LECTURE 20: SERIES (II)

Let's continue our series extravaganza! Today's goal is to prove the celebrated Ratio and Root Tests and to compare them.

1. The Root Test

Video: Root Test Proof

Among all the convergence tests, the root test is the best one, or at least better than the ratio test. Let me remind you how it works:

> **Example 1:**
>
> Use the root test to figure out if the following series converges:
> $$\sum_{n=0}^{\infty} \frac{n}{3^n}$$

Let $a_n = \frac{n}{3^n}$, then the root test tells you to look at:

$$|a_n|^{\frac{1}{n}} = \left| \frac{n}{3^n} \right|^{\frac{1}{n}} = \frac{n^{\frac{1}{n}}}{3^{n\left(\frac{1}{n}\right)}} = \frac{n^{\frac{1}{n}}}{3} \overset{n \to \infty}{\to} \frac{1}{3} = \alpha < 1$$

Therefore $\sum a_n$ converges absolutely.

Since $\lim_{n\to\infty} |a_n|^{\frac{1}{n}}$ doesn't always exist, we need to replace this with $\limsup_{n\to\infty} |a_n|^{\frac{1}{n}}$, which always exists. Therefore, we obtain the root

test:

Root Test

Consider $\sum a_n$ and let

$$\alpha = \limsup_{n \to \infty} |a_n|^{\frac{1}{n}}$$

(1) If $\alpha < 1$, then $\sum a_n$ converges absolutely (that is $\sum |a_n|$ converges)

(2) If $\alpha > 1$, then $\sum a_n$ diverges

(3) If $\alpha = 1$, then the root test is inconclusive, meaning that you'd have to use another test

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Proof of (1): ($\alpha < 1 \Rightarrow$ converges absolutely)

Main Idea: Since $\limsup_{n \to \infty} |a_n|^{\frac{1}{n}} = \alpha < 1$, then for large n we will have $|a_n|^{\frac{1}{n}} \leq \alpha$. So $|a_n| \leq \alpha^n$ and therefore $\sum |a_n| \leq \sum \alpha^n$, which is a geometric series that converges, since $\alpha < 1$.

However, keep in mind that this is just an (incorrect) idea, which we'll have to make precise below.
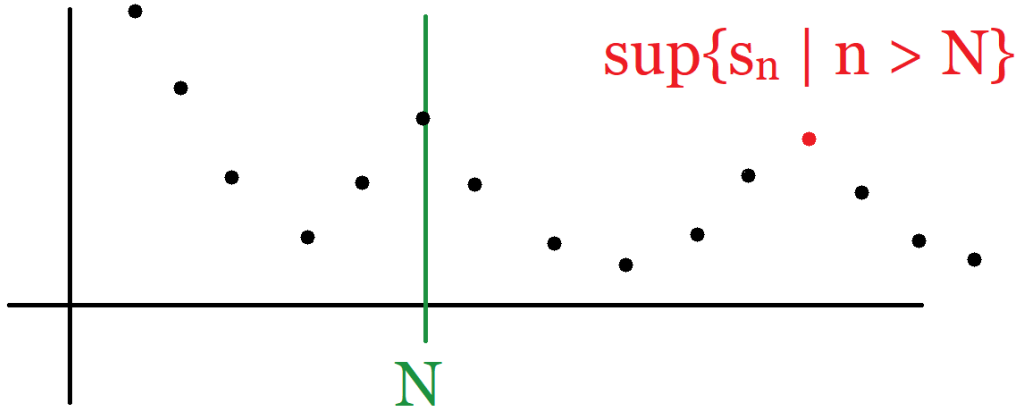
The proof itself is similar to the Pre-Ratio Test in section 12.

Since $\alpha < 1$, let $\epsilon > 0$ be such that $\alpha < \alpha + \epsilon < 1$. This is because we need a little bit of wiggle room between α and 1 (it also shows why the case $\alpha = 1$ doesn't work).

Now by definition of $\limsup$, we have

$$\limsup_{n \to \infty} |a_n|^{\frac{1}{n}} = \lim_{N \to \infty} \sup \left\{ |a_n|^{\frac{1}{n}} \mid n > N \right\} = \alpha$$

Hence, by definition of a limit with ϵ as above, there is N_1 such that if $N > N_1$, then

$$\left| \sup \left\{ |a_n|^{\frac{1}{n}} \mid n > N \right\} - \alpha \right| < \epsilon$$

Upshot: Since the above is true for *all* $N > N_1$, it is in particular true for *some* N $(> N_1)$.

Analogy: If everyone passes a class, then at least some student passes the class

With that N, we then get:

$$\left| \sup \left\{ |a_n|^{\frac{1}{n}} \mid n > N \right\} - \alpha \right| < \epsilon$$

$$\Rightarrow \sup \left\{ |a_n|^{\frac{1}{n}} \mid n > N \right\} - \alpha < \epsilon$$

$$\Rightarrow \sup \left\{ |a_n|^{\frac{1}{n}} \mid n > N \right\} < \alpha + \epsilon$$

But then, by definition of sup (think of it like a max), we get that for all $n > N$ (so $n \geq N + 1$) we have:

$$|a_n|^{\frac{1}{n}} < \alpha + \epsilon \Rightarrow |a_n| < (\alpha + \epsilon)^n$$

And, in particular:

$$\sum_{n=N+1}^{\infty} |a_n| \leq \sum_{n=N+1}^{\infty} (\alpha + \epsilon)^n = \sum_{n=1}^{\infty} r^n$$

Where $r = \alpha + \epsilon < 1$. But the latter is just a geometric series with $|r| < 1$ and therefore converges. Hence, by the comparison test,

$$\sum_{n=N+1}^{\infty} |a_n| \text{ converges}$$

And therefore:

$$\sum_{n=1}^{\infty} |a_n| = \underbrace{|a_1| + |a_2| + \cdots + |a_N|}_{\text{Finitely many terms}} + \underbrace{\sum_{n=N+1}^{\infty} |a_n|}_{\text{Converges}} \text{ converges}$$

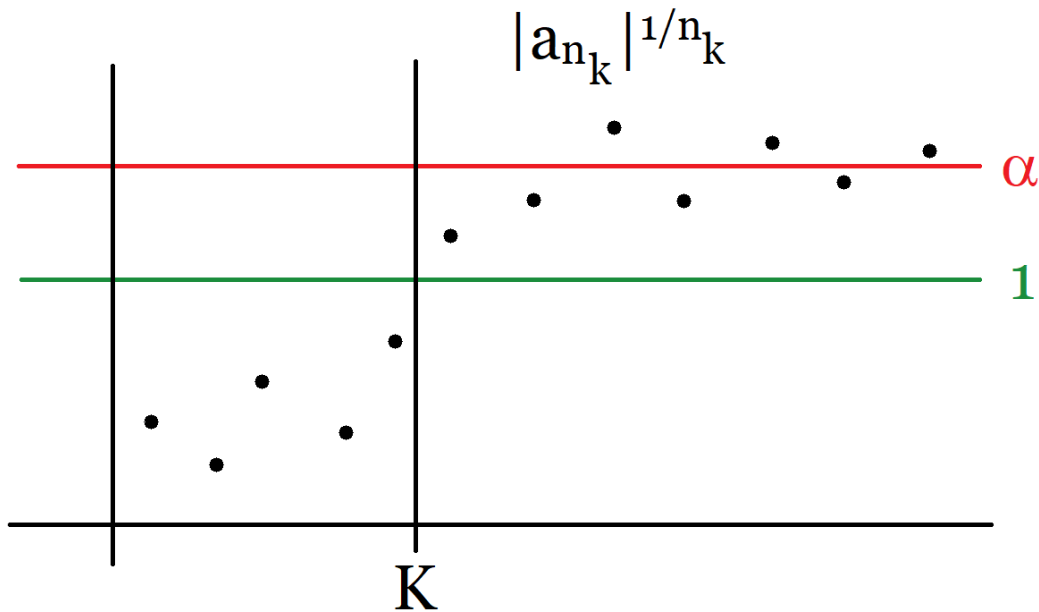Here $\sum a_n$ converges absolutely ✓

Proof of (2): $(\alpha > 1 \Rightarrow \text{diverges})$

Even easier! Remember that for any sequence (s_n), there is a subsequence (s_{n_k}) converging to $\limsup_{n\to\infty} s_n$.

Therefore here there is a subsequence $|a_{n_k}|^{\frac{1}{n_k}}$ of $|a_n|^{\frac{1}{n}}$ converging to $\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = \alpha > 1$
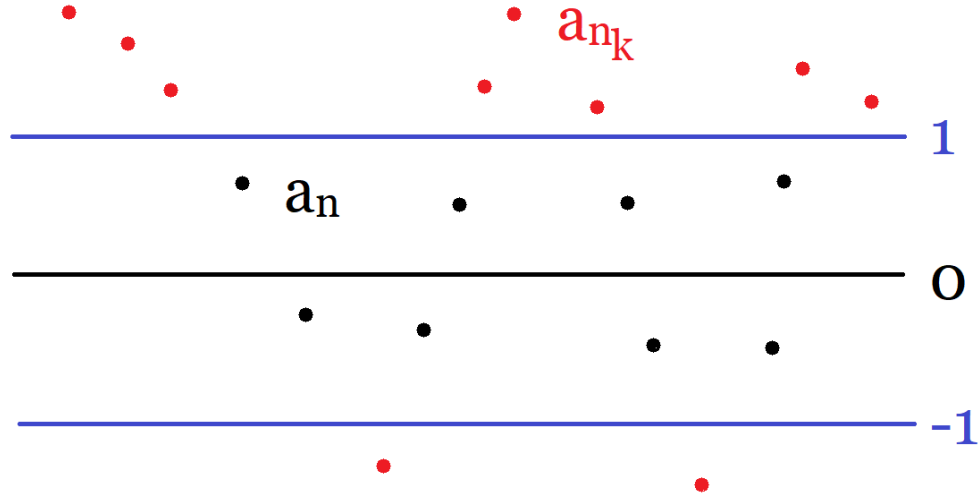
But this means that for all k large enough, we must have[1]

$$|a_{n_k}|^{\frac{1}{n_k}} > 1 \Rightarrow |a_{n_k}| > 1^{n_k} = 1$$

Now *if* $a_n \to 0$, then in particular for the subsequence a_{n_k} above, we must have $a_{n_k} \to 0$. But for this cannot happen since $|a_{n_k}| > 1$ for every $k \Rightarrow\Leftarrow$

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[1]Here's a proof if you want: Let $\epsilon = \alpha - 1$, then since $|a_{n_k}|^{\frac{1}{n_k}} \to \alpha$, there is K such that if $k > K$, then $\left||a_{n_k}|^{\frac{1}{n_k}} - \alpha\right| < \epsilon = \alpha - 1$, and therefore in particular $|a_{n_k}|^{\frac{1}{n_k}} - \alpha > -(\alpha - 1) = 1 - \alpha$ and therefore for $k > K$, we have $|a_{n_k}|^{\frac{1}{n_k}} > 1$

(In the picture above, (a_n) cannot converge to 0 because (a_{n_k}) (in red) doesn't even get close to 0.

Therefore $a_n \nrightarrow 0$, and so $\sum a_n$ diverges by the divergence test. ✓

Proof of (3)**:** All we need to do is find two series with $\alpha = 1$, one of which converges absolutely, and the other one diverges.

Consider $\sum_{n=1}^{\infty} \frac{1}{n}$. Then

$$|a_n|^{\frac{1}{n}} = \left(\frac{1}{n}\right)^{\frac{1}{n}} = \frac{1}{n^{\frac{1}{n}}} \to \frac{1}{1} = 1$$

So $\alpha = \limsup_{n \to \infty} |a_n|^{\frac{1}{n}} = 1$ but $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (because it's a $1-$series)
Now consider $\sum_{n=1}^{\infty} \frac{1}{n^2}$. Then

$$|a_n|^{\frac{1}{n^2}} = \left(\frac{1}{n}\right)^{\frac{1}{n^2}} = \frac{1}{n^{\frac{2}{n}}} = \frac{1}{\left(n^{\frac{1}{n}}\right)^2} \to \frac{1}{1} = 1$$

So $\alpha = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = 1$ but $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (because it's a $2-$series)

Therefore, even though $\alpha = 1$ in both case, in the first case the series diverges, and in the second case case, the series converges (absolutely)

✓ □

2. THE RATIO TEST

Video: Ratio Test Proof

Now, on the other side of the spectrum is the ratio test:

> **Example 2:**
>
> Use the ratio test to figure out if the following series converges:
>
> $$\sum_{n=0}^{\infty} \frac{n}{3^n}$$

This time look at ratios of successive terms:

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{n+1}{3^{n+1}}}{\frac{n}{3^n}} = \left(\frac{3^n}{3^{n+1}}\right)\left(\frac{n+1}{n}\right) = \left(\frac{1}{3}\right)\left(\frac{n+1}{n}\right) \to \frac{1}{3} < 1$$

Therefore the series converges absolutely.

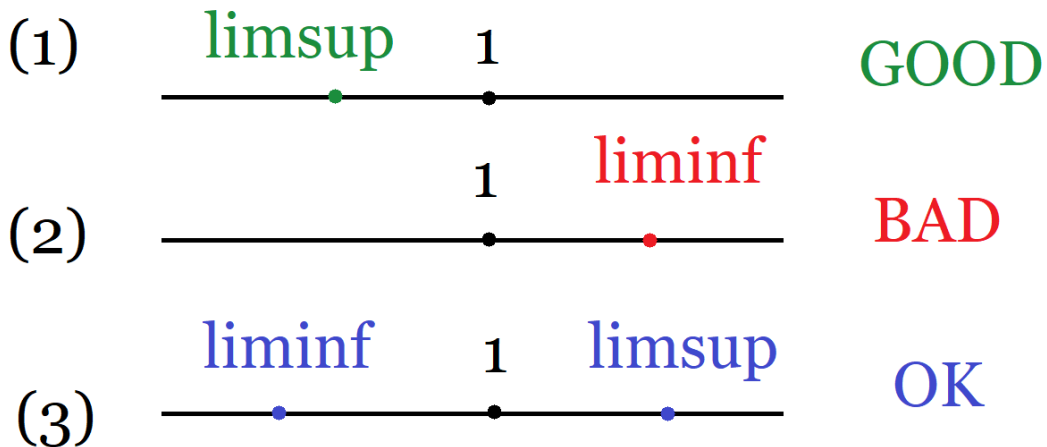**Note:** The ratio test is **excellent** for series involving $n!$, like $\sum \frac{1}{n!}$ for instance.

Here again, since $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ might not exist, we need to replace the limit with $\lim\sup$ and $\lim\inf$:

Ratio Test:

Consider $\sum a_n$. Then:

(1) If $\lim\sup_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$, then $\sum a_n$ converges absolutely.

(2) If $\lim\inf_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1$, then $\sum a_n$ diverges

(3) If $\lim\inf_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \le 1 \le \lim\sup_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$, then the ratio test is inconclusive.

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(1)   limsup   1   GOOD

(2)   1   liminf   BAD

(3)   liminf   1   limsup   OK

Note: Think of $\lim\sup$ as the worst possible limit and $\lim\inf$ as the best possible limit. In (1) we're saying that the worst possible limit is < 1, so not bad at all, in which case the series converges. In (2), the best possible limit is > 1, which is already bad, in which case the series diverges. In, (3), we can't really say anything: The best possible

limit is good (≤ 1) and the worst possible limit is bad (≥ 1), which is pretty typical.

Proof: Muuuuuch easier than the proof of the root test, because we've already done the hard part in section 12 ☺

Recall: Pre-Ratio Test

$$\liminf_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \leq \liminf_{n\to\infty} |a_n|^{\frac{1}{n}} \leq \limsup_{n\to\infty} |a_n|^{\frac{1}{n}} \leq \limsup_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

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(1) If $\limsup_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$, then, in the above, we get

$$\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} \leq \limsup_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$$

So

$$\alpha =: \limsup_{n\to\infty} |a_n|^{\frac{1}{n}} < 1$$

And therefore by the **root** test, we conclude that $\sum a_n$ converges absolutely ✓

(2) If $\liminf_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| > 1$, then, in the above, we get:

$$\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} \geq \liminf_{n\to\infty} |a_n|^{\frac{1}{n}} \geq \liminf_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| > 1$$

Therefore:

$$\alpha =: \limsup_{n\to\infty} |a_n|^{\frac{1}{n}} > 1$$

And hence by the **root** test, $\sum a_n$ diverges. ✓

(3) Finally, just as before, we need to find two series $\sum a_n$ with $\liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \le 1 \le \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ and such that the first series is divergent and the second series is absolutely convergent.

Consider $\sum \frac{1}{n}$, then

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{1}{n+1}}{\frac{1}{n}} = \frac{n}{n+1} \to 1$$

Therefore

$$\liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1 \le 1 \le 1 = \liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

But $\sum \frac{1}{n}$ diverges (because it's a $1-$series)

Now consider $\sum \frac{1}{n^2}$, then

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \frac{n^2}{(n+1)^2} = \left(\frac{n}{n+1}\right)^2 \to 1$$

Therefore

$$\liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1 \le 1 \le 1 = \liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

But $\sum \frac{1}{n^2}$ converges absolutely (because it's a $2-$series) ✓   □

Note: Already in the proof above we can see that the root test is more powerful than the ratio test, because we used the root test to prove the ratio test!

In fact, notice that the above proof shows that if a series $\sum a_n$ converges (or diverges) using the ratio test, then it must converge (or diverge) using the root test as well, so the root test is more general and powerful than the ratio test.

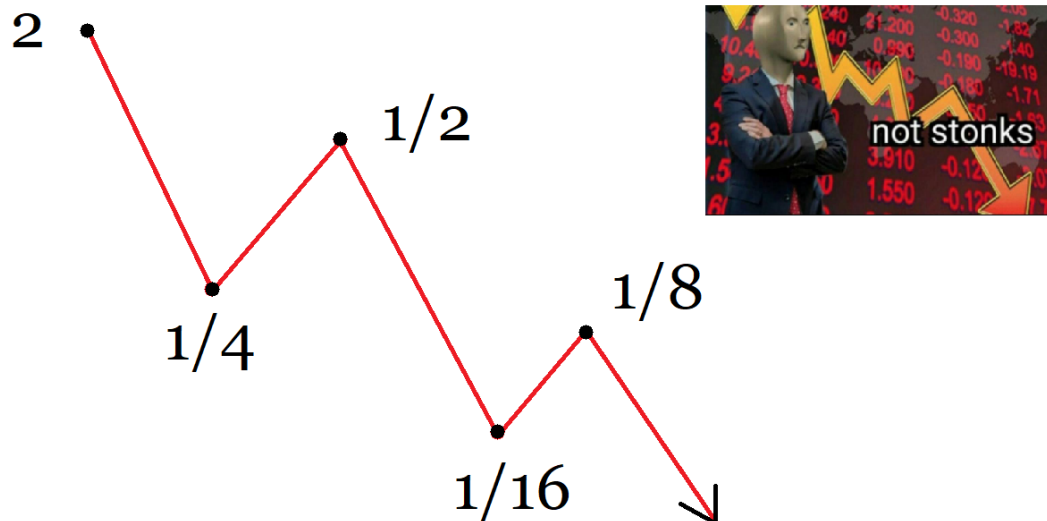## 3. Root Test > Ratio Test

Video: Ratio Test Vs Root Test

As another illustration of why the root test is better than the ratio test, consider the following:

Example 3:

Does the following series converge?

$$\sum_{n=0}^{\infty} 2^{(-1)^n - n} = 2 + \frac{1}{4} + \frac{1}{2} + \frac{1}{16} + \frac{1}{8} + \dots$$
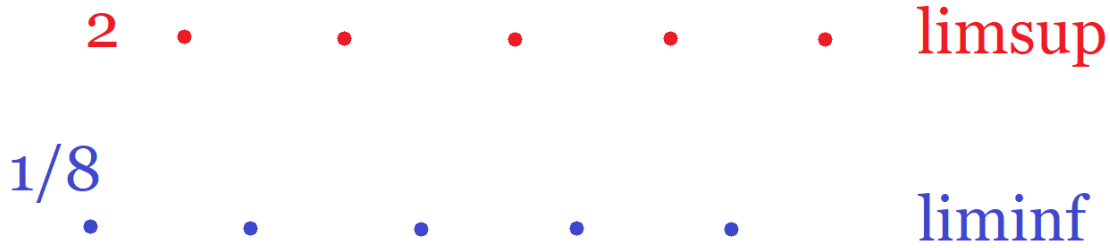
This is what I'd like to call the stock market series, or the *Not Stonks* series:

Let's try to apply both the ratio test and the root test to this series, in order to see who wins.
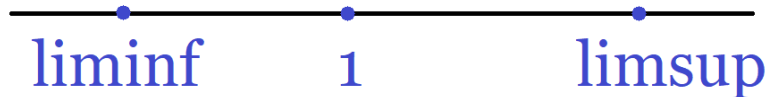
Ratio Test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{2^{(-1)^{n+1}-(n+1)}}{2^{(-1)^n-n}}$$

$$= 2^{(-1)^{n+1}-\cancel{n}-1-(-1)^n+\cancel{n}}$$

$$= 2^{-(-1)^n-(-1)^n-1}$$

$$= 2^{-((-1)^n+(-1)^n+1)}$$

$$= 2^{-(2(-1)^n+1)}$$

$$= \left(\frac{1}{8}, 2, \frac{1}{8}, 2, \frac{1}{8}, 2, \ldots\right)$$

2 • • • • • limsup

$1/8$

• • • • • liminf

Therefore $\liminf_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{8}$ and $\limsup_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 2$ and so:

$$\liminf_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \le 1 \le \limsup_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$

liminf   1   limsup

So we are in the third case of the ratio test (the *typical* case, neither good nor bad), and so the ratio test is **inconclusive**.

Root Test:

$$|a_n|^{\frac{1}{n}} = \left(2^{(-1)^n - n}\right)^{\frac{1}{n}} = 2^{\frac{(-1)^n}{n} - 1} \to 2^{0-1} = 2^{-1} = \frac{1}{2} < 1$$

(Here we used $\frac{(-1)^n}{n} \to 0$ by the squeeze theorem, since it is squeezed between $-\frac{1}{n}$ and $\frac{1}{n}$)

Hence

$$\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = \frac{1}{2} < 1$$

And therefore by the root test, $\sum a_n$ converges absolutely.

> **Summary:**
>
> The Root test is strictly better than the ratio test:
>
> If $\sum a_n$ converges (or diverges) by the ratio test, then it converges (or diverges) by the root test as well.
>
> But there are examples of series (like the above) which converge (or diverge) by the root test, but for which the ratio test is inconclusive.

Root Test

$\cdot \sum 2^{(-1)^n - n}$

Ratio Test
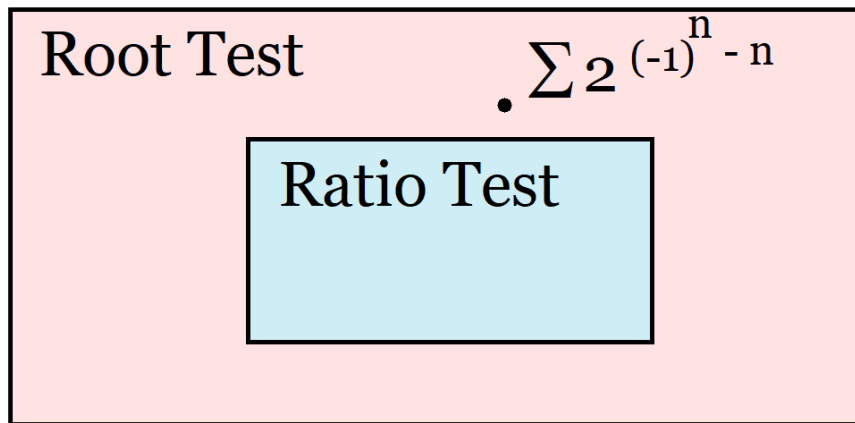
4. ROOT TEST PITFALL

Video: Root Test Pitfall

That said, don't get *too* overexcited, the root test doesn't *always* work. In particular, don't think that just because you see something to the power of n, you have to apply the root test!

> **Example 4:**
>
> Does the following series converge?
> $$\sum_{n=0}^{\infty} \left(\frac{2}{(-1)^n - 3} \right)^n$$

First try: Let's try using the root test:

$$|a_n|^{\frac{1}{n}} = \left| \frac{2}{(-1)^n - 3} \right| = \left(1, \frac{1}{2}, 1, \frac{1}{2}, \ldots \right)$$

Careful: Just because it alternates it doesn't *necessarily* mean that the root test is inconclusive! You really have to look at the limsup to figure out if it's inconclusive or not:

$$\alpha = \limsup_{n \to \infty} |a_n|^{\frac{1}{n}} = 1$$

1 $\quad\bullet\qquad\bullet\qquad\bullet\qquad\bullet\qquad\bullet\qquad$ limsup

1/2 $\bullet\qquad\bullet\qquad\bullet\qquad\bullet$
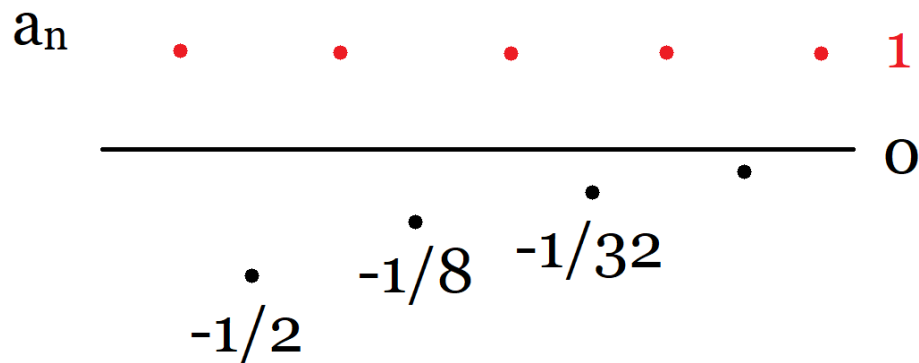
Therefore the root test is inconclusive, and we'll have to try another method.

Note: The ratio test would also be inconclusive (by the pre-ratio test), so we'll have to try to find another way of doing this:

Second try: Look at the sequence (a_n) itself!

$$a_n = \left(\frac{2}{(-1)^n - 3}\right)^n = \left(1, -\frac{1}{2}, 1, -\frac{1}{8}, 1, \left(-\frac{1}{2}\right)^5, 1, \left(-\frac{1}{2}\right)^7, 1, \ldots\right)$$

a_n $\quad\bullet\qquad\bullet\qquad\bullet\qquad\bullet\qquad\bullet\quad$ **1**

$\underline{\qquad\qquad\qquad\qquad\qquad\qquad}$ **0**

$\bullet\quad$ -1/32

-1/8

$\bullet$

-1/2

Notice that every other term of a_n is 1, hence $a_n \nrightarrow 0$, and therefore $\sum a_n$ diverges by the divergence test.