Quantified Boolean Formulas: (Solving and) Proof Complexity

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Propositional Satisfiability

- SAT: Satisfiability.
  eg. Is there an assignment to x, y, z satisfying all the clauses $(x \vee y \vee z), (x \vee \neg\, y \vee \neg\, z), (\neg\, x \vee y \vee \neg\, z), (\neg\, x \vee \neg\, y \vee z)$?
- Quintessential NP-complete problem.
- Very hard – in theory.

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  In practice – a solved problem! Many good SAT solvers around.
- Ambitious ongoing programs to design good solvers for problems harder than SAT.
- Focus of this talk: QBF.

QBF: Quantified Boolean Formulas

- We consider QBFs that are
  - totally quantified (no unbound variables),
  - in prenex form,
  - with inner propositional formula in CNF.
- e.g. Is this formula true?

$$\underbrace{\exists e\ \forall u\ \exists c\ \exists d}_{\text{quantifier prefix}}\quad \underbrace{(\neg e \vee c)(e \vee d)(\neg u \vee c)(u \vee d)(\neg c \vee \neg d)}_{\text{matrix}}$$

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- QBF subsumes SAT. eg. Is this QBF true?

$$\exists x \exists y \exists z (x \vee y \vee z) \wedge (x \vee \neg y \vee \neg z) \wedge (\neg x \vee y \vee \neg z) \wedge (\neg x \vee \neg y \vee z)$$

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- QBF more succinctly expressive than SAT; PSPACE-complete.

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  Runs of QBF solver on false QBFs provide proofs of falsity.

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  no short runs of QBF solver.
- Proof Complexity –
  Formalising sound-and-complete proof systems underlying solvers;
  Demonstrating limitations by proving lower bounds.

QBF semantics: The two-player evaluation game

- QBF $Q\vec{x} \cdot F(x)$
- Two players, $P_\exists$ and $P_\forall$, step through quantifier prefix left-to-right. $P_\exists$ picks values for $\exists$ variables, $P_\forall$ for $\forall$ variables.

  Assignment constructed on a run: $\tilde{a}$.

  $P_\exists$ wins a run of the game if $F(\tilde{a})$ true. Otherwise $P_\forall$ wins.
- $Q\vec{x} \cdot F(x)$ true if and only if $P_\exists$ has a winning strategy. (model, Skolem function, ...)
- $Q\vec{x} \cdot F(x)$ false if and only if $P_\forall$ has a winning strategy. (countermodel, Herbrand function, ...)

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Focus in this talk on refutations: proving false QBF s false.

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- To achieve soundness:
  - Addition rules must preserve (some/every) $P_\exists$ winning strategy.
  - Finally derive empty clause $\square$.
    (This defeats every potential $P_\exists$ strategy.)
- To achieve completeness:
  - From a $P_\forall$ winning strategy, use rules to derive $\square$.

Early SAT solvers

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  The DPLL algorithm (Davis-Putnam, Davis-Logemann-Loveland).
  - Decide / guess the value of some variables.
  - Propagate values in unit clauses.
  - Backtrack on falsified clauses.

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- Correctness:
  DPLL run reporting UNSAT yields refutation in Resolution.

$$\frac{x \vee A \qquad \overline{x} \vee B}{A \vee B}$$

Lifting a SAT solver to QBF

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- Do not propagate universal variables – these values are not controlled by $P_\exists$ and so cannot be forced.
- Make decisions in quantifier-block order (level-order).
- While learning, perform reductions wherever possible – remove universal literals from clauses if removal sound.

Treatment of Universal Variables

- Must be treated differently; $\exists x \forall y [x = y]$ false but $\exists x \exists y [x = y]$ true.
- Allow adversarial instantiation. When is this sound?
- Allow usage as resolution pivot – is this sound?

The Reduction rule

- When adversarial instantiation is sound, and how to implement it:
  - In the two-player game, if a partial run results in a purely universal clause, $P_\forall$ can/will falsify it.
  - If a derived clause has a "trailing" (rightmost) universal literal, $P_\exists$ must be prepared for $P_\forall$ to adversarially falsify it.

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  Universal reduction.

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  For other types of lines:
  $\frac{\ell(x_1, x_2, \ldots x_m, u)}{\ell(x_1, x_2, \ldots x_m, b)}$ ($\mathrm{var}(u)$ is universal, right of all x_i; $b \in \{\top, \bot\}$)

Reduction-based refutational proof systems

- A sound and complete proof system: Q-Res.
  [KleineBüning,Karpinski,Flögel 1995] Two rules:
  * Resolution: (super-sound) $\dfrac{x \vee A \qquad \overline{x} \vee B}{A \vee B}$ ($\mathrm{var}(x)$ is existential)
  * Universal reduction: (super-sound)
    $\dfrac{A \vee u}{A}$ ($\mathrm{var}(u)$ is universal, and right of all variables in A)

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- The QURes proof system (a.k.a. Res+$\forall$Red):
  Resolution + Universal Reduction. [vanGelder 2012]
- For (essentially) any line-based propositional proof system P,
  QBF proof system $P + \forall$Red: [Beyersdorff,Bonacina,Chew,Pich 2016,20]
  - CP+$\forall$Red [Beyersdorff,Chew,M,Shukla 2016]
  - $\mathcal{C}$-Frege + $\forall$Red
  - EFrege + $\forall$Red
  - OBDD($\wedge$, weaken) + $\forall$Red [Mengel,Slivovsky 2021]
  - $\vdots$

A resolution+reduction (QRes) refutation

$\exists e_1 \; \forall u_1 \; \exists c_1 \; \exists d_1 \quad \exists e_2 \; \forall u_2 \; \exists c_2 \; \exists d_2$

$(\bar{e}_1 \; c_1)$ $(e_1 \; d_1)$ $(\bar{e}_2 \; c_2)$ $(e_2 \; d_2)$ $(\bar{c}_1 \; \bar{d}_1 \; \bar{c}_2 \; \bar{d}_2)$

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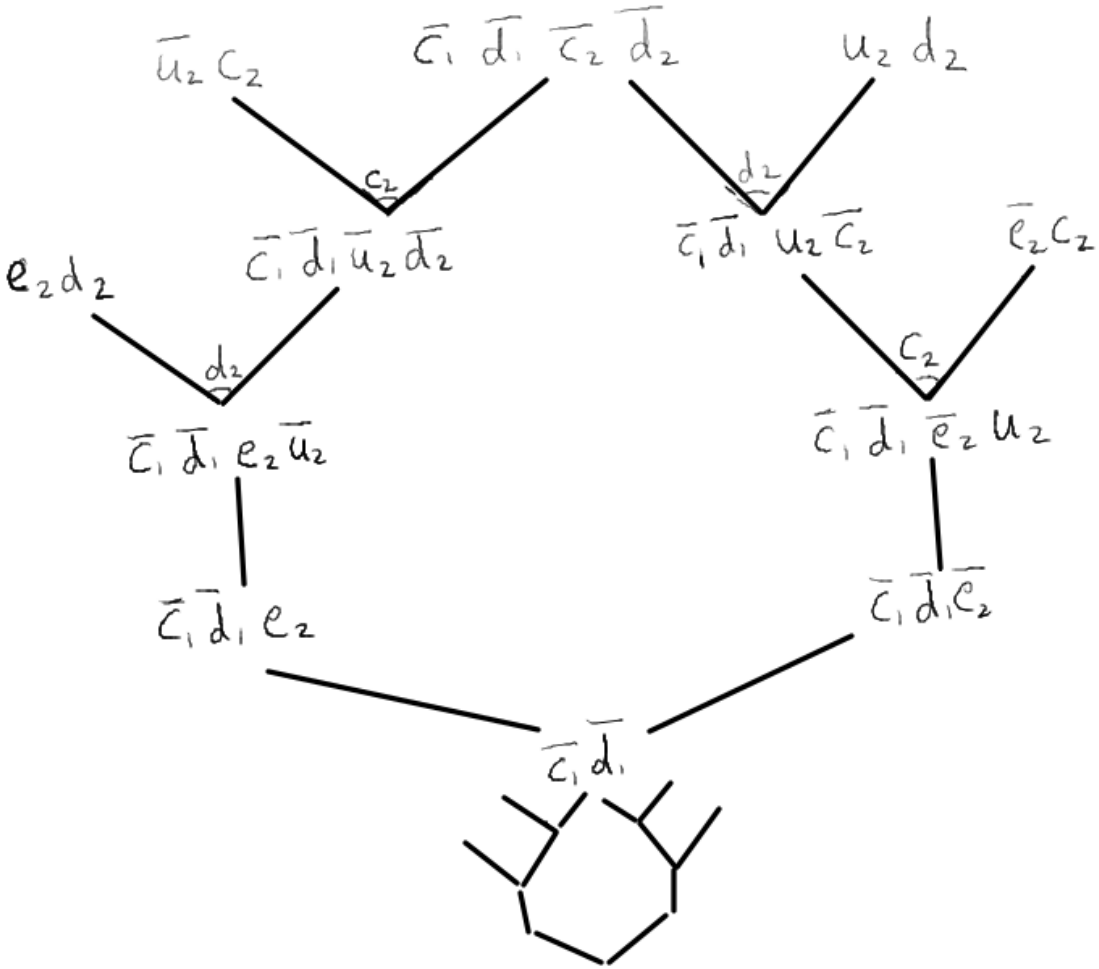

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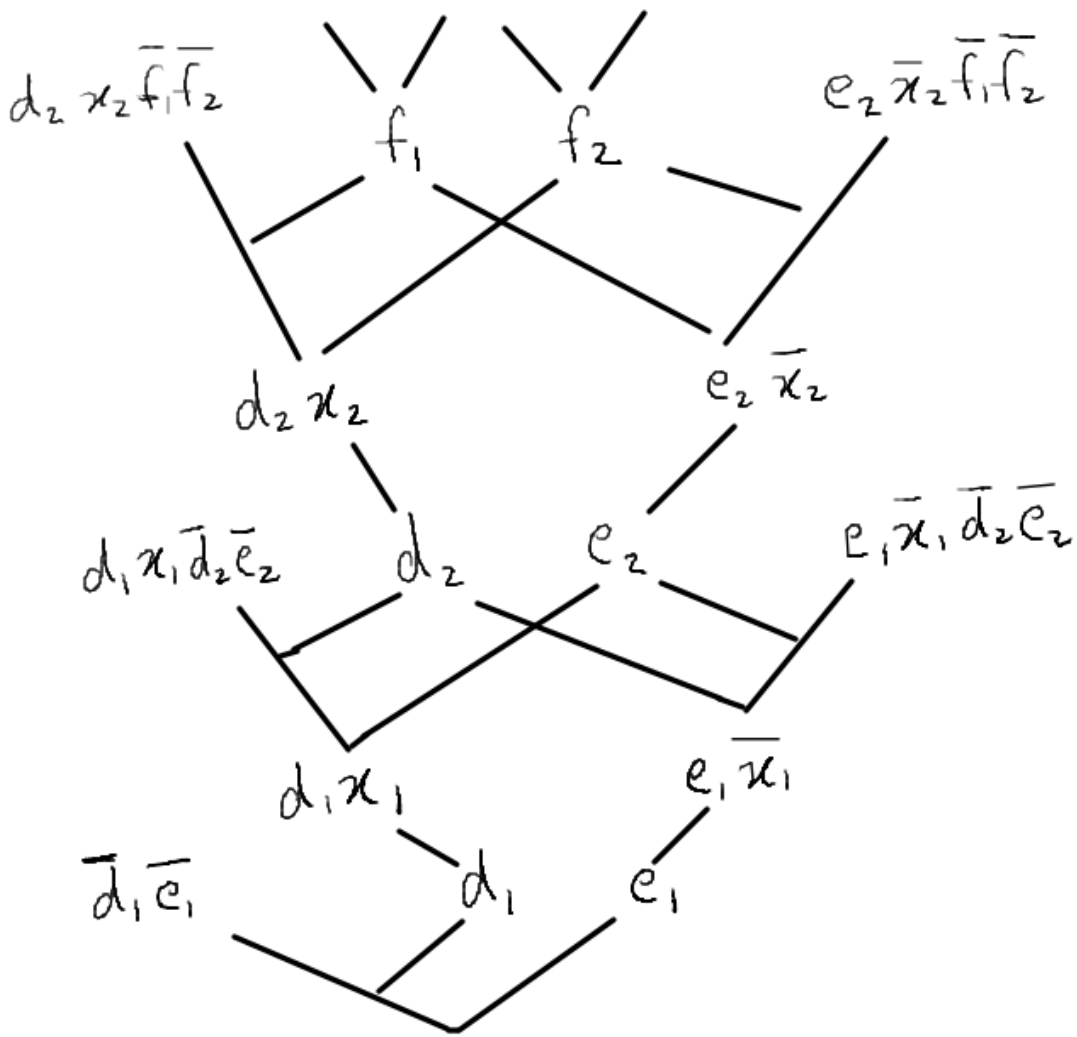

The Expansion framework

Unfolding the universal quantifier:

$$\forall u Q\vec{x} \cdot F(u, \vec{x}) \text{ is true}$$

$$\Updownarrow$$

$$[Q\vec{x} \cdot F(0, \vec{x})] \wedge [Q\vec{x} \cdot F(1, \vec{x})] \text{ is true}$$

$$\Updownarrow$$

$$Q\vec{x}^{u/0} Q\vec{x}^{u/1} \cdot \left[F(0, \vec{x}^{u/0}) \wedge F(1, \vec{x}^{u/1})\right] \text{ is true}$$

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- Expand the initial formula judiciously, on the fly.

  Then use standard resolution.
- Formula size blow-up? Not always too bad.

Limited Expansion

$$\exists (X_{i,j})_{n\times n} \forall z \exists a_1 \dots a_n b_1 \dots b_n \begin{array}{ll} (X_{ij}\ z\ a_i) & i,j \in [n] \\ (\overline{X}_{ij}\ \overline{z}\ b_j) & i,j \in [n] \\ (\overline{a}_1 \dots \overline{a}_n) & \\ (\overline{b}_1 \dots \overline{b}_n) & \end{array}$$

Initially $2n^2 + 2$ clauses.

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Initially $2n^2 + 2$ clauses. After expansion, $2n^2 + 4$ clauses.
Only $2n^2 + 2$ expanded clauses really needed.

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Expansion-based systems

- $\forall$Exp+Res [Janota,Marques-Silva 2015]
  - For each existential variable, specify the copy – values of all universals to the left.

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  - For each existential variable, specify the copy – values of all universals to the left.
- IR (Instantiation and Resolution) [Beyersdorff,Chew,Janota 2014]
  - When using an axiom, for each existential variable, specify a set of copies – values of all universals to the left, in this axiom.
  - At any stage, further restrict the set of copies by extending values to more universals

Reusing Expansion

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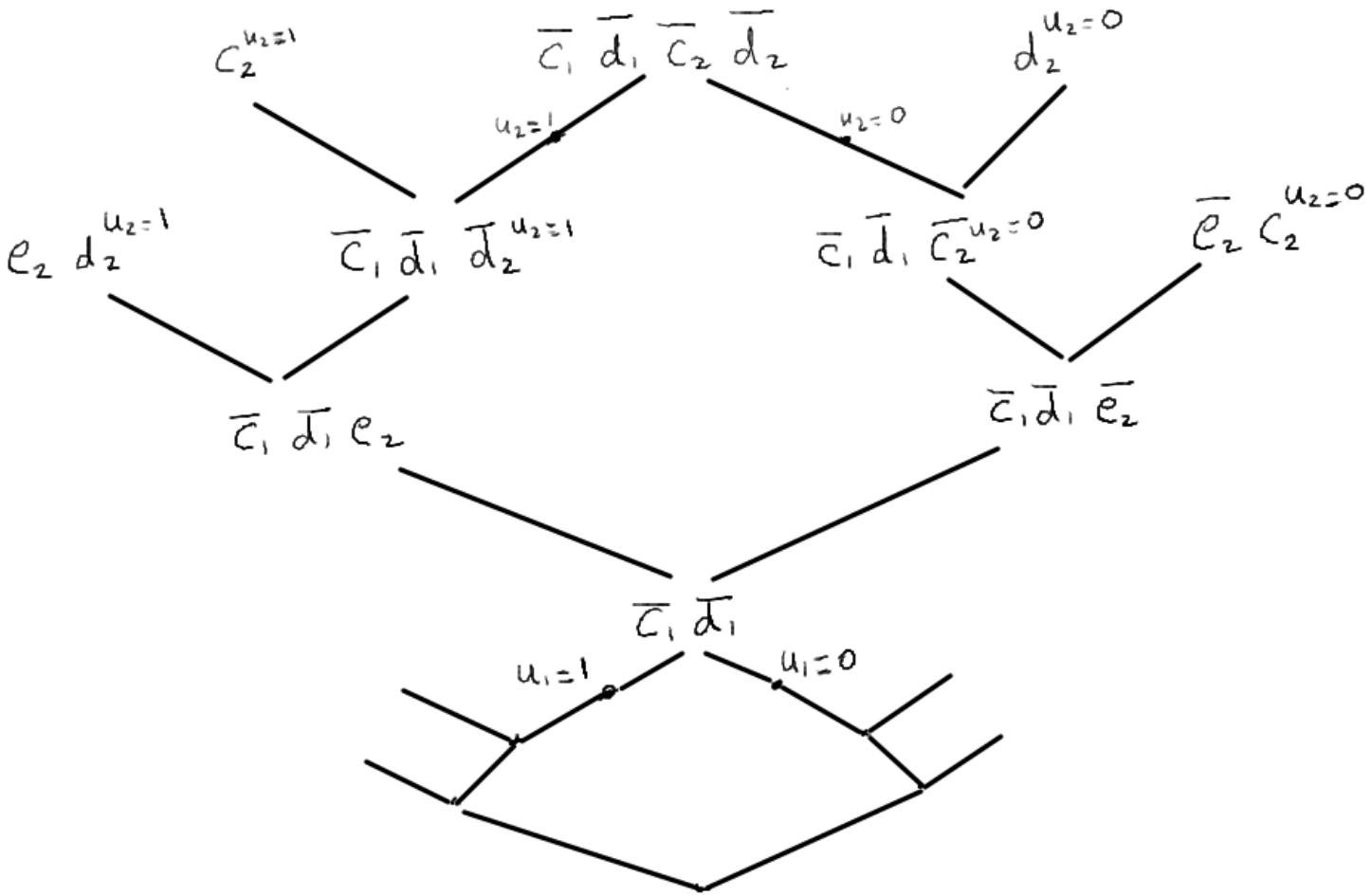

The CDCL paradigm in SAT solvers

- SAT solving: revolutionized by **CDCL** Conflict-driven Clause Learning
  - Start as in DPLL.
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- QCDCL: adaptation to QBF.

The Merging paradigm

- QBF proof systems based on universal reduction, expansion:
  None of these systems properly explains QCDCL runs.

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  $$\dfrac{\dfrac{x \vee u \qquad \neg x \vee \neg u}{u^*}}{\square}$$
- Long-Distance QResolution LD-Q-Res.
  - Allow u and $\neg u$ to be combined into u^*, provided u right of pivot.
  - Disallow resolution with pivot x if $u < x$ and antecedents contain "conflicting" $u, \neg u, u^*$.

An LDQRes refutation

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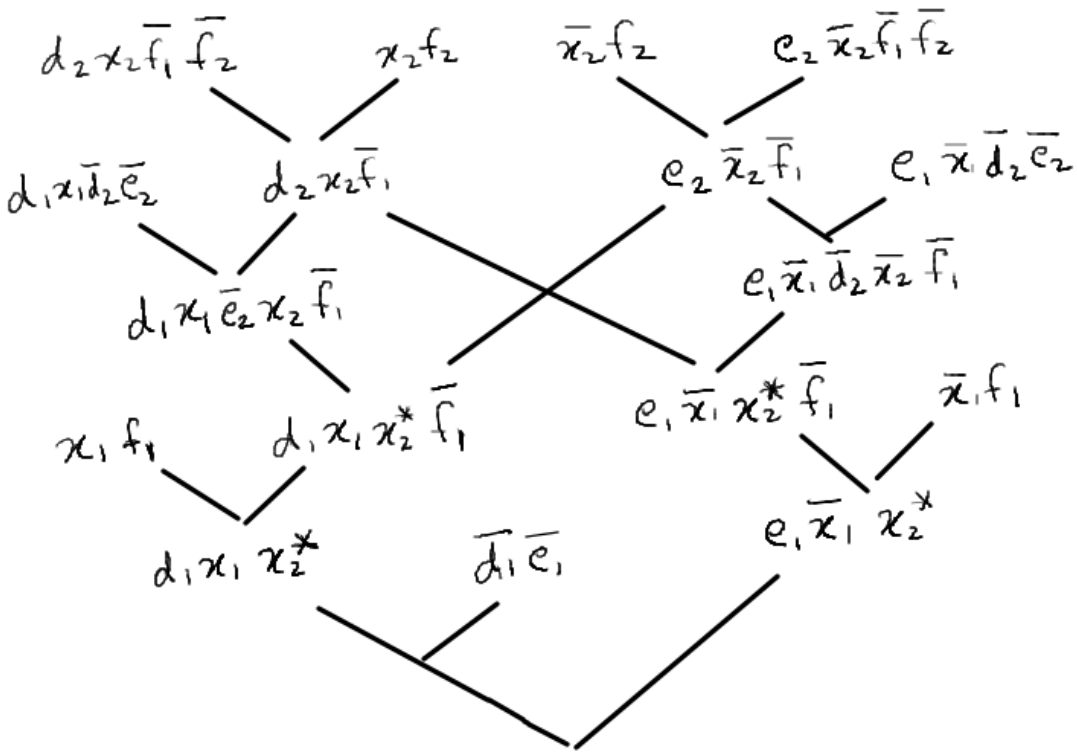

Wider applicability of Merging

Proof-theoretically, merging of literals is sound in other settings too.

- Q-Res with merging $\rightarrow$ LD-Q-Res.
  [Zhang,Malik 2002],[Balabanov,Jiang 2012],[Egly,Lonsing,Widl 2013]
  (underlies, but is more powerful than, QCDCL.)
- QU-Res with merging on existential pivots $\rightarrow$LQU-Res.
- QU-Res with merging on all pivots $\rightarrow$ LQU$^+$-Res.
  [Balabanov,Jiang,Widl 2014]
- merging in annotated expansions in IR $\rightarrow$ IRM.
  [Beyersdorff,Chew,Janota 2014,15,19]

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A somewhat different use of merging: MergeRes

Separate out partial strategies of $P_\forall$ player; perform resolution on existential sub-clauses while building up the strategies.
[Beyersdorff,Blinkhorn,M JAR20]

Discarding spurious dependencies

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- $\exists x \exists y \forall u \exists z \quad [x = z] \wedge [y = u]$ is false. But
  - The only winning strategy for $P_\forall$ is $u = \neg y$.
  - $P_\forall$ has no use for info about x.

Solvers and proof systems with dependency schemes

- Can the *structure of the clauses*
  – (clause-variable incidence, clause-pivot-clause connectivity, ...) –
  help identify such useless "dependencies"?

  Yes; some can be identified.
  (Proving soundness becomes even more complicated.)

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- Does identifying them help with shorter solver runs/proofs?

  Not always, but often enough.

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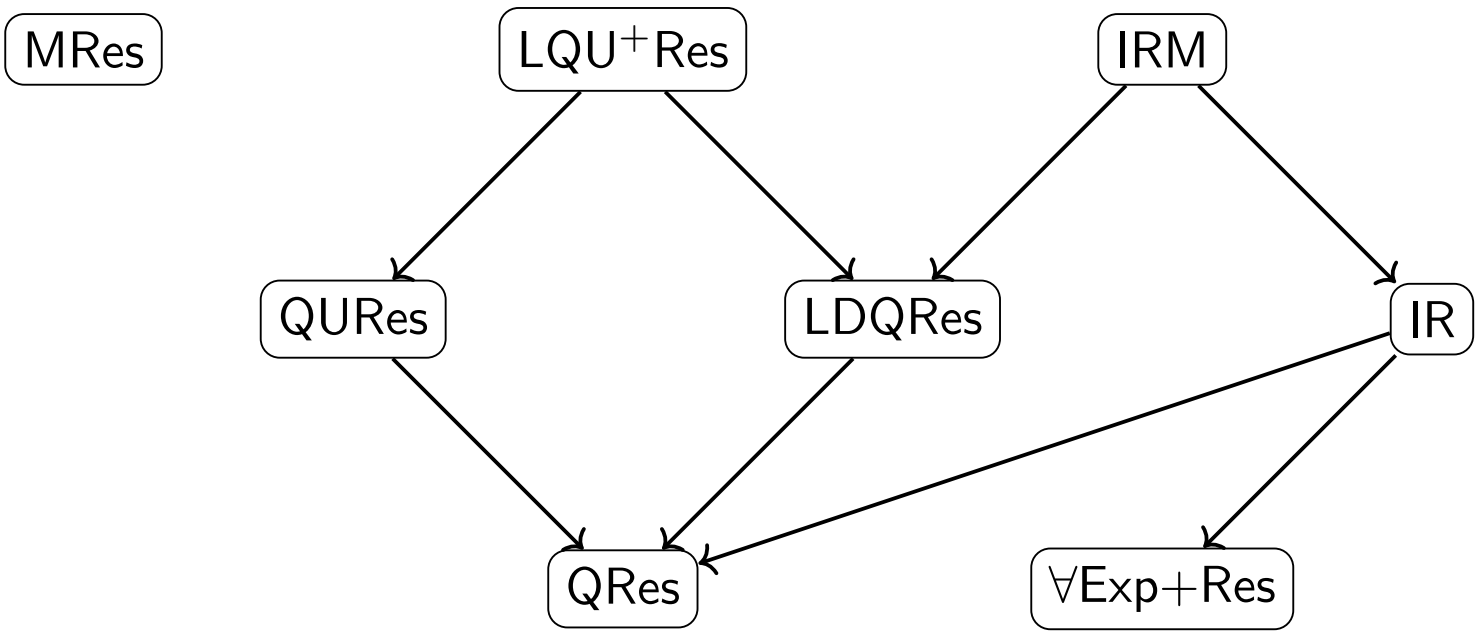
  (Proving soundness becomes even more complicated.)
- Does identifying them help with shorter solver runs/proofs?

  Not always, but often enough.
- Dependency Schemes: Trivial, Standard, Reflexive resolution paths, Tautology-free resolution paths, Implication-free resolution paths.
  [Biere,Lonsing 2010] [Samer,Szeider 2009] [Slivovsky,Szeider 2012,16] [Beyersdorff,Blinkhorn,Peitl 2020,21] ...

The simulation order



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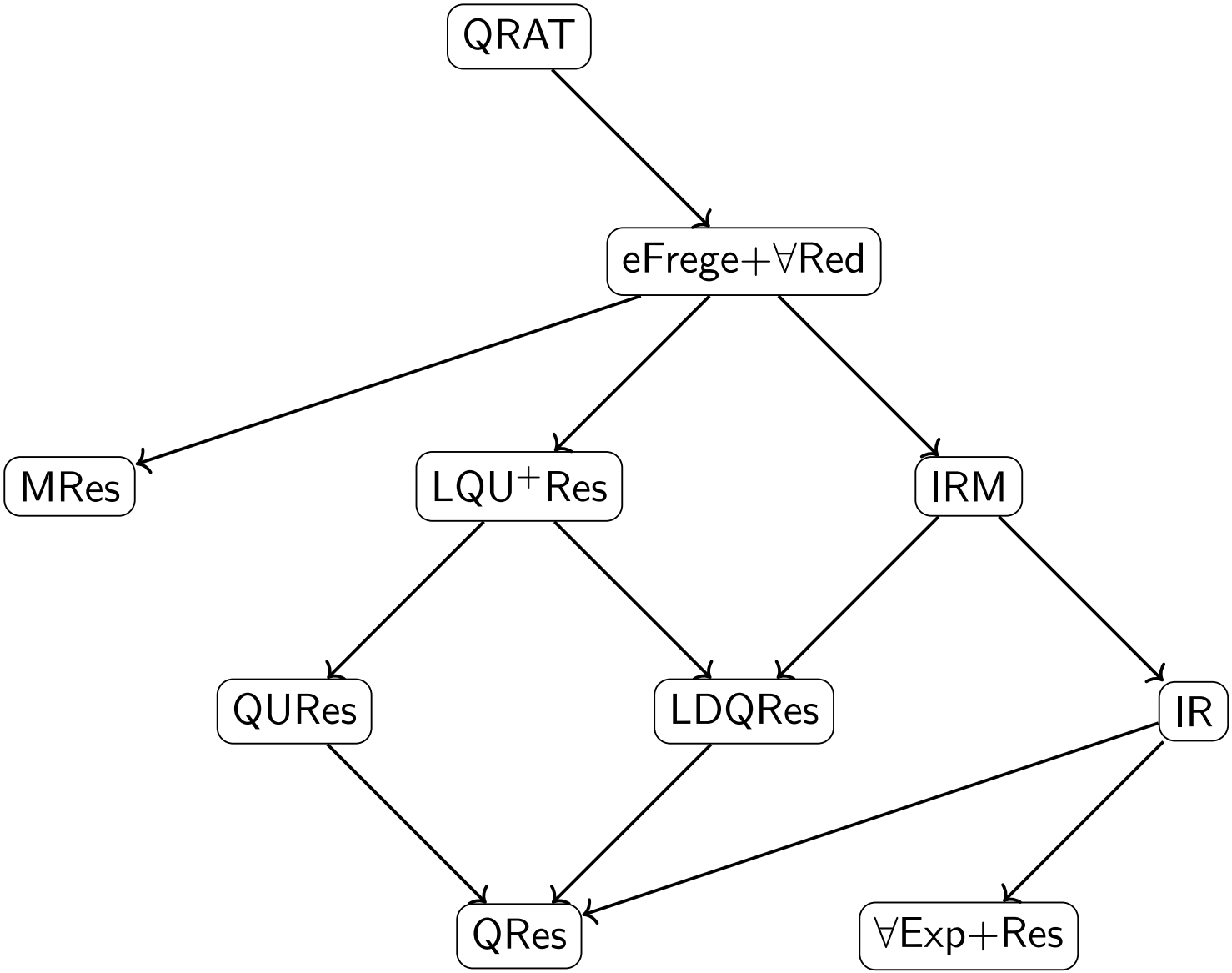

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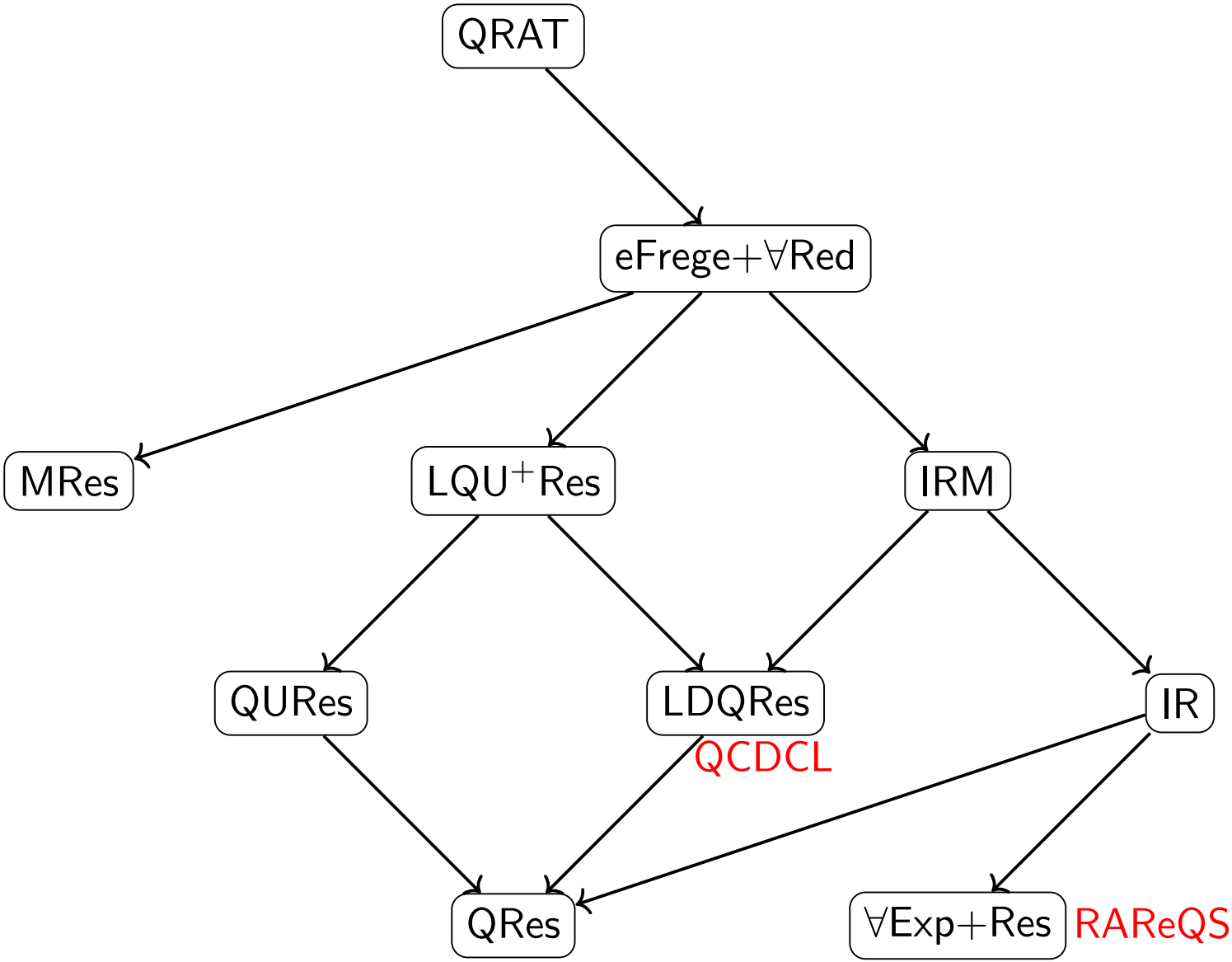

Proving lower bounds for QBF proof systems

- Each QBF proof system P, restricted to QBFs with no universal variables, is a propositional proof system P'.
- Hardness in P' implies hardness in P.
- A QBF hard for P may contain subformulas hard for P', in a somewhat obfuscated way. Is its hardness in P due to P' hardness, or due to the obfuscation, or both?

Discounting propositional hardness

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  How to formalise this?

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  Let a QBF solver use a SAT solver as a subroutine.

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- eg in $P + \forall$Red systems, allow entailment for free;
  count only reduction steps.
  Lower bounds on this count are "genuine" QBF lower bounds.
  [Chen 2016] [Beyersdorff,Hinde,Pich 2017]

Lifting propositional lower bound techniques

- Short proofs can be made narrow.
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  Works for most resolution-based QBF systems.
  [Beyersdorff,Chew,M,Shukla 2017]
- Random formulas are hard.
  Difficult to formalise: what is "random enough"?
  (A fully random k-CNF matrix will, w.h.p., have a purely universal clause, so trivial to refute in any reduction-based system!)
  Works for some structured randomness, but uses new idea.

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- If the winning strategy is hard to compute in the relevant circuit model, then all proofs in the proof system must be large.

Used for many lower bounds.

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- Cost of a formula: [Beyersdorff,Blinkhorn,Hinde 2018,19].
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All these measures can be large even if winning strategy trivial to compute.

Summary

- Correspondence between QBF solvers and QBF proof systems not yet clean-and-clear. In fact, far from clear.
- Proof-theoretic studies establish provable limitations of some solvers.
- Proof-theoretic studies can suggest new heuristics.

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Thank You!