

## LECTURE 10: MONOTONE SEQUENCES

### 1. MONOTONE SEQUENCE THEOREM

**Video:** Monotone Sequence Theorem

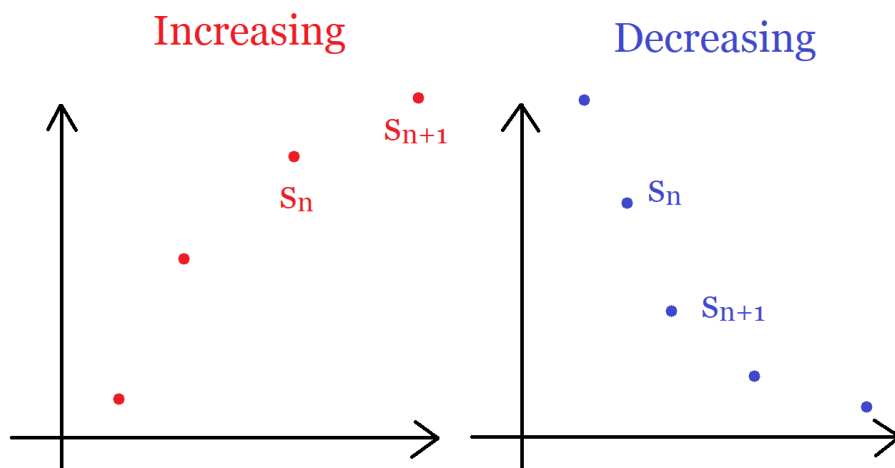
Notice how *annoying* it is to show that a sequence explicitly converges, and it would be nice if we had some easy general theorems that guarantee that a sequence converges.

#### Definition:

$(s_n)$  is **increasing** if  $s_{n+1} > s_n$  for each  $n$

$(s_n)$  is **decreasing** if  $s_{n+1} < s_n$  for each  $n$

If either of the above holds, we say that  $(s_n)$  is **monotonic**.



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Date: Monday, April 20, 2020.

**Examples:**  $s_n = \sqrt{n}$  is increasing,  $s_n = \frac{1}{n}$  is decreasing,  $s_n = (-1)^n$  is neither increasing nor decreasing.

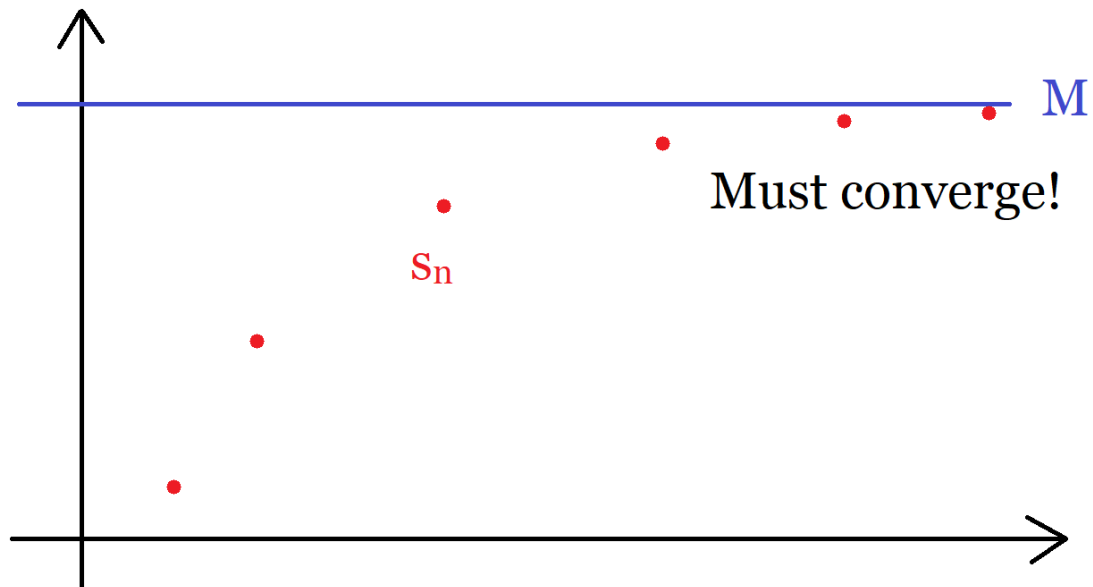
The following theorem gives a **very** elegant criterion for a sequence to converge, and explains why monotonicity is so important.

**Monotone Sequence Theorem:**

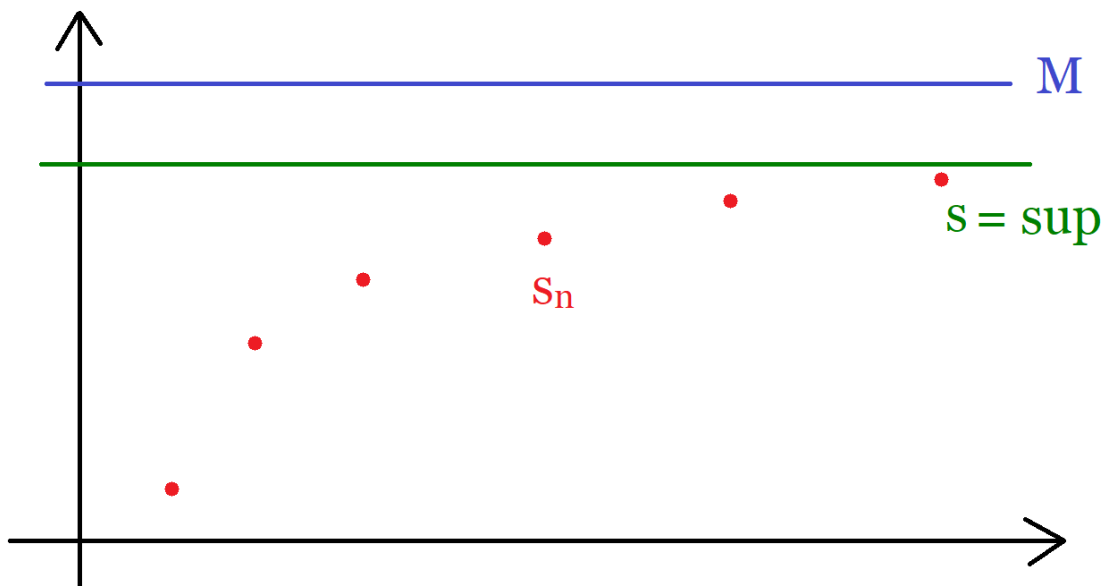
$(s_n)$  is increasing and bounded above, then  $(s_n)$  converges.

**Note:** The same proof works if  $(s_n)$  is nondecreasing ( $s_{n+1} \geq s_n$ )

**Intuitively:** If  $(s_n)$  is increasing and has a ceiling, then there's no way it cannot converge. In fact, try drawing a counterexample, and you'll see that it doesn't work!



**WARNING:** If  $(s_n)$  is bounded above by  $M$ , it does **NOT** mean that  $s_n$  converges to  $M$ , as the following picture shows. But what *is* true in this case is that  $s_n$  converges to  $s$  where  $s$  is the sup of all the  $s_n$ .



**Proof:** Elegant interplay between the concept of sup (section 4) and the concept of convergence (section 8).

**STEP 1:** Consider

$$S = \{s_n \mid n \in \mathbb{N}\}$$

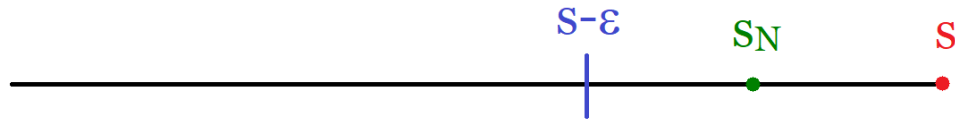
Since  $s_n \leq M$  for all  $M$ ,  $S$  is bounded above, hence  $S$  has a least upper bound  $s = \sup(S)$ .

**Claim:**  $\lim_{n \rightarrow \infty} s_n = s$ .

**STEP 2:** Let  $\epsilon > 0$  be given.

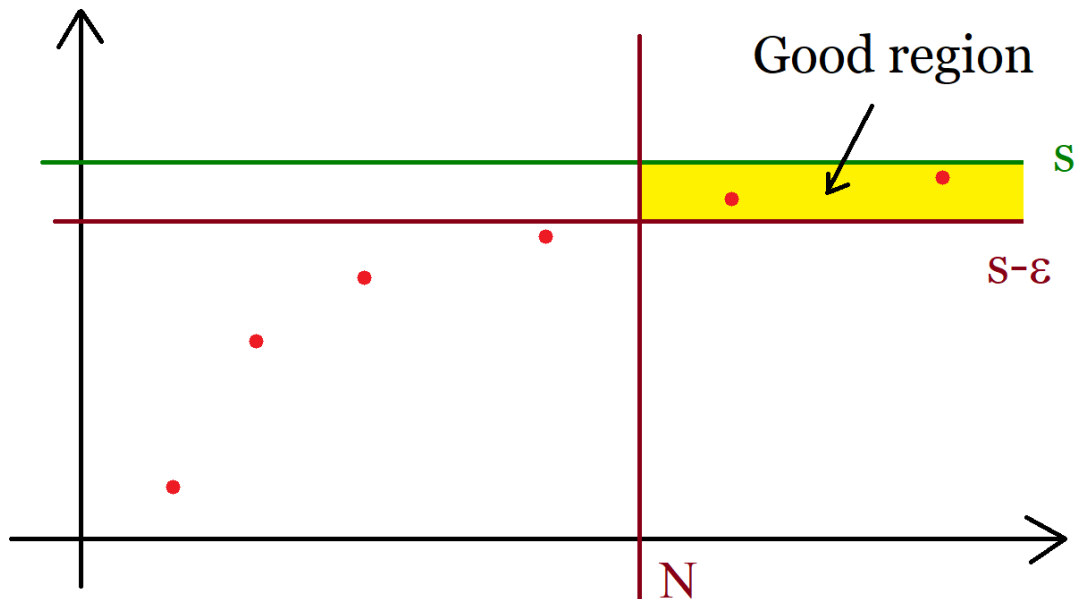
We need to find  $N$  such that if  $n > N$ , then  $|s_n - s| < \epsilon$ .

Consider  $s - \epsilon < s$ . By definition of a sup, this means that there is  $s_N \in S$  such that  $s_N > s - \epsilon$



But then, for that  $N$ , if  $n > N$ , since  $s_n$  is increasing, we have

$$s_n - s > s_N - s > -\epsilon$$



On the other hand, since  $s = \sup(S)$  by definition of sup, we have  $s_n \leq s$  for all  $s$  and so

$$s_n - s \leq s - s = 0 < \epsilon$$

Therefore we get

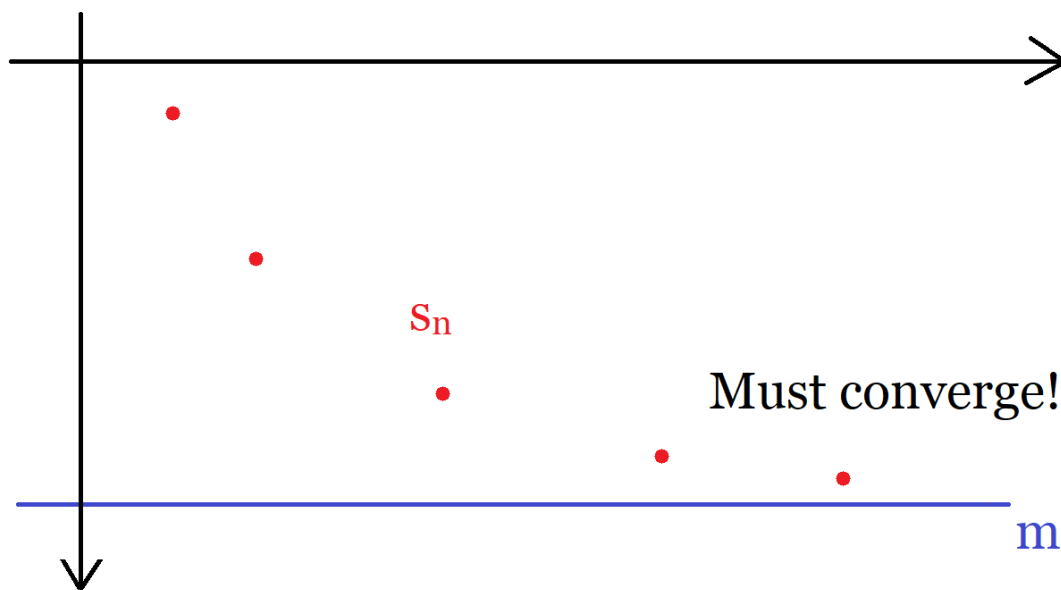
$$-\epsilon < s_n - s < \epsilon \Rightarrow |s_n - s| < \epsilon$$

And so  $(s_n)$  converges to  $s$  □

Of course, by considering  $-s_n$  we get the following corollary:

**Corollary:**

$(s_n)$  is decreasing and bounded below, then  $(s_n)$  converges.



**Why?** In that case  $(-s_n)$  is increasing and bounded above, so converges to  $s$ , and therefore  $(s_n)$  converges to  $-s$  (or repeat the above

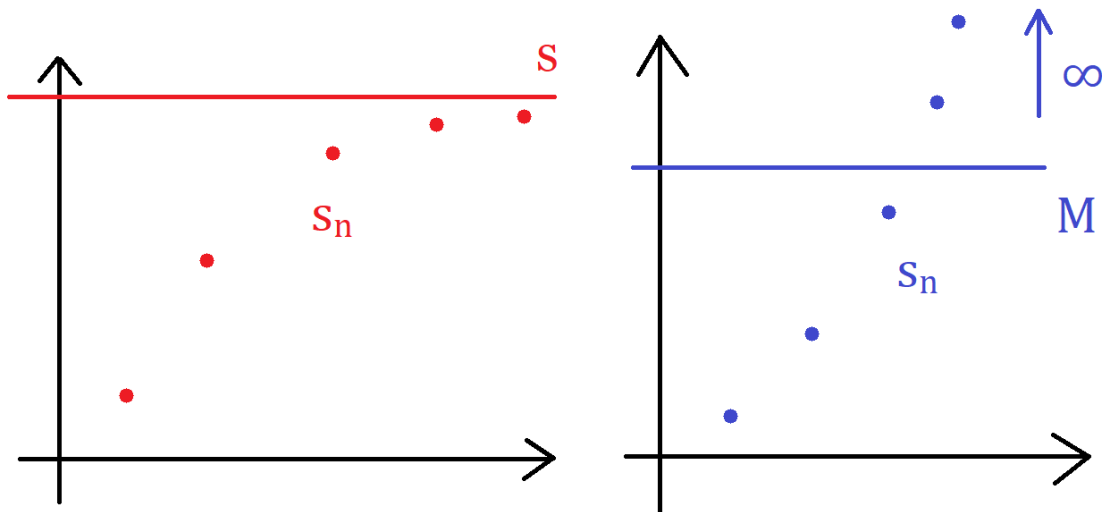
proof, but with inf) □

**In fact:** We don't even need  $(s_n)$  to be bounded above, provided that we allow  $\infty$  as a limit.

**Theorem:**

$(s_n)$  is increasing, then it either converges or goes to  $\infty$

So there are really just 2 kinds of increasing sequences: Either those that converge or those that blow up to  $\infty$ .

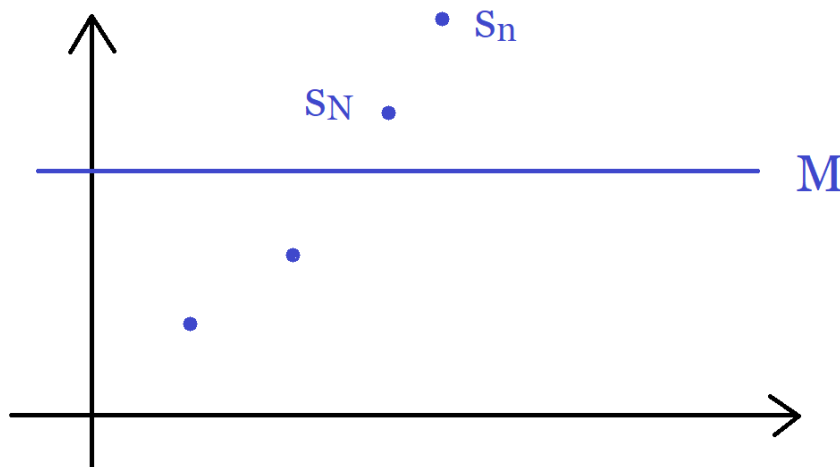


**Proof:**

**Case 1:**  $(s_n)$  is bounded above, but then by the Monotone Sequence Theorem,  $(s_n)$  converges ✓

**Case 2:**  $(s_n)$  is not bounded above, and we claim that  $\lim_{n \rightarrow \infty} s_n = \infty$ .

Let  $M > 0$  be given, want to find  $N$  such that if  $n > N$ , then  $s_n > M$ .



Notice first of all that there is  $N$  such that  $s_N > M$ , because otherwise  $s_N \leq M$  for all  $N$  and so  $M$  would be an upper bound for  $(s_n)$ .

With that  $N$ , if  $n > N$ , then since  $(s_n)$  is increasing, we get  $s_n > s_N = M$ , so  $s_n > M$  and hence  $s_n$  goes to  $\infty$  ✓  $\square$

Finally, notice that the proof of the Monotone Sequence Theorem uses the Least-Upper Bound Property (because we defined  $\sup$ ), but in fact something even more awesome is true:

### Cool Fact:

The Least Upper Bound Property is **equivalent** to the Monotone Sequence Theorem! (WOW)

## 2. DECIMAL EXPANSIONS

**Video:** Decimal Expansions

There is a more natural (but less elegant) construction of  $\mathbb{R}$  than using cuts that you're probably more acquainted with, namely decimal expansions.

**Motivation:** What does it mean for  $\pi = 3.1415 \dots$ ?

Notice:

$$\begin{aligned}\pi &= 3 + \frac{1}{10} + \frac{4}{100} + \frac{1}{1000} + \dots \\ &= 3 + \frac{1}{10} + \frac{4}{10^2} + \frac{1}{10^3} + \dots \\ &= k + \frac{d_1}{10} + \frac{d_2}{10^2} + \frac{d_3}{10^3} + \dots\end{aligned}$$

Now consider the following sequence  $(s_n)$

$$\begin{aligned}s_0 &= 3 = k \\ s_1 &= 3.1 = k + \frac{d_1}{10} \\ s_2 &= 3.14 = k + \frac{d_1}{10} + \frac{d_2}{10^2} \\ s_3 &= 3.141 = k + \frac{d_1}{10} + \frac{d_2}{10^2} + \frac{d_3}{10^3} \\ s_n &= 3.1415 \dots d_n = k + \frac{d_1}{10} + \frac{d_2}{10^2} + \dots + \frac{d_n}{10^n}\end{aligned}$$

Notice that  $(s_n)$  is bounded above by  $4 = k + 1$  and moreover,  $(s_n)$  is increasing (since we're only adding positive terms), therefore by the monotone sequence theorem,  $(s_n)$  is converging to  $s$ , and this limit is what we call



$$\pi = 3.1415\cdots = k.d_1d_2d_3\cdots$$

So, in some sense, it is reasonable to define real numbers as follows:

**Definition:**

$\mathbb{R}$  is the set of all numbers of the form

$$k.d_1d_2\cdots$$

Where  $k \in \mathbb{Z}$  and each  $d_i$  is a digit between 0 and 9 (and  $\cdots$  is to be understood in the limit sense as above)

Of course, this leaves many questions to be unanswered, such as: “Does *every* real number (such as  $\sqrt{2}$ ) even have a decimal expansion?” or “How can you show that a rational number is a real number?” Those questions are answered in section 16 (which we unfortunately won’t cover ☺)

More importantly, how would you show that  $\mathbb{R}$  (as constructed above) has the least-upper bound property? In some sense, as hard as cuts seem, they make proving this property much easier!

Even worse, there is actually a glitch in the above definition. For this we need the following formula, which you might remember from calculus (for a proof, see exercise 9.18)

**Geometric Series:**

If  $|r| < 1$ , then

$$\lim_{n \rightarrow \infty} 1 + r + r^2 + \cdots + r^n = \frac{1}{1 - r}$$

$$\begin{aligned}
0.99999 \dots &= \frac{9}{10} + \frac{9}{10^2} + \frac{9}{10^3} + \dots \\
&= \frac{9}{10} \left( 1 + \frac{1}{10} + \frac{1}{10^2} + \dots \right) \\
&= \frac{9}{10} \left( \frac{1}{1 - \frac{1}{10}} \right) \\
&= \frac{9}{10} \left( \frac{1}{\frac{9}{10}} \right) \\
&= 1
\end{aligned}$$

So actually we have  $0.99999 \dots = 1.00000 \dots$ , so both of those decimal expansions actually represent the same real number! So the above construction is bad in the sense that different decimal expansions might give you the same number. This is different for cuts; different cuts actually give different real numbers!

There is an easy way to get around that, actually: In the above construction, simply throw away decimal expansions that end with an infinite string of 9's. That is, in the above definition, consider just the decimal expansions that don't end with 9's.

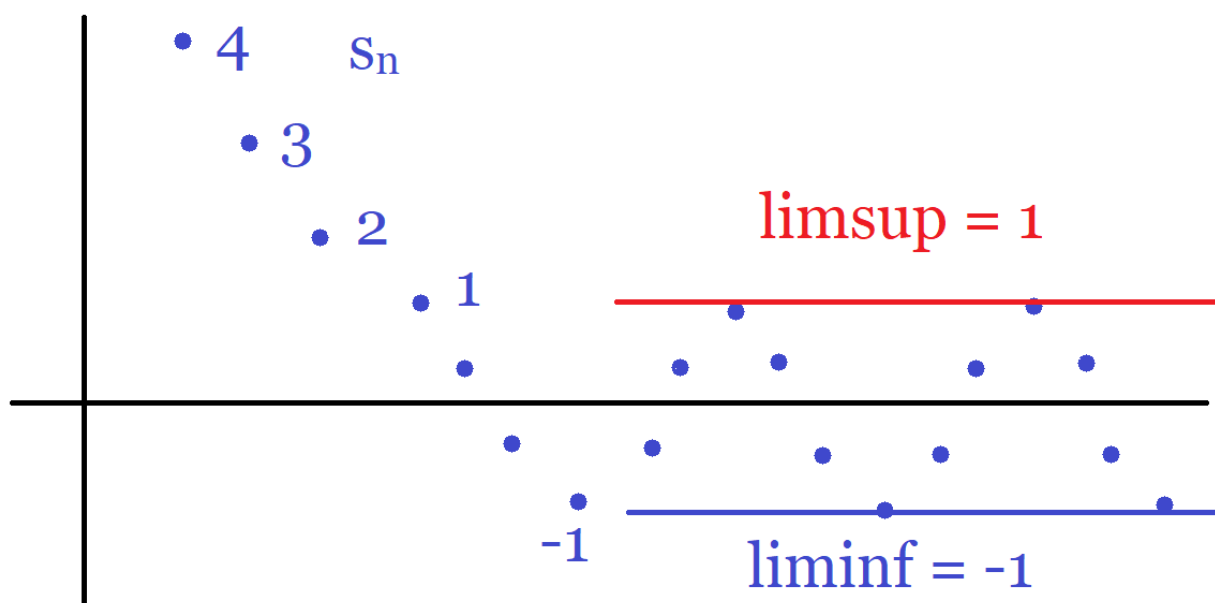
### 3. $\limsup$

**Video:** What is  $\limsup$ ?

Finally, let me discuss the second most important concept in analysis (after sup of course): The  $\limsup$ . Because so far we talked about convergent sequences. But in reality, a lot of sequences don't converge! How do we deal with them?

**Note:** For the following, assume that  $(s_n)$  is bounded (but see below).

Consider the following example:



Even though  $(s_n)$  doesn't converge, we would like to say that the largest possible limit ( $= \limsup$ ) of  $(s_n)$  is 1 and the smallest possible limit ( $= \liminf$ ) of  $(s_n)$  is  $-1$ .

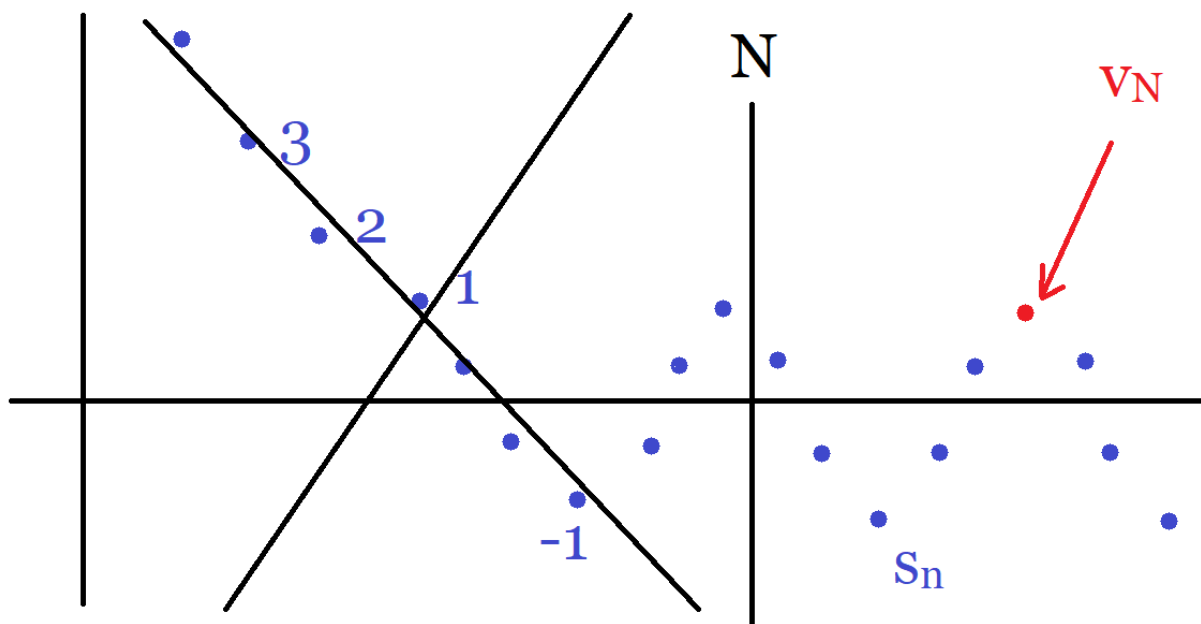
Notice that the  $\limsup$  is **NOT** the same as the  $\sup$ . In this example, the  $\sup$  is 4 but the  $\limsup$  is 1.

The idea is as follows: The  $\limsup$  of  $s_n$  is essentially the  $\sup$  of  $s_n$ , but for *large* values of  $n$ .

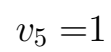
To make this more precise: Given  $N$ , define the following helper sequence  $(v_N)$  by:

$$v_N = \sup \{s_n \mid n > N\}$$

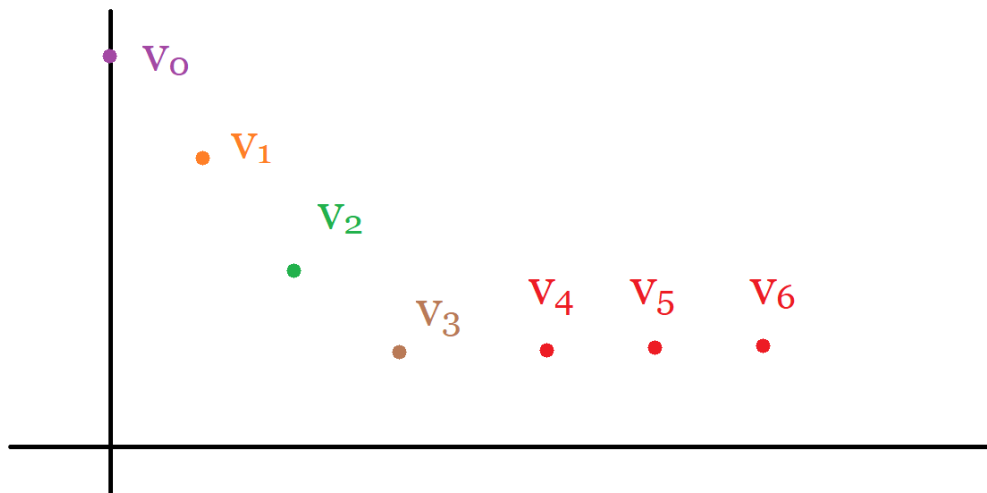
Namely, you look at the largest value of  $s_n$ , but *after*  $N$ . You ignore what's happening before  $N$ .



$(v_N)$  isn't just some random sequence, but actually has some nice properties! For this, let's plot a couple of values of  $v_N$



Notice that the values of  $v_N$  seem to stabilize!

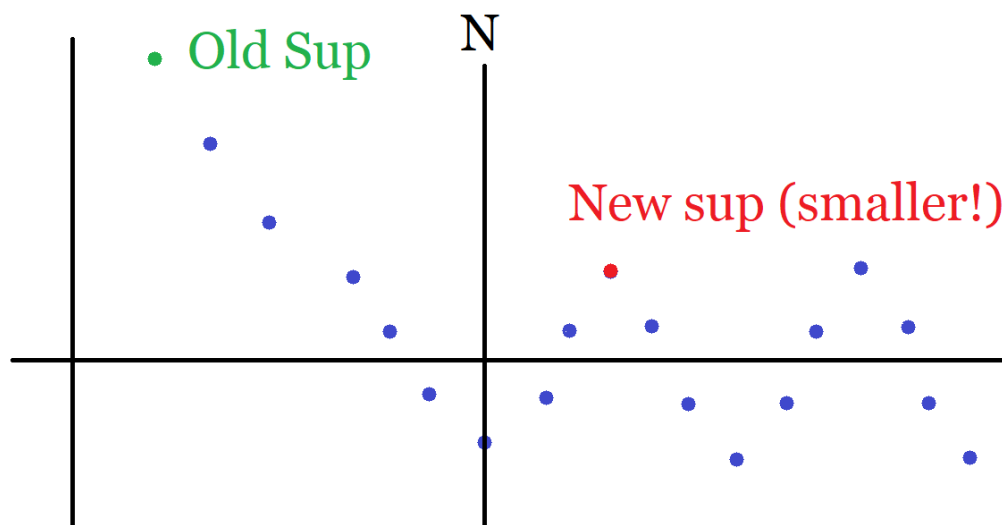


Although things generally don't always stabilize, what *is* true is that:

**Fact:**

$(v_N)$  is a decreasing sequence

**Why?** For example, notice that for  $v_0 = \sup \{s_n \mid n > 0\}$  you have lots of values of  $s_n$  to compare, and here in fact the sup is 4. But for  $v_N = \sup \{s_n \mid n > N\}$  you have much fewer values to compare, so the sup cannot be as big as the original one!



**Analogy:** Suppose you have a class of 10 students and the highest score on an exam is 98. Now if 5 students drop, then the highest score now isn't necessarily as big any more, since some of the dropped student may have had very good scores.

Not only is  $(v_N)$  decreasing, but it's also bounded below (since  $(s_n)$  is). Therefore, by the *Monotone Sequence Theorem*,  $(v_N)$  must exist:

**Fact:**

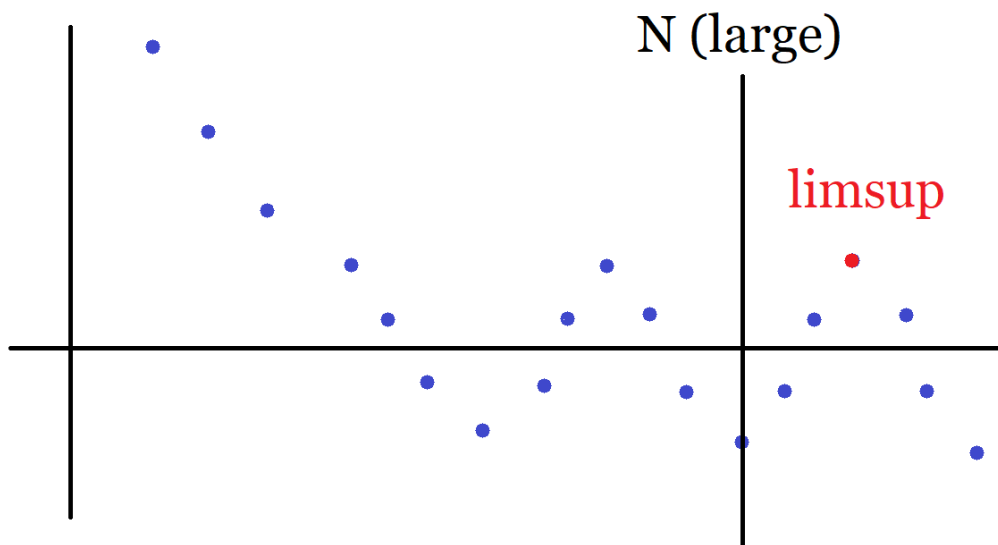
$\lim_{N \rightarrow \infty} (v_N)$  exists

And it is *that* limit that we call  $\limsup$ :

**Definition:**

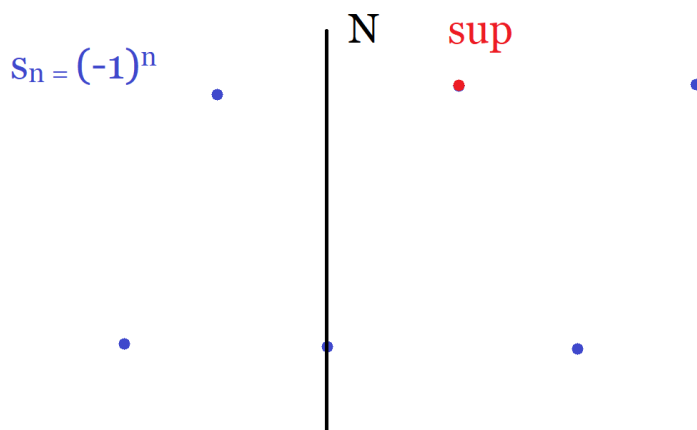
$$\limsup_{n \rightarrow \infty} s_n = \lim_{N \rightarrow \infty} v_N = \lim_{N \rightarrow \infty} \sup \{s_n \mid n > N\}$$

**Interpretation:** All this means is that the  $\limsup$  is the  $\sup$  of  $s_n$  but for *large* values of  $n$ , so it's really essentially the largest possible limit of  $s_n$



### Example:

Find  $\limsup_{n \rightarrow \infty} s_n$  where  $s_n = (-1)^n$





Notice that for *every*  $N$  (not necessarily large),

$$v_N = \sup \{s_n \mid n > N\} = 1$$

And therefore

$$\limsup_{n \rightarrow \infty} s_n = \lim_{N \rightarrow \infty} \sup \{s_n \mid n > N\} = \lim_{N \rightarrow \infty} 1 = 1$$

And why is  $\limsup$  **SO** important? Because even though  $\lim_{n \rightarrow \infty} s_n$  doesn't always exist, we have:

**Upshot:**

$\limsup_{n \rightarrow \infty} s_n$  **ALWAYS** exists!

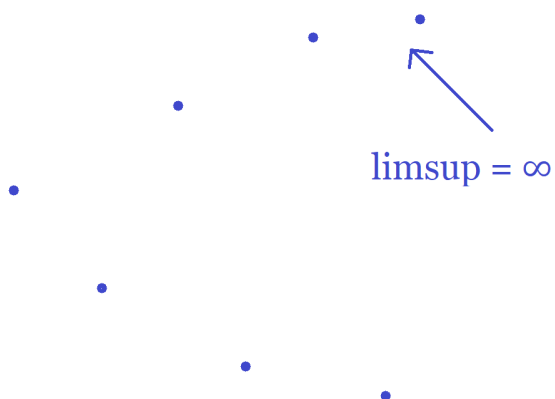
And it's **GOOD** for things to exist! For example, notice how useful it is for  $\sup(S)$  to always exist (we used that a LOT), and same goes for  $\limsup$ .

**Note:** So far we assumed that  $(s_n)$  is bounded, but even if it's not we say that:

**Definition:**

If  $(s_n)$  is not bounded above, then we define

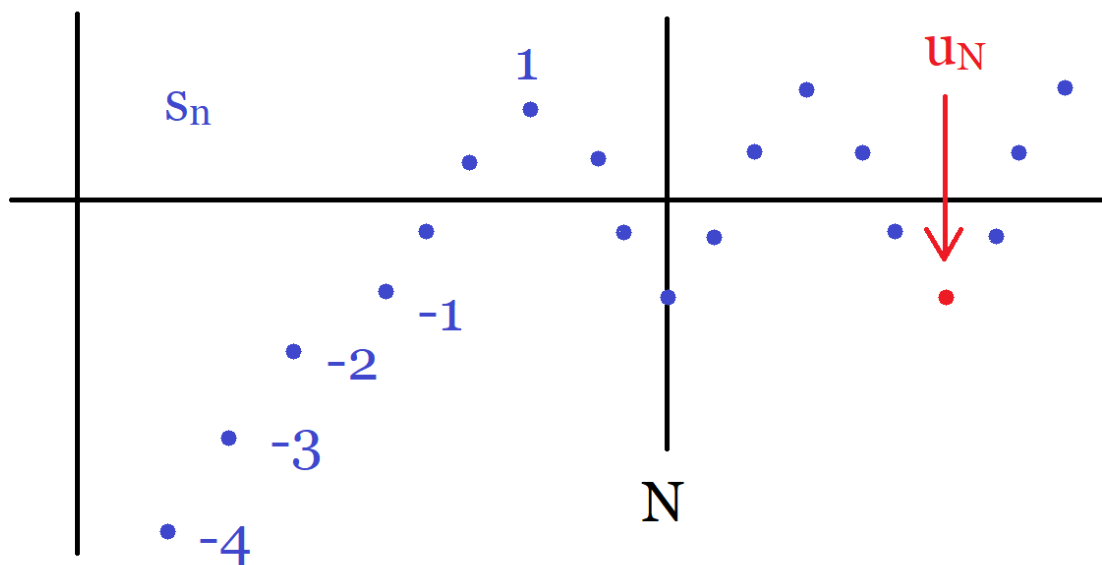
$$\limsup_{n \rightarrow \infty} s_n = \infty$$



#### 4. $\liminf$

Everything that we said for  $\limsup$  can be defined analogously with  $\liminf$ .

Consider this time the following sequence:

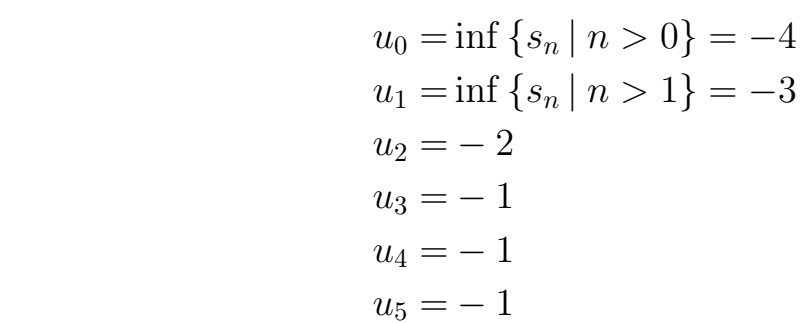


$$u_N = \text{inf} \{s_n \mid n > N\}$$

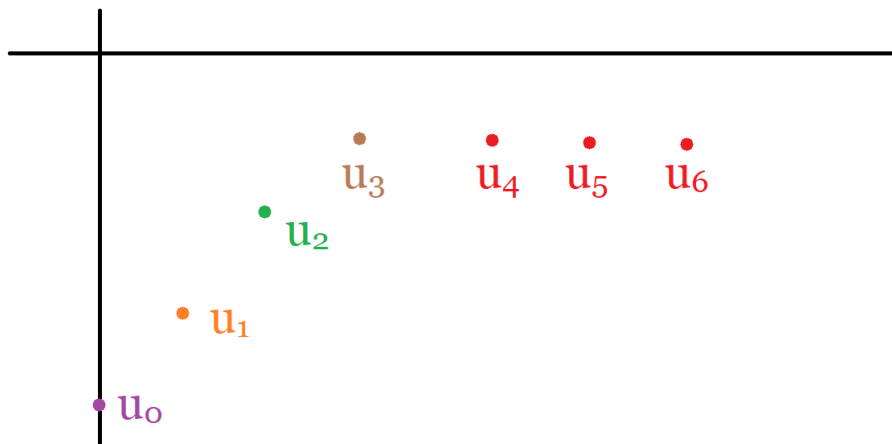
(So this time you look at the *smallest* value of  $s_n$  after  $N$ )  
 Let's plot a couple of values of  $u_N$

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$$u_5 = -1$$

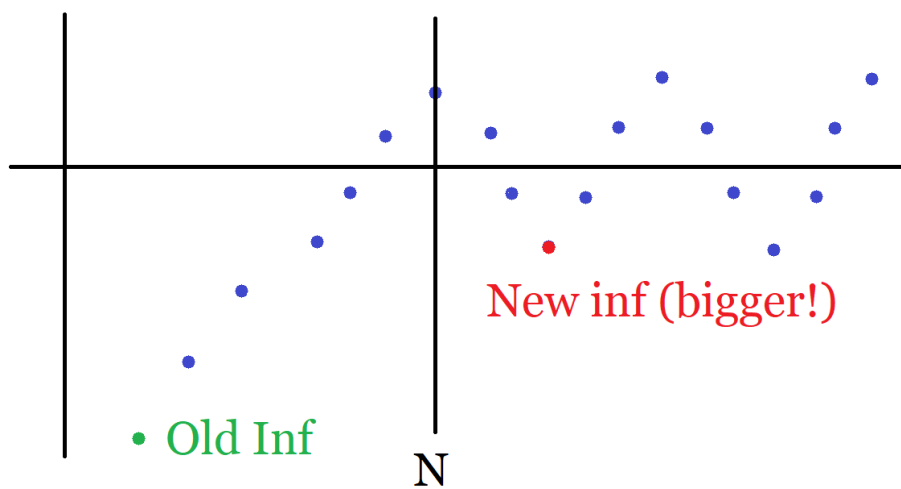


And just as before, we have:

**Fact:**

$(u_N)$  is an increasing sequence

**Why?** Again, it's because we have fewer and fewer values to compare, which causes to increase the inf.



**Analogy:** If you have 10 students and the lowest score is 20%. Now suppose 5 (bad) students dropped. Then the lowest score is now (probably) higher.

And since  $(u_N)$  is increasing and bounded above, by the Monotone Sequence Theorem we get that  $(u_N)$  converges and in particular:

**Definition:**

$$\liminf_{n \rightarrow \infty} s_n = \lim_{N \rightarrow \infty} u_N = \lim_{N \rightarrow \infty} \inf \{s_n \mid n > N\}$$

**Example: (not in the video)**

Find  $\liminf_{n \rightarrow \infty} s_n$  where  $s_n = (-1)^n$

Notice that for *every*  $N$  (not necessarily large),

$$u_N = \inf \{s_n \mid n > N\} = -1$$

And therefore

$$\liminf_{n \rightarrow \infty} s_n = \lim_{N \rightarrow \infty} \inf \{s_n \mid n > N\} = \lim_{N \rightarrow \infty} -1 = -1$$

And just as before:

**Upshot:**

$\liminf_{n \rightarrow \infty} s_n$  **ALWAYS** exists!

Finally:

**Definition:**

If  $(s_n)$  is not bounded below, then we define

$$\liminf_{n \rightarrow \infty} s_n = -\infty$$

5.  $\liminf$  vs  $\limsup$ 

The good news is that we never have to deal with  $\liminf$  explicitly, because we have the following identity:

**Fact:**

$$\liminf_{n \rightarrow \infty} s_n = -\limsup_{n \rightarrow \infty} (-s_n)$$

**Why?** Recall that for any set  $S$  we have:

$$\inf(S) = -\sup(-S)$$

Now if  $N$  is given, let  $S = \{s_n \mid n > N\}$ , then  $-S = \{-s_n \mid n > N\}$  and the above identity becomes

$$\inf \{s_n \mid n > N\} = -\sup \{-s_n \mid n > N\}$$

Finally take  $\lim_{N \rightarrow \infty}$  on both sides:

$$\begin{aligned} \lim_{N \rightarrow \infty} \inf \{s_n \mid n > N\} &= \lim_{N \rightarrow \infty} -\sup \{-s_n \mid n > N\} \\ \lim_{N \rightarrow \infty} \inf \{s_n \mid n > N\} &= -\lim_{N \rightarrow \infty} \sup \{-s_n \mid n > N\} \\ \liminf_{n \rightarrow \infty} s_n &= -\limsup_{n \rightarrow \infty} (-s_n) \quad \square \end{aligned}$$