f(x)$



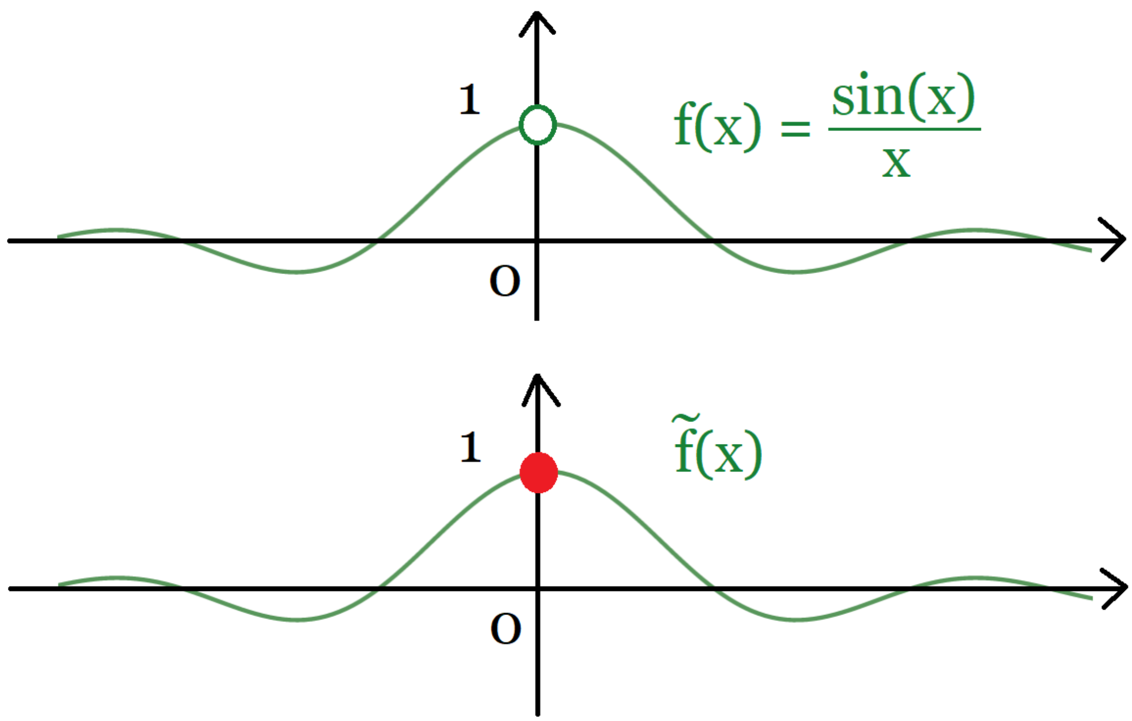

So next time you ask for an extension on an assignment, ask for a *continuous* extension ☺

**Example (not in the video):**

Let $f(x) = \frac{\sin(x)}{x}$ for $x \neq 0$, then $\tilde{f} : \mathbb{R} \to \mathbb{R}$ defined by:

$$\tilde{f}(x) = \begin{cases} \frac{\sin(x)}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

Is a continuous extension of $f$.



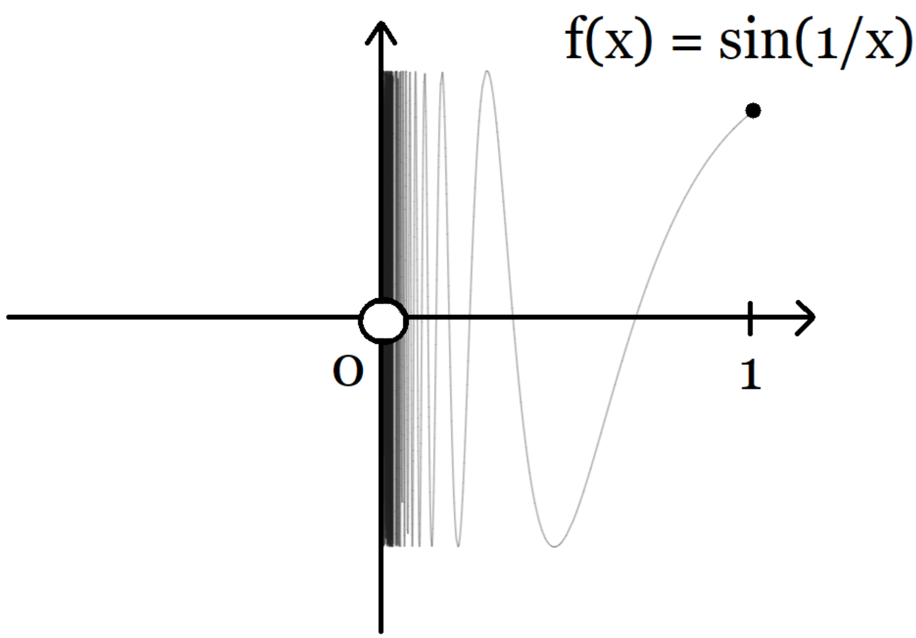

**Note:** The reason $\tilde{f}$ is continuous is because (from Calculus)

$$\lim_{x\to 0} \frac{\sin(x)}{x} \to 1$$

This limit is beyond the scope of the course, but check out this video for a really elegant geometric proof: $\frac{\sin(x)}{x}$ as $x$ goes to 0. It turns out that $\tilde{f}$ is uniformly continuous on $\mathbb{R}$ (see HW) but that's a pure coincidence.

**Example 2:**

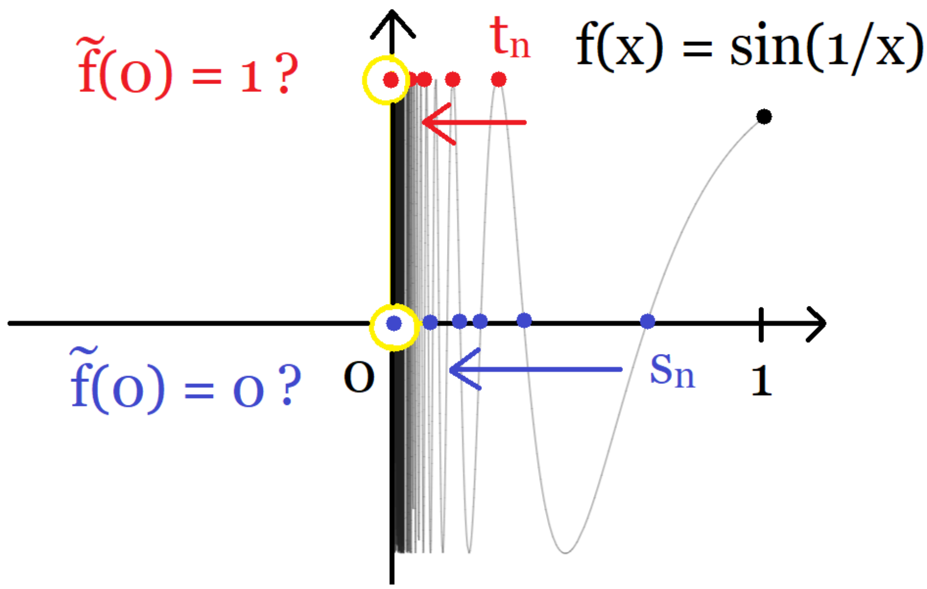
This time consider $f(x) = \sin\left(\frac{1}{x}\right)$ on $(0, 1]$.




Notice that, since $x \neq 0$, $f$ is continuous on $(0, 1]$.

Can we extend $f$ to a continuous function $\tilde{f}$?

**NO** No matter how we define $\tilde{f}(0)$, $\tilde{f}$ will not be continuous on $[0, 1]$




**Why?** (not in the Video) Let $s_n = \frac{1}{\pi n} \to 0$. If $\tilde{f}$ were continuous at 0, then:

$$\tilde{f}(0) = \lim_{n\to\infty} \tilde{f}(s_n) = \lim_{n\to 0} f(s_n) = \sin\left(\frac{1}{s_n}\right) = \sin(\pi n) = 0$$

(Here we used $\tilde{f} = f$ on $(0,1]$)

On the other hand, let $t_n = \frac{1}{\frac{\pi}{2}+2\pi n} \to 0$. Then

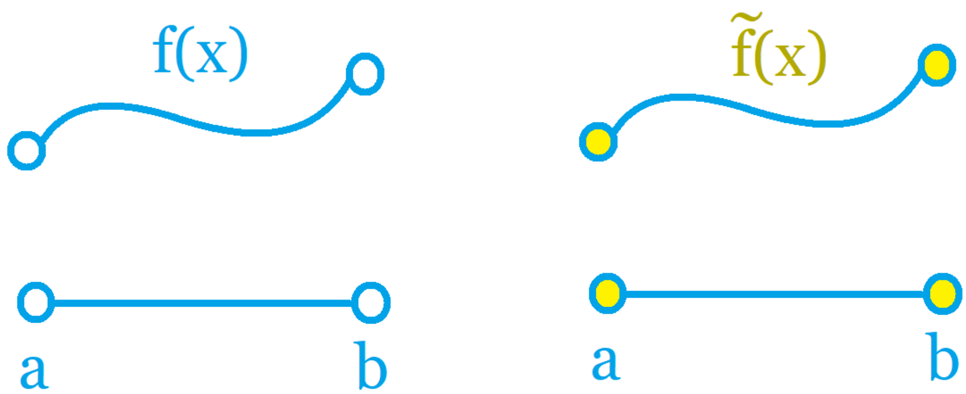
$$\tilde{f}(0) = \lim_{n\to\infty} \tilde{f}(t_n) = \lim_{n\to 0} f(t_n) = \sin\left(\frac{1}{t_n}\right) = \sin\left(\frac{\pi}{2} + 2\pi n\right) = 1$$

Which contradicts $\tilde{f}(0) = 0 \Rightarrow\Leftarrow$. Hence $\tilde{f}$ cannot exist $\square$

Why did Example 1 work but Example 2 fail? The key difference is uniform continuity: In Example 1, $f(x) = x\sin\left(\frac{1}{x}\right)$ is uniformly continuous on $(0,1]$ (see below), whereas in Example 2, $f(x) = \sin\left(\frac{1}{x}\right)$ is not uniformly continuous on $(0,1]$

**Fact 1:**

Suppose $f : (a,b) \to \mathbb{R}$ is continuous. If $f$ has a continuous extension $\tilde{f}$ on $[a,b]$, then $f$ is uniformly continuous.




**Note:** This fact works for *any* subset of $[a,b]$, not just $(a,b)$. The main reason this proof works is because $[a,b]$ is compact.

**Application:**

Let $f(x) = x\sin\left(\frac{1}{x}\right)$ on $(0,1]$ (from Example 1). Then $f$ has a continuous extension $\tilde{f}(x)$ on $[0,1]$, so $x\sin\left(\frac{1}{x}\right)$ is uniformly continuous on $(0,1]$.

By the same reasoning, $f(x) = \frac{\sin(x)}{x}$ is uniformly continuous on $(0,1)$.

**Proof:** Suppose $f$ has a continuous extension $\tilde{f}$ on $[a, b]$. Then since $\tilde{f}$ is continuous on $[a, b]$ (by definition) and $[a, b]$ is compact, $\tilde{f}$ is uniformly continuous on $[a, b]$ (see last time). Therefore, in particular, $\tilde{f} = f$ is uniformly continuous on $(a, b)$ $\square$

**Question:** Is the converse true? That is, if $f$ is uniformly continuous, does $f$ have a continuous extension? **YES!**

**Fact 2:**

If $f : (a, b) \to \mathbb{R}$ is uniformly continuous on $(a, b)$, then $f$ has a continuous extension $\tilde{f}$ on $[a, b]$

**Note:** There is a **LOT** of flexibility here: You can replace $[a, b]$ as above by $(a, b)$, $[a, b)$, $(a, \infty)$ etc.

**Note:** The main reason this proof works is because $(a, b)$ is *dense* in $[a, b]$, that is the closure of $(a, b)$ is $[a, b]$. In fact, the same proof works if you replace $(a, b)$ by $\mathbb{Q}$ and $[a, b]$ with $\mathbb{R}$.

**Application:**

$f(x) = \sin\left(\frac{1}{x}\right)$ from Example 2 is **NOT** uniformly continuous on $(0, 1]$ because $f$ has no continuous extension on $[a, b]$

**Proof:** The proof is magical! We'll do some wishful thinking that actually works.

**STEP 1:** Suppose $f$ is uniformly continuous on $(a, b)$. Since on $(a, b)$, $\tilde{f}(x) =: f(x)$ is continuous, all we really need to do is define $\tilde{f}(a)$ and show $\tilde{f}$ is continuous at $a$ (the case $\tilde{f}(b)$ is similar)

**Main Idea:**

**If** $\tilde{f}$ were continuous at $a$, **then** for any sequence $(s_n)$ in $(a,b)$ with $s_n \to a$, we would have

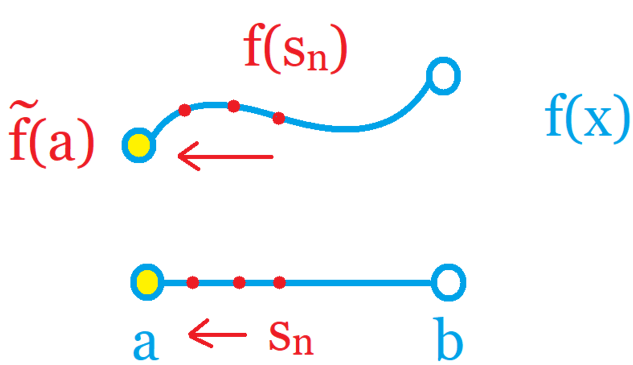
$$\lim_{n\to\infty} f(s_n) = \lim_{n\to\infty} \tilde{f}(s_n) = \tilde{f}(a)$$

(Here we used $s_n \in (a,b)$ and $\tilde{f} = f$ on $(a,b)$)

The idea is then to *define* $\tilde{f}(a)$ as:

$$\tilde{f}(a) =: \lim_{n\to\infty} f(s_n)$$

Where $(s_n)$ is any sequence in $(a,b)$ converging to $a$



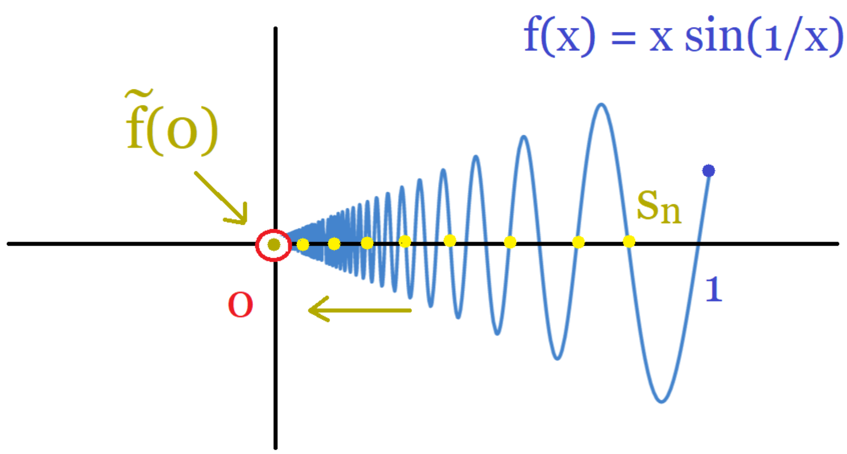
**Example: (not in the video)**

Take again $f(x) = x\sin\left(\frac{1}{x}\right)$ from Example 1. What is $\tilde{f}(0)$?

Let $s_n = \frac{1}{\pi n} \to 0$. Then, by the above, we have

$$\tilde{f}(0) = \lim_{n\to\infty} f(s_n) = \lim_{n\to\infty} s_n \sin\left(\frac{1}{s_n}\right) = \lim_{n\to\infty}\left(\frac{1}{\pi n}\right)\underbrace{\sin(\pi n)}_{0} = 0$$

Therefore $\tilde{f}(0) = 0$




The definition above seems too good to be true! We're *literally* defining $\tilde{f}(a)$ in such a way that it solves our problem. It turns out that it actually works. But in order to make sure that $\tilde{f}(a)$ is well-defined, we need to answer the following questions:

(1) Does $f(s_n)$ even converge? (otherwise $\lim f(s_n)$ makes no sense)

(2) More importantly: Is the above limit independent of the choice of the sequence $(s_n)$ used?

**STEP 2:**

**Claim 1:** If $(s_n)$ is a sequence in $(a, b)$ that converges to $a$, then $f(s_n)$ converges

**Proof of Claim 1:** Since $(s_n)$ converges, $(s_n)$ is Cauchy, and therefore, since $f$ is uniformly continuous, by the previous section, $f(s_n)$ is Cauchy, and therefore $f(s_n)$ converges ✓

**STEP 3:**

**Claim 2:** Suppose $(s_n)$ and $(t_n)$ are two sequences in $(a, b)$ converging to $a$, then

$$\lim_{n\to\infty} f(s_n) = \lim_{n\to\infty} f(t_n)$$

(This shows that the definition $\tilde{f}(a)$ above does not depend on the choice of $(s_n)$)

**Proof of Claim 2:** Suppose $(s_n)$ and $(t_n)$ both converge to $a$.

Here's a neat idea: let's *interlace* the two sequences $(s_n)$ and $(t_n)$ to get a new sequence $(u_n)$:

$$(u_n) = (s_1, t_1, s_2, t_2, \dots)$$

**Claim 3:** $(u_n)$ converges to $a$

(See optional proof below)

Since $u_n \to a$ and $f$ is continuous,

$$f(u_n) = (f(s_1), f(t_1), f(s_2), f(t_2), \dots)$$

converges to some $s \in \mathbb{R}$. Therefore, any subsequence of $f(u_n)$ converges to $s$ as well.

But $f(s_n) = (f(s_1), f(s_2), \dots)$ is a subsequence of $f(u_n)$, and hence converges to $s$. Similarly $f(t_n) = (f(t_1), f(t_2), \dots)$ is a subsequence of $f(u_n)$, hence converges to $s$ as well.

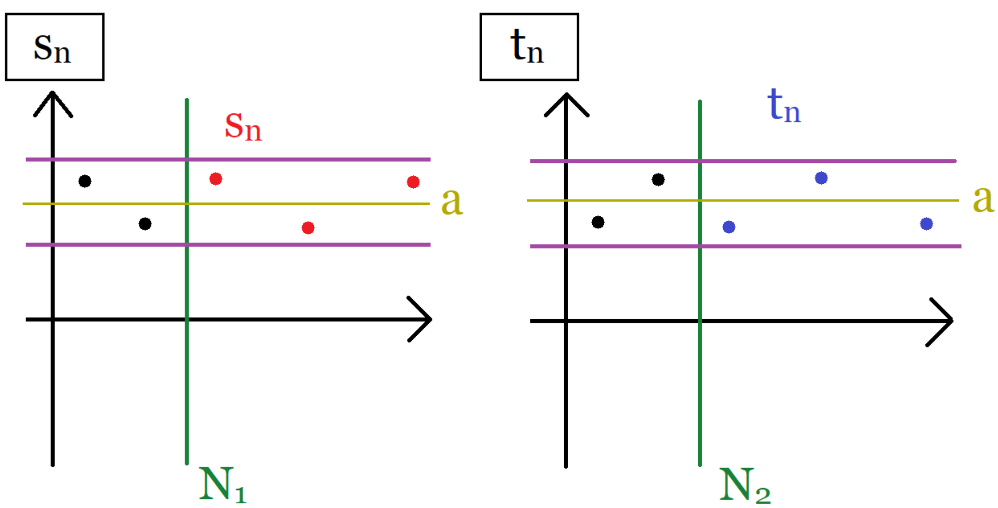
Therefore

$$\lim_{n\to\infty} f(s_n) = s = \lim_{n\to\infty} f(t_n)\checkmark$$

**Proof of Claim 3:** (optional, not in the video)

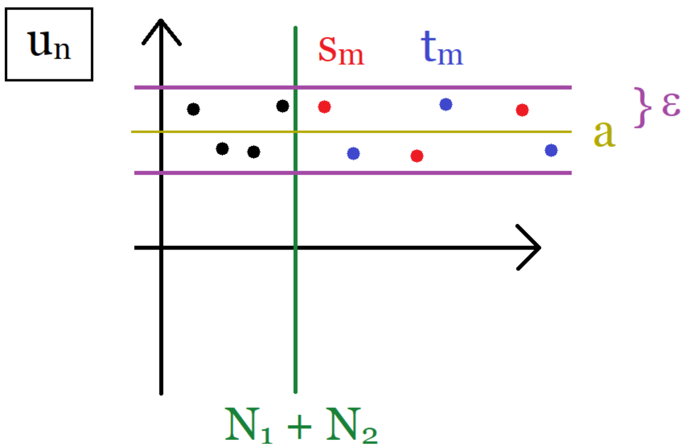
Let $\epsilon > 0$ be given.

Since $s_n \to a$, there is $N_1$ such that if $n > N_1$, then $|s_n - a| < \epsilon$, and since $t_n \to a$, there is $N_2$ such that if $n > N_2$, then $|t_n - a| < \epsilon$.



Let $N = N_1 + N_2$

Then if $n > N$, either $u_n = s_m$ for some $m > N_1$ in which case $|u_n - a| = |s_m - a| < \epsilon$; or $u_n = t_m$ for some $m > N_2$, in which case $|u_n - a| = |t_m - a| < \epsilon$ as well ✓




## STEP 4:

Define

$$\tilde{f}(a) =: \lim_{n\to\infty} f(s_n)$$

Where $(s_n)$ is any sequence in $(a, b)$ converging to $a$

By STEP 2 and STEP 3, $\tilde{f}(a)$ is well-defined.

It is enough to check that $\tilde{f}$ is continuous at $x = a$

This is mathemagical: Let $(s_n)$ be a sequence in $[a,b]$ converging to $a$, we need to show $\tilde{f}(s_n) \to \tilde{f}(a)$

WLOG, assume $s_n \in (a,b)$ for all $n$, because, since we only care about things close to $a$, we may assume $s_n < b$, and if $s_n = a$, then $\tilde{f}(s_n) = \tilde{f}(a)$ anyway.

Therefore $(s_n)$ is a sequence in $(a,b)$ converging to $a$, and therefore:

$$\lim_{n\to\infty} \tilde{f}(s_n) = \lim_{n\to\infty} f(s_n) = \tilde{f}(a) \checkmark$$

(where, in the first step, we used $s_n \in (a,b)$ and in the second step we used the DEFINITION of $\tilde{f}(a)$)

Hence $\tilde{f}$ is a continuous extension of $f$ ✓ □

## 3. Uniform Continuity and Derivatives

**Video:** Uniform Continuity and Derivatives

I really saved the best for last, because here is the most useful way to show that $f$ is uniformly continuous.

**Fact:**

If $f$ is continuous on $[a,b]$ (and differentiable on $(a,b)$).

Suppose $f'$ is bounded, that is there is $M > 0$ such that $|f'(x)| \leq M$ for all $x \in (a,b)$.

Then $f$ is uniformly continuous on $[a,b]$

**Note:** The same result holds if you replace $[a, b]$ by $(a, b)$ or by $[a, \infty)$ or variations thereof; and even for $\mathbb{R}$ (Otherwise there would be no point in proving this fact because continuous functions on $[a, b]$ are automatically uniformly continuous)

**Example 1:**

Let $f(x) = \frac{1}{x}$ on $[2, \infty)$ (continuous). Then $f'(x) = -\frac{1}{x^2}$ and therefore, for all $x \in (2, \infty)$ have

$$|f'(x)| = \left|-\frac{1}{x^2}\right| = \frac{1}{x^2} \leq \frac{1}{2^2} = \frac{1}{4} = M$$

Therefore $f$ is uniformly continuous on $[2, \infty)$

**Example 2:**

Let $f(x) = \sin(x)$ on $\mathbb{R}$

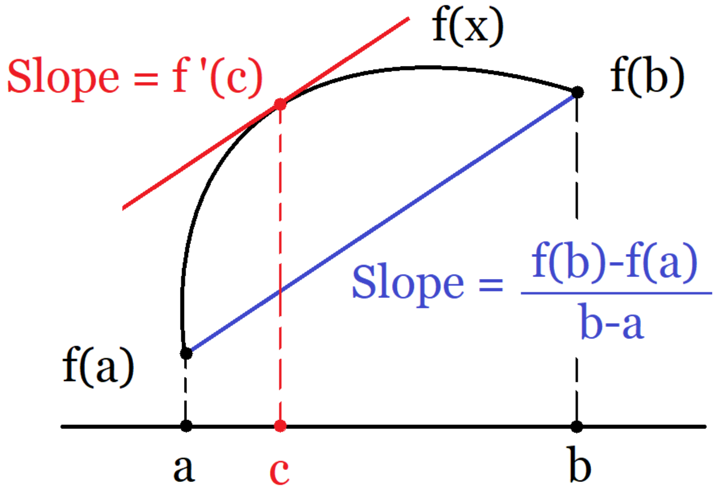
Then for all $x$, $|f'(x)| = |\cos(x)| \leq 1 = M$, hence $\sin(x)$ is uniformly continuous on $\mathbb{R}$

The proof of this uses the Mean Value Theorem from Calculus (which you'll cover in Math 140B)

**Mean Value Theorem:**

If $f$ is continuous on $[a, b]$ and differentiable on $(a, b)$, then there is $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$




In other words, the slope $f'(c)$ of the tangent line of $f$ $c$ is equal to the slope $\frac{f(b)-f(a)}{b-a}$ of the secant line of $f$. In other words, there is a point $c$ in $(a,b)$ at which the tangent line of $f$ and the secant line are parallel.

**Note:** In case you're interested, here's a proof of the Mean Value Theorem: MVT Proof

**Proof of Fact:** Suppose $|f'(x)| \leq M$ for all $x$.

Let $\epsilon > 0$ be given and let $\delta = \frac{\epsilon}{M}$.

Then if $x, y \in (a,b)$ and $|x-y| < \delta$, so by the Mean Value Theorem with $x$ and $y$, there is $c$ between $x$ and $y$ such that

$$\frac{f(y)-f(x)}{y-x} = f'(c) \Rightarrow f(y)-f(x) = f'(c)(y-x)$$

But then we get

$$|f(y)-f(x)| = \underbrace{|f'(c)|}_{\leq M} |y-x| \leq M\,|y-x| < M\left(\frac{\epsilon}{M}\right) = \epsilon \checkmark$$

Hence $f$ is uniformly continuous on $[a, b]$ $\square$

Congratulations!!! We are now officially done with the material of the course! ☺ In the next lecture, we will review the main concepts of the course in a series called *The Essence of Analysis*.