

LECTURES ON THE TRACE IN A FINITE VON NEUMANN ALGEBRA

by

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Suppose that R is a von Neumann algebra with center C and unitary group U . By a center-valued trace on R we mean a norm continuous linear mapping T from R into C such that

- (i) $T(U*AU) = T(A)$ ($A \in R$, $U \in U$);
- (ii) $T(C) = C$ ($C \in C$);
- (iii) $T(CA) = CT(A)$ ($C \in C$, $A \in R$);
- (iv) $T(A)$ is positive if A , in R , is positive.

It is relatively easy to prove that, if such a mapping T exists, then T is unique and R is finite (in a sense explained in §5). In fact, every finite von Neumann algebra has a center-valued trace T , and T is ultraweakly continuous; but the question of existence of T is much harder than that of uniqueness.

The existence of a trace was proved at first for a factor (that is, a von Neumann algebra with center consisting of scalar multiples of the identity). With R a finite factor, Murray and von Neumann [8] constructed a "dimension function", defined on the projections in R , and from this produced a scalar-valued function τ on R , which they called a "relative trace". They showed that τ has properties (i), ..., (iv) above, and that $\tau(A+B) = \tau(A) + \tau(B)$ whenever A, B are commuting self-adjoint operators in R . In [9] they established the

unrestricted additivity of τ (thus showing that τ is a trace in the sense defined above), by an intricate and indirect argument. In view of this history, the basic difficulty in this area of the subject is sometimes referred to as the problem of "additivity of the trace". From a present-day point of view, the nomenclature is perhaps a little misleading: the basic problem is to prove the existence of a (linear) trace, rather than to prove the linearity of an existing relative trace.

The rather involved argument used in [8, 9] was somewhat simplified and extended by Dixmier [2] and Kadison [6, 7], so as to apply to any finite von Neumann algebra. Recently, F. J. Yeadon [14] found an entirely different proof, based on results of Akemann [1] and Takesaki [13] concerning the predual of an operator algebra, together with the Ryll-Nardzewski fixed point theorem [12, 10]. In this proof, the argument which establishes the existence of the trace yields at the same time its ultraweak continuity. In contrast with the earlier proofs, no use is made of (any variant of) a dimension function; but extensive use is made of properties of the predual which were not required (at the time, not available) in the earlier proofs.

In these notes, we develop the theory of von Neumann algebras, *ab initio*, up to the point where the existence of the trace can be proved by Yeadon's method. The treatment is not quite self-contained; the notes were designed as a supplement to [11], and in §5 we quote without proof some results which are

readily accessible from [3] or [11]. However, this is the only case in which we state without proof results concerning von Neumann algebras, which are needed for our later work. It is assumed that the reader is familiar with Banach space theory, with the spectral theorem for a bounded self-adjoint Hilbert space operator, and with the functional calculus associated with such an operator (continuous functions suffice, for our purposes).

In §1 we prove some simple results concerning Banach spaces. These are used, in §2, to establish the existence and elementary properties of the predual of an ultraweakly closed linear space of operators; the treatment in this section is similar to that given in [3]. In §3 we discuss some of the elementary properties of von Neumann algebras, and prove the double commutant theorem. We show, in §4, that a norm continuous linear functional on a von Neumann algebra is ultraweakly continuous if and only if it is completely additive on projections. This was proved first by Takesaki [13] (in a slightly different form); the treatment given in §4 is based on [5]. By means of Takesaki's result, we derive a necessary and sufficient condition (due to Akemann [1]) that a bounded subset in the predual of a von Neumann algebra should be relatively weakly compact. It is perhaps worth remarking that only the sufficiency part of this theorem is required in our subsequent work. In §5, we use Akemann's result just cited, and the Ryll-Nardzewski

fixed point theorem, in an account of Yeadon's proof [14] of the existence of the trace in a finite von Neumann algebra.

1. Auxiliary results on linear spaces

Lemma 1.1. If E is a real or complex vector space, $p_1, p_2, \dots, p_n$ are seminorms on E , f is a linear functional on E and

$$|f(x)| \leq p_1(x) + p_2(x) + \dots + p_n(x)$$

for each x in E , then there exist linear functionals $f_1, f_2, \dots, f_n$ on E such that

$$f = f_1 + f_2 + \dots + f_n, \quad |f_j(x)| \leq p_j(x)$$

when $x \in E$ and $1 \leq j \leq n$.

Proof. With E^n the vector space of all n -tuples $(x_1, x_2, \dots, x_n)$ of elements of E , we define a seminorm p on E^n , a subspace D of E^n and a linear functional g on D by

$$p(x_1, x_2, \dots, x_n) = p_1(x_1) + p_2(x_2) + \dots + p_n(x_n),$$

$$D = \{(x, x, \dots, x) : x \in E\},$$

$$g(x, x, \dots, x) = f(x) \quad (x \in E).$$

Since $|g(x, x, \dots, x)| \leq p(x, x, \dots, x)$ for each element $(x, x, \dots, x)$

of D , it follows from the Hahn-Banach theorem that g extends to a linear functional h on E^n for which $|h(v)| \leq p(v)$ whenever $v \in E^n$. We define a linear functional f_j on E by

$$f_j(x) = h(0, \dots, 0, x, 0, \dots, 0)$$

(where, on the right hand side, x appears in the j th. position).

Then, for each x in E ,

$$\begin{aligned} |f_j(x)| &= |h(0, \dots, 0, x, 0, \dots, 0)| \\ &\leq p(0, \dots, 0, x, 0, \dots, 0) = p_j(x), \end{aligned}$$

and

$$\begin{aligned} f(x) &= g(x, x, \dots, x) = h(x, x, \dots, x) \\ &= h(x, 0, \dots, 0) + h(0, x, 0, \dots, 0) + \dots + h(0, \dots, 0, x) \\ &= f_1(x) + f_2(x) + \dots + f_n(x). \end{aligned}$$

If E is a Banach space and F is a linear subspace of the Banach dual space E^* , we write $\sigma(E, F)$ for the weak topology induced by F on E . Hence $\sigma(E, F)$ is the locally convex topology defined by the family $\{p_f : f \in F\}$ of seminorms on E , where $p_f(x) = |f(x)|$ ($x \in E$).

Lemma 1.2. Suppose that E_1 is the closed unit ball in a Banach space E , F is a linear subspace of the Banach dual space E^* , $\bar{F}$ is the norm closure of F in E^* and f is a linear functional on E .

- (i) f is $\sigma(E, F)$ continuous on E if and only if $f \in F$.
- (ii) f is $\sigma(E, F)$ continuous on E_1 if and only if $f \in \overline{F}$.
- (iii) The topologies $\sigma(E, F)$ and $\sigma(E, \overline{F})$ coincide on E_1 .
- (iv) If F is a norm closed subspace of E^* and f is $\sigma(E, F)$ continuous on E_1 , then f is $\sigma(E, F)$ continuous on E .

Proof. Clearly f is $\sigma(E, F)$ continuous if $f \in F$.
 Conversely, suppose that f is $\sigma(E, F)$ continuous. There exist $g_1, \dots, g_n$ in F such that

$$|f(x)| \leq p_{g_1}(x) + \dots + p_{g_n}(x) \quad (x \in E).$$

By Lemma 1.1, there exist linear functionals $f_1, \dots, f_n$ on E such that $f = f_1 + \dots + f_n$ and

$$|f_j(x)| \leq p_{g_j}(x) = |g_j(x)| \quad (x \in E).$$

If $x \in E$ and $g_j(x) = 0$, then $f_j(x) = 0$; so $f_j = 0$ if $g_j = 0$. If $g_j \neq 0$, we can choose x_j in E such that $g_j(x_j) = 1$. For each x in E ,

$$g_j(x - g_j(x)x_j) = g_j(x) - g_j(x)g_j(x_j) = 0;$$

so $0 = f_j(x - g_j(x)x_j) = f_j(x) - f_j(x_j)g_j(x)$. Thus $f_j = f_j(x_j)g_j \in F$, and $f = f_1 + \dots + f_n \in F$.

- (ii) If $f \in \overline{F}$ there is a sequence (f_n) of elements of F such that $\|f - f_n\| \rightarrow 0$. The restriction $f|_{E_1}$ is the uniform

limit of the sequence $(f_n|_{E_1})$ of $\sigma(E, F)$ continuous functions, so $f|_{E_1}$ is $\sigma(E, F)$ continuous.

Conversely, suppose that f is $\sigma(E, F)$ continuous on E_1 . This implies that f is norm continuous on E_1 , and therefore norm continuous on E . Given any positive ε , the $\sigma(E, F)$ continuity at 0 of $f|_{E_1}$ entails the existence of $g_1, \dots, g_n$ in F with the following property: $|f(x)| < \varepsilon$ whenever $x \in E_1$ and $p_{g_1}(x) + \dots + p_{g_n}(x) < 1$. We claim that

$$(1.2.1) \quad |f(x)| \leq \varepsilon \|x\| + \|f\| [p_{g_1}(x) + \dots + p_{g_n}(x)] \quad (x \in E).$$

To prove this, it suffices (by homogeneity) to consider the case in which $\|x\| = 1$; and (1.2.1) then follows easily by considering separately two possibilities, $p_{g_1}(x) + \dots + p_{g_n}(x) \geq 1$ and $p_{g_1}(x) + \dots + p_{g_n}(x) < 1$.

From (1.2.1) and Lemma 1.1, $f = f_1 + f_2$ where the linear functionals f_1, f_2 on E satisfy

$$|f_1(x)| \leq \varepsilon \|x\|, \quad |f_2(x)| \leq \|f\| [p_{g_1}(x) + \dots + p_{g_n}(x)],$$

for all x in E . Thus $\|f_1\| \leq \varepsilon$, f_2 is $\sigma(E, F)$ continuous, and thus $f_2 \in F$, by (i). We have now proved that, if $\varepsilon > 0$, there exists f_2 in F with $\|f - f_2\| \leq \varepsilon$; so $f \in \bar{F}$.

(iii) This follows at once from (ii).

(iv) This is an immediate consequence of parts (i) and (ii) of the lemma.

Suppose that E and F are Banach spaces and $\langle \cdot, \cdot \rangle$ is a bounded bilinear functional on $E \times F$. The equation $\phi_f(e) = \langle e, f \rangle$ defines a bounded linear functional ϕ_f on E , when $f \in F$; and the mapping $f \rightarrow \phi_f$ from F into the dual space E^* is bounded and linear. If this last mapping is an isometric isomorphism from F onto E^* , we say that F can be identified (up to isometric isomorphism) with E^* by means of the bilinear functional $\langle \cdot, \cdot \rangle$.

Theorem 1.3. Suppose that B is a Banach space, B_* is a norm closed subspace of the dual space B^* , and B can be identified (up to isometric isomorphism) with the dual space of B_* by means of the bilinear functional $\langle \cdot, \cdot \rangle$ on $B_* \times B$ defined by

$$\langle f, x \rangle = f(x) \quad (f \in B_*, \quad x \in B).$$

If M is a $\sigma(B, B_*)$ closed subspace of B and M_* is the subspace $\{f|_M : f \in B_*\}$ of M^* , then

- (i) M_* is a norm closed subspace of M^* ;
- (ii) given g in M_* and $\varepsilon (> 0)$, there exists f in B_* such that $g = f|_M$ and $\|f\| < \|g\| + \varepsilon$;
- (iii) M can be identified (up to isometric isomorphism) with the dual space of M_* by means of the bilinear functional $\langle \cdot, \cdot \rangle$ on $M_* \times M$ defined by

$$\langle g, y \rangle = g(y) \quad (g \in M_*, \quad y \in M).$$

Proof. With $N = \{f \in B_* : f|_M = 0\}$, N is a norm closed subspace of B_* . Since M is $\sigma(B, B_*)$ closed,

$$(1.3.1) \quad M = \{x \in B : f(x) = 0 \text{ for each } f \text{ in } N\}.$$

The linear mapping $f \rightarrow f|_M$ carries B_* onto M_* , is norm decreasing, has kernel N , and so induces a norm decreasing one to one linear mapping

$$(1.3.2) \quad f + N \rightarrow f|_M$$

from the quotient Banach space B_*/N onto M_* .

We assert that the mapping (1.3.2) is isometric. For this, suppose $f_0 \in B_*$ and $\|f_0 + N\| = 1$; that is, $\text{dist}(f_0, N) = 1$. By the Hahn Banach theorem there is a bounded linear functional ϕ on B_* such that $\|\phi\| = \phi(f_0) = 1$ and $\phi(f) = 0$ for every f in N . Since B can be identified with the dual space of B_* , ϕ corresponds to an element x of B such that $\|x\| = f_0(x) = 1$ and $f(x) = 0$ for each f in N . By (1.3.1), $x \in M$; so $\|f_0|_M\| \geq f_0(x) = 1$. This proves our assertion that the mapping (1.3.2) is isometric. Since B_*/N is complete, it follows that M_* is complete, and is therefore a norm closed subspace of M^* . This proves (i).

Given g in M_* , $g = f_0|_M$ for some f_0 in B_* , and $\|g\| = \|f_0 + N\| = \text{dist}(f_0, N)$. Given $\epsilon (> 0)$ there exists f_1 in N such that $\|f_0 + f_1\| < \|g\| + \epsilon$. Since $f_1|_M = 0$, we have $\|f\| < \|g\| + \epsilon$ and $f|_M = g$, where $f = f_1 + f_2 \in B_*$. This proves (ii).

For each x in M , the linear functional $\phi_x: g \rightarrow g(x)$ on M_* is bounded, with

$$\|\phi_x\| = \sup \{|g(x)| : g \in M_*, \|g\| < 1\}.$$

By (ii),

$$\|\phi_x\| = \sup \{|f(x)| : f \in B_*, \|f\| < 1\} = \|x\|$$

(the last equality follows from the fact that $x \in B$ and B can be identified with the dual space of B_*). Thus the linear mapping $x \rightarrow \phi_x$, from M into the dual space of M_* , is isometric. Given any bounded linear functional ϕ on M_* , the linear functional $f \rightarrow \phi(f|M)$ on B_* is bounded, and vanishes on N . Hence there is an element x of B such that $\phi(f|M) = f(x)$ for all f in B_* , and $f(x) = 0$ whenever $f \in N$. By (1.3.1), $x \in M$; so $\phi(f|M) = (f|M)(x) = \phi_x(f|M)$ for all f in B_* . Thus $\phi = \phi_x$, and (iii) is proved.

2. The linear space $B(H)$

Let H be a Hilbert space and denote by $B(H)$ the set of all bounded linear operators from H into itself. In this section we consider $B(H)$ as a Banach space with the usual operator norm

$$\|T\| = \sup\{\|Tx\| : x \in H, \|x\| \leq 1\} \quad (T \in B(H)).$$

With x and y in H , the equation

$$\omega_{x,y}(T) = \langle Tx, y \rangle \quad (T \in B(H))$$

defines a bounded linear functional $\omega_{x,y}$ on $B(H)$, with $\|\omega_{x,y}\| = \|x\| \|y\|$. The set of all finite linear combinations of linear functionals $\omega_{x,y}$ is a linear subspace $B(H)_{\sim}$ of the Banach dual space $B(H)^*$. The norm closure of $B(H)_{\sim}$ in $B(H)^*$ will be denoted by $B(H)_*$.

The topology $\sigma(B(H), B(H)_{\sim})$ on $B(H)$ is called the weak operator topology, while $\sigma(B(H), B(H)_*)$ is called the ultraweak topology. Clearly the weak operator topology is Hausdorff. It is coarser than the ultraweak topology, and this in turn is coarser than the norm topology. The following results is an immediate consequence of Lemma 1.2.

- Lemma 2.1. (i) $B(H)_{\sim}$ is the set of all weak operator continuous linear functionals on $B(H)$.
 (ii) $B(H)_*$ is the set of all ultraweakly continuous linear functionals on $B(H)$.
 (iii) A linear functional on $B(H)$ is ultraweakly continuous if and only if it is weak operator continuous on the unit ball of $B(H)$.
 (iv) The weak operator and ultraweak topologies coincide on the unit ball of $B(H)$.

Lemma 2.2. $B(H)$ can be identified (up to isometric isomorphism) with the Banach dual space of $B(H)_*$ by means of the bilinear functional $\langle \cdot, \cdot \rangle$ on $B(H)_* \times B(H)$ which is defined by

$$\langle f, T \rangle = f(T) \quad (f \in B(H)_*, T \in B(H)).$$

Proof. For a fixed T in $B(H)$, the linear functional $\phi_T : f \rightarrow f(T)$ on $B(H)_*$ is bounded, with $\|\phi_T\| \leq \|T\|$. Moreover,

$$\begin{aligned} \|T\| &= \sup\{|\langle Tx, y \rangle| : x, y \in H, \|x\| = \|y\| = 1\} \\ &= \sup\{|\omega_{x,y}(T)| : x, y \in H, \|x\| = \|y\| = 1\} \\ &= \sup\{|\phi_T(\omega_{x,y})| : x, y \in H, \|x\| = \|y\| = 1\} \\ &\leq \|\phi_T\|, \end{aligned}$$

since $\|\omega_{x,y}\| = 1$ when $\|x\| = \|y\| = 1$. It follows that the linear mapping $T \rightarrow \phi_T$ is an isometric isomorphism from $B(H)$ into the Banach dual space of $B(H)_*$.

If ϕ is a bounded linear functional on $B(H)_*$, the equation

$$L(x, y) = \phi(\omega_{x,y})$$

defines a sesqui-linear functional L on $H \times H$; and L is bounded since

$$|L(x, y)| \leq \|\phi\| \|\omega_{x,y}\| = \|\phi\| \|x\| \|y\|.$$

Hence there is a T in $B(H)$ such that $L(x,y) = \langle Tx,y \rangle$, for each x and y in H . The last equation can be written as $\phi(\omega_{x,y}) = \omega_{x,y}(T) = \phi_T(\omega_{x,y})$. Since finite linear combinations of $\omega_{x,y}$'s are everywhere dense in $B(H)_*$, it follows from norm continuity and linearity of ϕ and ϕ_T that $\phi = \phi_T$. Hence the mapping $T \mapsto \phi_T$ carries $B(H)$ onto the dual space of $B(H)_*$.

Corollary 2.3. The unit ball of $B(H)$ is compact in the weak operator topology.

Proof. When $B(H)$ is identified with the dual space of $B(H)_*$, as in Lemma 2.2, the weak $*$ topology on $B(H)$ is the ultraweak topology. It follows that the unit ball of $B(H)$ is ultraweakly compact, and hence weak operator compact by Lemma 2.1 (iv).

Theorem 2.4. Suppose that M is an ultraweakly closed subspace of $B(H)$, $M_\sim$ is the set of all weak operator continuous linear functionals on M , and M_* is the set of all ultraweakly continuous linear functionals on M .

- (i) $M_\sim = \{f|M : f \in B(H)_\sim\}$.
- (ii) $M_* = \{f|M : f \in B(H)_*\}$.
- (iii) $M_\sim$ and M_* are subspaces of the Banach dual space M^* , and M_* is the norm closure of $M_\sim$.
- (iv) If $g \in M_*$ and $\epsilon > 0$, there is an f in $B(H)_*$ such that $g = f|M$ and $\|f\| < \|g\| + \epsilon$.

(v) M can be identified (up to isometric isomorphism) with the Banach dual space of M_* by means of the bilinear functional $\langle \cdot, \cdot \rangle$ on $M_* \times M$ which is defined by

$$\langle g, T \rangle = g(T) \quad (g \in M_*, T \in M).$$

(vi) If g is a linear functional on M , then $g \in M_*$ if and only if g is weak operator continuous on the unit ball of M .

Proof. Parts (i) and (ii) follow from the Hahn-Banach theorem, together with Lemma 2.1 (i),(ii). In view of (ii) and Lemma 2.2, parts (iv) and (v) of the present theorem follow immediately from Theorem 1.3, as does the fact that M_* is a norm closed subspace of M^* . By (i) and (ii), the norm decreasing linear mapping $f \rightarrow f|_M$ carries $B(H)_*$ onto M_* , and carries $B(H)_\sim$ onto $M_\sim$. Since $B(H)_\sim$ is norm dense in $B(H)_*$, it follows that $M_\sim$ is norm dense in M_* . Since, also, M_* is closed, this proves (iii).

From (i) and (ii), the weak operator and ultraweak topologies on M are $\sigma(M, M_\sim)$ and $\sigma(M, M_*)$ respectively. Since M_* is the norm closure of $M_\sim$, part (vi) of the theorem follows from Lemma 1.2(ii).

With the notation used in the statement of Theorem 2.4, we call M_* the predual of M .

Remark 2.5. When M is a $*$ subalgebra of $B(H)$ and contains the identity operator I , in addition to satisfying the other hypotheses of Theorem 2.4, the conclusion of part (iv) of that theorem can be strengthened; each g in M_* has the form $f|M$, with f in $B(H)_*$ and $\|f\| = \|g\|$. For a proof of this see, for example, [3: Exercice 10, p.68] or [11: Lemma 7, p.132].

The strong operator topology on $B(H)$ is the locally convex topology defined by the family $\{q_x : x \in H\}$ of seminorms on $B(H)$, where

$$q_x(T) = \|Tx\| \quad (x \in H, T \in B(H)).$$

It is clear that the strong operator topology is finer than the weak operator topology, but coarser than the norm topology.

Lemma 2.6. A linear functional f on $B(H)$ is weak operator continuous if and only if it is strong operator continuous. A convex subset of $B(H)$ is weak operator closed if and only if it is strong operator closed.

Proof. It suffices to prove the first statement in the lemma, since the second follows from the first by the Hahn-Banach theorem.

If f is strong operator continuous, there exist (non-zero) $x_1, \dots, x_n$ in H such that

$$|f(T)| \leq q_{x_1}(T) + \dots + q_{x_n}(T) \quad (T \in B(H)).$$

By Lemma 1.1, $f = f_1 + \dots + f_n$, where $f_1, \dots, f_n$ are linear functionals on $B(H)$ and

$$|f_j(T)| \leq q_{x_j}(T) = \|Tx_j\| \quad (T \in B(H)).$$

If $S, T \in B(H)$ and $Sx_j = Tx_j$, then

$$|f_j(S) - f_j(T)| = |f_j(S-T)| \leq \|(S-T)x_j\| = 0;$$

so $f_j(S) = f_j(T)$. Moreover, each vector in $B(H)$ has the form Tx_j , for some T in $B(H)$. It follows that the mapping $\phi_j : Tx_j \rightarrow f_j(T)$ is well-defined, and is a linear functional on H . Furthermore, ϕ_j is bounded, since

$$|\phi_j(Tx_j)| = |f_j(T)| \leq \|Tx_j\|.$$

Hence there is a vector y_j in H such that $\phi_j(x) = \langle x, y_j \rangle$ for each x in H ; and

$$\begin{aligned} f(T) &= \sum_{j=1}^n f_j(T) = \sum_{j=1}^n \phi_j(Tx_j) \\ &= \sum_{j=1}^n \langle Tx_j, y_j \rangle = \sum_{j=1}^n \omega_{x_j, y_j}(T). \end{aligned}$$

Thus $f = \sum \omega_{x_j, y_j}$. This shows that strong operator continuity of f implies weak operator continuity; and the reverse implication is apparent.

Remark 2.7. With M a subspace of $B(H)$ and f a linear functional on M , it follows from Lemma 2.6 and the Hahn-Banach

theorem that f is weak operator continuous if and only if it is strong operator continuous.

3. The $*$ algebra $B(H)$

Let H be a Hilbert space. In this section we review some of the properties of $B(H)$ as an associative linear algebra with (the usual) involution $*$, defined by

$$\langle Tx, y \rangle = \langle x, T^*y \rangle \quad (T \in B(H); \quad x, y \in H).$$

The mapping $T \rightarrow T^*$ is continuous in any one of the norm, ultraweak or weak operator topologies. For a fixed T_0 in $B(H)$, the mappings $T \rightarrow T_0 T$ and $T \rightarrow T T_0$ are continuous in each of the topologies just mentioned, and also in the strong operator topology. The mapping $(S, T) \rightarrow ST$ is continuous, from $B(H) \times B(H)$ to $B(H)$ in the norm topology, and from $B(H)_1 \times B(H)$ to $B(H)$ in the strong operator topology, where $B(H)_1$ is the unit ball of $B(H)$. The proofs are all elementary, and are left as an exercise for the reader.

By a $*$ algebra over H we mean a subalgebra R of $B(H)$, such that $T^* \in R$ whenever $T \in R$. The weak operator closure R^- of a $*$ algebra R coincides with the strong operator closure, and is itself a $*$ algebra, by virtue of Lemma 2.6 and the continuity properties set out in the preceding paragraph. We shall see later (Corollary 3.3) that R^- is also the ultra-weak closure of R .

A von Neumann algebra over H is a weak operator closed $*$ algebra R over H which contains the identity operator I on H . It is apparent that such an algebra R contains the self-adjoint and skew-adjoint parts

$$\frac{1}{2}(T + T^*), \quad \frac{1}{2}i(T^* - T)$$

of each of its elements T . With A a self-adjoint element of R , the spectral projections of A lie in R , since they are weak operator limits of polynomials in A . Moreover, A is the norm limit of certain finite linear combinations of its spectral projections, by the spectral theorem. From this it follows that a von Neumann algebra R is the norm closed linear span of the projections it contains.

Suppose A is a self-adjoint element of a von Neumann algebra R and f is a complex-valued continuous function on the spectrum $\text{sp}(A)$ of A . The operator $f(A)$, corresponding to f in the functional calculus for A , is the norm limit of a sequence of polynomials in A ; so $f(A) \in R$. This implies, in particular, that R contains the positive and negative parts of each of its self-adjoint elements.

We assert that each element T in a von Neumann algebra R is a finite linear combination of unitary operators in R . To prove this, it suffices to consider the case in which $T = T^*$ and $\|T\| \leq 1$. The spectrum $\text{sp}(T)$ is then a subset of the

compact interval $[-1,1]$, and we can define a continuous complex-valued function f on $\text{sp}(T)$ by $f(t) = t+i/(1-t^2)$. Since

$$t = \frac{1}{2}[f(t) + \overline{f(t)}], \quad f(t)\overline{f(t)} = \overline{f(t)}f(t) = 1$$

for each t in $\text{sp}(T)$, it follows that the operator $U = f(T)$ satisfies

$$U \in R, \quad T = \frac{1}{2}(U+U^*), \quad UU^* = U^*U = I.$$

Although stated here for von Neumann algebras, the results of this and the preceding paragraph apply also to a norm closed $*$ algebra over H which contains the identity operator.

With F a subset of $B(H)$, we define

$$F^* = \{T^* : T \in F\},$$

$$F' = \{S \in B(H) : ST = TS \text{ for each } T \text{ in } F \cup F^*\};$$

and we call F' the commutant of F . It is clear that F' is a von Neumann algebra. If X is a closed subspace of H and E is the projection from H onto X , it is not difficult to verify that $E \in F'$ if and only if $T(X) \subseteq X$ for each T in $F \cup F^*$.

We now state and prove von Neumann's "double commutant theorem"; we write R'' in place of $(R')'$.

Theorem 3.1. If R is a $*$ algebra over H and contains the identity operator I on H , then R'' is the strong operator closure of R .

Proof. It is apparent that $R \subseteq R''$ and that R'' is strong operator closed; so it suffices to prove that each element A of R'' lies in the strong operator closure of R . Each strong operator neighborhood N_0 of A contains a set N of the form

$$(3.1.1) \quad N = \{S \in B(H) : \|Sx_j - Ax_j\| < \epsilon \quad (j = 1, \dots, n)\},$$

where ϵ is a fixed positive real number and $x_1, \dots, x_n$ are fixed elements of H . It suffices to prove that R meets N .

We consider first the case in which $n = 1$. With x in H , the norm closure $\overline{Rx}$ of the linear subspace $Rx = \{Tx : T \in R\}$ is invariant under each operator in R , and contains x . Hence the projection E from H onto $\overline{Rx}$ is in R' , $Ex = x$, and so

$$Ax = AEx = EAx \in \overline{Rx}.$$

It follows that there is an element T in R such that $\|Tx - Ax\| < \epsilon$.

Let H_0 be the Hilbert direct sum of n copies of H ; that is, H_0 is the Hilbert space of all n -tuples $(y_1, \dots, y_n)$ of elements of H , with the inner product defined by

$$\langle (y_1, \dots, y_n), (z_1, \dots, z_n) \rangle = \sum_1^n \langle y_j, z_j \rangle.$$

There is a natural isomorphism between the $*$ algebra $B(H_0)$ and the $*$ algebra of all $n \times n$ matrices with entries in $B(H)$.

With S in $B(H)$, let $\tilde{S}$ be the operator in $B(H_0)$ whose matrix has S in each diagonal position, and zeros elsewhere. One verifies without difficulty that the set

$$R_0 = \{\tilde{T} : T \in R\}$$

is a $*$ algebra over H_0 and contains the identity operator on H_0 . Routine matrix calculations show that R'_0 consists of those operators on H_0 whose matrices have all their entries in R' , and that $\tilde{A} \in R''_0$. Let x be the element $(x_1, \dots, x_n)$ of H_0 , where $x_1, \dots, x_n$ are the vectors occurring in the definition (3.1.1) of N . The argument used in the preceding paragraph now applies, with R_0 in place of R , to show that there is an element T_0 of R_0 such that $\|T_0 x - \tilde{A}x\| < \epsilon$. For some T in R , we have $T_0 = \tilde{T}$, and

$$\begin{aligned} \sum_1^n \|Tx_j - Ax_j\|^2 &= \|((T-A)x_1, \dots, (T-A)x_n)\|^2 \\ &= \|\tilde{T}x - \tilde{A}x\|^2 < \epsilon^2. \end{aligned}$$

Thus $T \in R \cap N$; and R meets N , as required.

We next state the "Kaplansky density theorem", but omit the proof, which can be found, for example, in [3 : pp.43,44] or [11 : pp.24-26].

Theorem 3.2. Suppose that A and B are $*$ algebras over H , such that $A \subseteq B$ and A is strong operator dense in B .

Let A_s and B_s denote the sets of all self-adjoint elements, of A and B respectively. Then A_1 is strong operator dense in B_1 , and $(A_s)_1$ is strong operator dense in $(B_s)_1$, where in each case the suffix 1 is used to denote the closed unit ball in the appropriate space of operators.

Corollary 3.3. A $*$ algebra A over H has the same closure in each of the ultraweak, weak operator and strong operator topologies.

Proof. The closure A^- of A in the weak operator topology coincides with the strong operator closure (and the same comment applies to the unit ball A_1 of A) by Lemma 2.6. It is clear that the ultraweak closure of A is a subset of A^- . To prove the reverse inclusion, suppose that $A \in A^-$ and $\|A\| \leq 1$. By Theorem 3.2, A lies in the strong (equivalently, weak) operator closure of A_1 . Since the weak operator and ultraweak topologies coincide on the unit ball of $B(H)$, A is in the ultraweak closure of A_1 , and so lies in the ultraweak closure of A .

We remark that neither Theorem 3.2, nor the part of Corollary 3.3 which depends on it, will be used in these notes. We have included these results only for completeness.

A linear functional f on a von Neumann algebra R is described as hermitian if $f(T)$ is real whenever $T = T^* \in R$.

Each linear functional g on $\mathcal{R}$ has the form $g_1 + ig_2$, with g_1 and g_2 hermitian; it suffices to take $g_1 = \frac{1}{2}(g+g^*)$ and $g_2 = \frac{1}{2}i(g^*-g)$, where

$$g^*(T) = \overline{g(T^*)} \quad (T \in \mathcal{R}).$$

Lemma 3.4. Suppose that ω is a hermitian bounded linear functional on a von Neumann algebra $\mathcal{R}$, and η is a real number.

(i) If $\omega(A) > \eta$ for some positive operator A in the unit ball of $\mathcal{R}$, then $\omega(E) > \eta$ for some projection E in $\mathcal{R}$.

(ii) If $|\omega(F)| \leq \eta$, for every projection F in $\mathcal{R}$, then $\|\omega\| \leq 4\eta$.

Proof. (i) By the spectral theorem there exists an orthogonal family $\{E_1, \dots, E_n\}$ of projections in $\mathcal{R}$, and $\lambda_1, \dots, \lambda_n$ in the interval $[0, 1]$, such that

$$\|A - \sum \lambda_j E_j\| < \|\omega\|^{-1}(\omega(A) - \eta).$$

Thus

$$\omega(A) - \sum_{j=1}^n \lambda_j \omega(E_j) < \omega(A) - \eta, \quad \sum_{j=1}^n \lambda_j \omega(E_j) > \eta.$$

Renumbering if necessary, we can arrange that

$$\omega(E_j) > 0 \quad (1 \leq j \leq m) \quad \text{and} \quad \omega(E_j) \leq 0 \quad (m < j \leq n).$$

With E the projection $E_1 + \dots + E_m$, we have

$$\omega(E) = \sum_{j=1}^m \omega(E_j) \geq \sum_{j=1}^m \lambda_j \omega(E_j) \geq \sum_{j=1}^n \lambda_j \omega(E_j) > \eta.$$

(ii) If $|\omega(F)| \leq \eta$, for every projection F in R , it follows from part (i) (applied to both ω and $-\omega$) that $|\omega(A)| \leq \eta$ for every positive operator A in the unit ball R_1 of R . Each A in R_1 has the form $A_1 - A_2 + i(A_3 - A_4)$, with $A_1, \dots, A_4$ positive elements of R_1 . Thus

$$|\omega(A)| \leq \sum_{j=1}^4 |\omega(A_j)| \leq 4\eta$$

A linear functional f on a von Neumann algebra R is described as positive if $f(A) \geq 0$ for each positive operator A in R . A positive linear functional f on R is hermitian, since each self-adjoint element of R can be expressed as the difference of two positive operators in R . Moreover, we have the Cauchy-Schwarz inequality

$$|f(T^*S)|^2 \leq f(S^*S) f(T^*T) \quad (S, T \in R),$$

which results from the fact that the equation

$$\langle S, T \rangle = f(T^*S)$$

defines a positive semi-definite inner product $\langle \cdot, \cdot \rangle$ on R .

Lemma 3.5. A norm continuous linear functional ω on a von Neumann algebra R is positive if and only if $\omega(E) \geq 0$ for each projection E in R .

Proof. A projection is a positive operator; and, by the spectral theorem, each positive operator in R can be

approximated in the norm topology by a finite linear combination $\sum \lambda_j E_j$ of projections in R , with each λ_j a positive real number.

4. Complete additivity and ultraweak continuity

A norm continuous linear functional ω on a von Neumann algebra R is completely additive if

$$\omega(\sum E_\alpha) = \sum \omega(E_\alpha),$$

for every orthogonal family (E_α) of projections in R . Since the sum $\sum E_\alpha$ converges in the ultraweak topology, it is apparent that ultraweak continuity of ω implies complete additivity. Our first objective in this section is to prove the reverse implication (Theorem 4.5). By means of this result we derive, in Theorem 4.7, a characterization of those subsets in the predual R_* which are relatively compact in the topology $\sigma(R_*, R)$.

Lemma 4.1. If ω is a hermitian and completely additive norm continuous linear functional on a von Neumann algebra R , and E is a projection in R , then R contains a subprojection F of E such that $\omega(F) \geq \omega(E)$ and the restriction $\omega|_{FRF}$ is a positive linear functional on the von Neumann algebra FRF .

Proof. Let (E_α) be a maximal orthogonal family of subprojections of E in R , such that $\omega(E_\alpha) < 0$ for each α .

With $F = E - \sum E_\alpha$, maximality of (E_α) implies that $\omega(G) \geq 0$ for each projection G in $\mathcal{FRF}$. Thus $\omega|_{\mathcal{FRF}}$ is a positive linear functional. By complete additivity of ω ,

$$\omega(F) = \omega(E) - \omega(\sum E_\alpha) = \omega(E) - \sum \omega(E_\alpha) \geq \omega(E).$$

Lemma 4.2. Suppose that a norm continuous linear functional ω on a von Neumann algebra $\mathcal{R}$ is both positive and completely additive, and E is a non-zero projection in $\mathcal{R}$. Then there is a non-zero subprojection F of E in $\mathcal{R}$, and a vector x , such that

$$|\omega(R)| \leq \|Rx\| \quad (R \in \mathcal{RF}).$$

Proof. If y is a vector such that $\|Ey\|^2 > \omega(E)$, we have

$$(\omega_y - \omega)(E) = \|Ey\|^2 - \omega(E) > 0.$$

Since $\omega_y - \omega$ is hermitian, completely additive and norm continuous, it follows from Lemma 4.1 that $\mathcal{R}$ contains a subprojection F of E with $(\omega_y - \omega)|_{\mathcal{FRF}}$ a positive linear functional and

$$(\omega_y - \omega)(F) \geq (\omega_y - \omega)(E) > 0.$$

The last inequality shows that $F \neq 0$. With R in $\mathcal{RF}$, we have $R^*R \in \mathcal{FRF}$, and so

$$0 \leq (\omega_y - \omega)(R^*R) = \|Ry\|^2 - \omega(R^*R).$$

The Cauchy-Schwarz inequality now gives

$$|\omega(R)|^2 \leq \omega(I) \omega(R^*R) \leq \omega(I) \|Ry\|^2.$$

This proves the lemma, with x an appropriate scalar multiple of y .

Lemma 4.3. If a norm continuous linear functional ω on a von Neumann algebra R is both positive and completely additive, then ω is ultraweakly continuous.

Proof. Let $(F_j)_{j \in J}$ be a maximal orthogonal family of non-zero projections in R with the following property: there is a vector x_j in R such that $|\omega(R)| \leq \|Rx_j\|$ whenever $R \in RF_j$. If $\sum F_j \neq I$, it follows from Lemma 4.2 that there is a vector x and a non-zero subprojection F of $I - \sum F_j$ in R , such that $|\omega(R)| \leq \|Rx\|$ whenever $R \in RF$. This projection F can be added to the family (F_j) , contradicting maximality. Hence $\sum_{j \in J} F_j = I$.

We want to prove that $\omega \in R_*$. Since R_* is norm closed in R^* , it suffices to show that, given any positive ϵ , there is an ω_1 in R_* satisfying $\|\omega - \omega_1\| \leq \epsilon$. By complete additivity of ω , $\sum_{j \in J} \omega(F_j)$ converges to $\omega(I)$; so there is a finite subset K of J such that

$$\omega\left(\sum_{j \in J \setminus K} F_j\right) = \sum_{j \in J \setminus K} \omega(F_j) \leq \epsilon^2 \|\omega\|^{-1}.$$

With $E = \sum_{j \in J \setminus K} F_j$, we have $\omega(E) \leq \epsilon^2 \|\omega\|^{-1}$ and $E + \sum_{j \in K} F_j = I$.

Thus $\omega = \omega_1 + \omega_2$, where

$$\omega_1(R) = \sum_{j \in K} \omega(RF_j), \quad \omega_2(R) = \omega(RE) \quad (R \in \mathcal{R}).$$

Since $|\omega_1(R)| \leq \sum_{j \in K} \|RF_j x_j\|$, ω_1 is strong operator continuous; so $\omega_1 \in \mathcal{R}_*$, by Lemma 2.6. Furthermore, by the Cauchy-Schwarz inequality,

$$\begin{aligned} |\omega_2(R)|^2 &= |\omega(RE)|^2 = |\omega((ER^*)^*E)|^2 \\ &\leq \omega(ER^*RE) \omega(E) \\ &\leq \|\omega\| \|R\|^2 \omega(E) \leq \varepsilon^2 \|R\|^2. \end{aligned}$$

Hence $\|\omega_2\| \leq \varepsilon$; that is, $\|\omega - \omega_1\| \leq \varepsilon$.

Lemma 4.4. Suppose that a, b, c are real numbers such that $a, b, ab - c^2$ are non-negative, and that $E, T \in \mathcal{B}(H)$ with E a projection and $\|T\| \leq 1$. Then the operator $aE + b(I-E) + c[ET(I-E) + (I-E)T^*E]$ is positive.

Proof. With x in H we have

$$\begin{aligned} &a\langle Ex, x \rangle + b\langle (I-E)x, x \rangle + c\langle ET(I-E)x, x \rangle + c\langle (I-E)T^*Ex, x \rangle \\ &= a\|Ex\|^2 + b\|(I-E)x\|^2 + 2c \operatorname{Re}\langle T(I-E)x, Ex \rangle \\ &= a\|Ex\|^2 + b\|(I-E)x\|^2 - 2|c| \|(I-E)x\| \|Ex\| \geq 0, \end{aligned}$$

since the quadratic form $as^2 + bt^2 - 2|c|st$ is positive semi-definite.

Theorem 4.5. A norm continuous linear functional on a

von Neumann algebra $\mathcal{R}$ is ultraweakly continuous if and only if it is completely additive.

Proof. It is apparent that ultraweak continuity implies complete additivity. In proving the converse implication we may suppose that ω is hermitian, since the hermitian and skew hermitian parts of a completely additive linear functional are again completely additive.

With ω hermitian and completely additive, and $\|\omega\| \leq 1$, let

$$(4.5.1) \quad \mu = \sup\{\omega(A) : A = A^* \in \mathcal{R}, \ 0 \leq A \leq I\},$$

so that $0 \leq \mu \leq \|\omega\| \leq 1$. Given ϵ such that $0 < \epsilon \leq 3/4$, there is a positive operator E_1 in the unit ball of $\mathcal{R}$ such that $\omega(E_1) > \mu - \epsilon$. By Lemma 3.4(i), we may suppose that E_1 is a projection; and by Lemma 4.1 we may suppose also that $\omega|_{E_1\mathcal{R}E_1}$ is a positive linear functional. Since $\omega|_{E_1\mathcal{R}E_1}$ is also completely additive, it is ultraweakly continuous by Lemma 4.3. With $E_2 = I - E_1$, ω is the sum of the four linear functionals $\omega_{j,k}$ ($j,k = 1,2$) defined by $\omega_{j,k}(R) = \omega(E_j\mathcal{R}E_k)$; and $\omega_{1,1} \in \mathcal{R}_*$. Note also that, if F is any projection in $E_2\mathcal{R}E_2$, then $\omega(F) < \epsilon$; for $E_1 + F$ is a projection in $\mathcal{R}$ and, by (4.5.1),

$$\mu \geq \omega(E_1 + F) = \omega(E_1) + \omega(F) > \mu - \epsilon + \omega(F).$$

We show next that $\|\omega_{1,2}\|$ and $\|\omega_{2,1}\|$ are small. For this,

suppose that T lies in the unit ball of $\mathcal{R}$ and define S in $\mathcal{R}$ by

$$S = (1-\epsilon)E_1 + \epsilon E_2 + \epsilon^{1/2}(1-\epsilon)^{1/2}(E_1 T E_2 + E_2 T^* E_1),$$

so that

$$I-S = \epsilon E_1 + (1-\epsilon)E_2 - \epsilon^{1/2}(1-\epsilon)^{1/2}(E_1 T E_2 + E_2 T^* E_1).$$

By Lemma 4.4, both S and $I-S$ are positive operators; so $0 \leq S \leq I$. By (4.5.1),

$$\begin{aligned} \mu &\geq \omega(S) \\ &= (1-\epsilon)\omega(E_1) + \epsilon\omega(E_2) + \epsilon^{1/2}(1-\epsilon)^{1/2}\omega(E_1 T E_2 + E_2 T^* E_1) \\ &\geq (1-\epsilon)(\mu-\epsilon) - \epsilon + 2\epsilon^{1/2}(1-\epsilon)^{1/2} \operatorname{Re} \omega(E_1 T E_2). \end{aligned}$$

Since $\mu \leq 1$ and $0 < \epsilon \leq \frac{3}{4}$, a little manipulation gives

$$\operatorname{Re} \omega_{1,2}(T) = \operatorname{Re} \omega(E_1 T E_2) \leq \frac{1}{2}(\mu+2)\epsilon^{1/2}(1-\epsilon)^{-1/2} \leq 3/\epsilon.$$

These inequalities are valid for each T in the unit ball of $\mathcal{R}$; so $\|\omega_{1,2}\| \leq 3/\epsilon$, and similarly $\|\omega_{2,1}\| \leq 3/\epsilon$. Thus

$$(4.5.2) \quad \|\omega - \omega_{1,1} - \omega_{2,2}\| = \|\omega_{1,2} + \omega_{2,1}\| \leq 6/\epsilon.$$

We show next that

$$(4.5.3) \quad \text{there is an } \omega_0 \text{ in } \mathcal{R}_* \text{ satisfying } \|\omega_{2,2} + \omega_0\| \leq 4\epsilon + 6/\epsilon.$$

This suffices to complete the proof of the theorem; for $\mathcal{R}_*$ is norm closed in $\mathcal{R}^*$, and it follows from (4.5.2) and (4.5.3) that

(for any preassigned positive ϵ) there is an element ω_* ($= \omega_{1,1} - \omega_0$) of R_* satisfying $\|\omega - \omega_*\| \leq 4\epsilon + 12\sqrt{\epsilon}$. It remains to prove (4.5.3).

The restriction $v = -\omega|_{E_2RE_2}$ is a completely additive linear functional on E_2RE_2 , such that $\|v\| \leq 1$ and $v(F) > -\epsilon$ for each projection F in E_2RE_2 . By applying to v the argument used above for ω , we deduce the existence of projections F_1 and F_2 in E_2RE_2 , with sum E_2 , satisfying the following conditions: if a linear functional $v_{j,k}$ on E_2RE_2 is defined (for $j,k = 1,2$) by $v_{j,k}(S) = v(F_jSF_k)$ ($S \in E_2RE_2$), then $v_{1,1}$ is ultraweakly continuous,

$$(4.5.4) \quad \|v - v_{1,1} - v_{2,2}\| \leq 6\sqrt{\epsilon},$$

and $v(F) < \epsilon$ for each projection F in F_2RF_2 . This last inequality, together with our earlier result in the opposite direction, shows that $|v(F)| < \epsilon$ for each projection F in F_2RF_2 . By Lemma 3.4 (ii), $\|v|_{F_2RF_2}\| \leq 4\epsilon$; whence

$$|v_{2,2}(S)| = |v(F_2SF_2)| \leq 4\epsilon\|F_2SF_2\| \leq 4\epsilon\|S\|$$

whenever $S \in E_2RE_2$, and so $\|v_{2,2}\| \leq 4\epsilon$. This, with (4.5.4), gives

$$\|v - v_{1,1}\| \leq 4\epsilon + 6\sqrt{\epsilon}.$$

With ω_0 defined by $\omega_0(R) = v_{1,1}(E_2RE_2)$ when $R \in R$, we have $\omega_0 \in R_*$ and

$$\begin{aligned}
|(\omega_0 + \omega_{2,2})(R)| &= |\nu_{1,1}(E_2 R E_2) + \omega(E_2 R E_2)| \\
&= |(\nu_{1,1} - \nu)(E_2 R E_2)| \\
&\leq \|\nu_{1,1} - \nu\| \|E_2 R E_2\| \\
&\leq (4\varepsilon + 6\sqrt{\varepsilon})\|R\|,
\end{aligned}$$

for each R in $\mathcal{R}$. This proves (4.5.3).

Theorem 4.6. If ω is a norm continuous linear functional on a von Neumann algebra $\mathcal{R}$, and the restriction $\omega|_A$ is ultraweakly continuous for each maximal abelian $*$ subalgebra A of $\mathcal{R}$, then ω is ultraweakly continuous.

Proof. With (E_α) an orthogonal family of projections in $\mathcal{R}$, there is a maximal abelian $*$ subalgebra A of $\mathcal{R}$ which contains each E_α . Since $\sum E_\alpha$ is ultraweakly convergent and $\omega|_A$ is ultraweakly continuous, we have $\omega(\sum E_\alpha) = \sum \omega(E_\alpha)$. This shows that ω is completely additive, and so ultraweakly continuous by Theorem 4.5.

Theorem 4.7. A bounded set K in the predual $\mathcal{R}_*$ of a von Neumann algebra $\mathcal{R}$ is relatively compact in $\mathcal{R}_*$, for the topology $\sigma(\mathcal{R}_*, \mathcal{R})$, if and only if the following condition is satisfied:

(*) if (E_n) is an orthogonal sequence of projections in $\mathcal{R}$, then $\omega(E_n) \rightarrow 0$, uniformly for ω in K .

Proof. Suppose first that the condition (*) is satisfied.

Since K is a bounded subset of $R_*(\subseteq R^*)$, its $\sigma(R^*, R)$ closure K_1 in R^* is $\sigma(R^*, R)$ compact. To prove that K is $\sigma(R_*, R)$ relatively compact in R_* it suffices to show that $K_1 \subseteq R_*$; for then $K \subseteq K_1 \subseteq R_*$, and K_1 is $\sigma(R_*, R)$ compact since $\sigma(R_*, R)$ is the restriction to R_* of the topology $\sigma(R^*, R)$.

With $(E_j)_{j \in J}$ an orthogonal family of projections in R , with sum E ,

$$(4.7.1) \quad \sum_{j \in J} \omega(E_j) = \omega(E) \quad (\omega \in K),$$

since $K \subseteq R_*$. If the convergence of the sum in (4.7.1) is not uniform, for ω in K , we can choose $\omega_1, \omega_2, \dots$ in K , disjoint finite subsets $J_1, J_2, \dots$ of J , and a positive δ , such that

$$\left| \sum_{j \in J_n} \omega_n(E_j) \right| \geq \delta \quad (n = 1, 2, \dots).$$

With F_n defined as $\sum_{j \in J_n} E_j$, (F_n) is an orthogonal sequence of projections in R and $|\omega_n(F_n)| \geq \delta$ for each n , contradicting (*). Thus the sum in (4.7.1) converges uniformly, for ω in K .

With ω in K_1 , there is a net (ω_α) of elements of K which converges to ω in the topology $\sigma(R^*, R)$. Since $\omega_\alpha(E) \rightarrow \omega(E)$ and $\omega_\alpha(E_j) \rightarrow \omega(E_j)$ for each j , while

$$\sum_{j \in J} \omega_\alpha(E_j) = \omega_\alpha(E)$$

for each α , it results from the uniformity of convergence just established that

$$\sum_{j \in J} \omega(E_j) = \omega(E).$$

This proves that ω is completely additive. By Theorem 4.5, $\omega \in R_*$; so $K_1 \subseteq R_*$, as required.

Conversely, suppose that K is $\sigma(R_*, R)$ relatively compact in R_* , but that the condition (*) is not satisfied. We shall in due course obtain a contradiction, so completing the proof of the theorem.

Since (*) fails, we can find a positive δ , an orthogonal sequence (E_n) of projections in R , and a sequence (ω_n) in K , such that

$$(4.7.2) \quad |\omega_n(E_n)| \geq 4\delta \quad (n = 1, 2, \dots).$$

Since $\sigma(R_*, R)$ is the weak topology on the Banach space R_* , the Eberlein-Šmulian theorem (see, for example [4 : p.430]) implies that (ω_n) has a subsequence which is $\sigma(R_*, R)$ convergent to an element ω of R_* . Passing to this subsequence (and the corresponding subsequence of (E_n)), we may suppose that (ω_n) is $\sigma(R_*, R)$ convergent to ω . Since $\omega \in R_*$ and (E_n) converges ultraweakly to 0, we have $\omega(E_n) \rightarrow 0$. By deleting a finite number of terms from the sequences (E_n) , (ω_n) , we can assume that $|\omega(E_n)| \leq \delta \quad (n = 1, 2, \dots)$.

With $\tau_n = \omega_n - \omega$, we have

$$(4.7.3) \quad \tau_n \in R_*, \quad |\tau_n(E_n)| \geq 3\delta \quad (n = 1, 2, \dots),$$

and (τ_n) is $\sigma(R_*, R)$ convergent to zero. For any bounded sequence (λ_n) of scalars and any τ in R_* , the series $\sum \lambda_n E_n$ converges ultraweakly to an element T of R , whence $\sum \lambda_n \tau(E_n)$ converges (to $\tau(T)$). It follows that

$$(4.7.4) \quad \sum_{n=1}^{\infty} |\tau(E_n)| < \infty \quad (\tau \in R_*).$$

We construct a strictly increasing sequence $(n(j))$ of positive integers with the following properties:

$$(4.7.5) \quad \left| \sum_{j=1}^{k-1} \tau_{n(k)}(E_{n(j)}) \right| < \delta \quad (k = 2, 3, \dots),$$

$$(4.7.6) \quad \sum_{j=n(k)+1}^{\infty} |\tau_{n(k)}(E_j)| < \delta \quad (k = 1, 2, \dots).$$

We start the process by taking $n(1) = 1$. Suppose now that $p > 1$, and $n(1), \dots, n(p-1)$ have been constructed in such a way that (4.7.5) holds when $2 \leq k \leq p-1$ and (4.7.6) holds when $1 \leq k \leq p-2$ (these conditions are satisfied, vacuously, in the case $p = 2$, with $n(1) = 1$). Since (τ_n) is $\sigma(R_*, R)$ convergent to zero, and the series $\sum |\tau_{n(p-1)}(E_j)|$ converges, the inequalities

$$\left| \sum_{j=1}^{p-1} \tau(E_{n(j)}) \right| < \delta, \quad \sum_{j=n}^{\infty} |\tau_{n(p-1)}(E_j)| < \delta$$

are both satisfied for all sufficiently large n . Thus (4.7.5) holds when $k = p$, and (4.7.6) holds when $k = p-1$, provided we take $n(p) (> n(p-1))$ sufficiently large. By this inductive

process we construct a sequence $(n(j))$ with the stated properties. From (4.7.6),

$$(4.7.7) \quad \sum_{j=k+1}^{\infty} |\tau_{n(k)}(E_{n(j)})| < \delta \quad (k = 1, 2, \dots).$$

With F the projection $\sum_1^{\infty} E_{n(j)}$ in R ,

$$\tau_{n(k)}(F) = \sum_{j=1}^{\infty} \tau_{n(k)}(E_{n(j)}) \quad (k = 1, 2, \dots),$$

since $\tau_{n(k)} \in R_*$. It now follows from (4.7.3), (4.7.5) and (4.7.7) that $|\tau_{n(k)}(F)| \geq \delta$ ($k = 1, 2, \dots$), contradicting our assertion that (τ_n) is $\sigma(R_*, R)$ convergent to zero. This completes the proof of the theorem.

5. The center-valued trace

In this section we prove the existence and main properties of the center-valued trace in a finite von Neumann algebra (Theorem 5.4 (ii)). Before embarking on the series of lemmas which leads up to the proof of this result, we review certain properties of von Neumann algebras which will be required.

With E and F projections in a von Neumann algebra R , we write $E \sim F$, and say that E is equivalent to F , if there is an element V in R (necessarily a partial isometry) such that $V^*V = E$, $VV^* = F$. It is readily verified that $\sim$ is indeed an equivalence relation, and that $QE \sim QF$ if $E \sim F$ and Q is a central projection in R . We write $E \precsim F$ if

$E \sim E_1 \leq F$, for some projection E_1 in R ; and $E \prec F$ means that $E \lesssim F$ but E, F are inequivalent. It can be shown that, if both $E \lesssim F$ and $F \lesssim E$, then $E \sim F$; so that $\lesssim$ induces a partial order on the equivalence classes of projections. The "comparison theorem" asserts that, if E, F are projections in R , then there exist central projections P, Q and R in R , such that $P + Q + R = I$, with the following property: if G is a non-zero central projection in R , then $GE \sim GF$ if $G \leq P$, $GE \prec GF$ if $G \leq Q$, and $GF \prec GE$ if $G \leq R$. One verifies easily that the relations $\sim$ and $\lesssim$ are completely additive, in the following sense: if (E_α) and (F_α) are orthogonal families of projections in R , indexed by the same set, and $E_\alpha \sim F_\alpha$ (respectively, $E_\alpha \lesssim F_\alpha$) for each α , then $\sum E_\alpha \sim \sum F_\alpha$ (respectively, $\sum E_\alpha \lesssim \sum F_\alpha$). These results can be found, for example, in [3 : pp.213-218] or [11 : pp.35-40].

With (E_α) a family of projections in $\mathcal{B}(H)$, there is a smallest projection $\bigvee E_\alpha$ which dominates each E_α , and a largest projection $\bigwedge E_\alpha$ dominated by each E_α . The range of $\bigvee E_\alpha$ is the closed subspace generated by $\bigcup E_\alpha(H)$, while $\bigwedge E_\alpha$ has a range $\bigcap E_\alpha(H)$. It is an immediate consequence of the double commutant theorem that a von Neumann algebra R which contains each E_α contains also $\bigvee E_\alpha$ and $\bigwedge E_\alpha$. With E and F projections in R , one can show that $E \vee F - F \sim E - E \wedge F$ (see, for example, [3 : Corollaire 1, p.216] or [11 : p.44]).

A von Neumann algebra R is said to be finite if no

projection in R , except I , is equivalent to I . It follows easily from the comparison theorem that, if E, E_1, F, F_1 are projections in a finite von Neumann algebra R , and $E_1 \leq E$, $F_1 \leq F$, $E \lesssim F$ and $F_1 \lesssim E_1$, then $E - E_1 \lesssim F - F_1$.

If R is a von Neumann algebra with unitary group U , and $A \in R$, we denote by $\kappa_R(A)$ the norm closed convex hull of the set $\{U^*AU : U \in U\}$. It is known that $\kappa_R(A)$ meets the center of R (see, for example, [3 : pp.251-253] or [11 : pp.92-98]).

Lemma 5.1. If E, F and $E_1, E_2, \dots$ are projections in a finite von Neumann algebra R , such that $E_k \lesssim F$, $E_k \leq E_{k+1}$ ($k = 1, 2, \dots$) and $\bigvee E_k = E$, then $E \lesssim F$.

Proof. We construct inductively an orthogonal sequence $F_1, F_2, \dots$ of subprojections of F such that

$$E_1 \sim F_1, \quad E_k - E_{k-1} \sim F_k \quad (k > 1).$$

Since $E_1 \lesssim F$, we can choose a suitable F_1 , so the process starts. Suppose now that $F_1, \dots, F_r$ have been constructed, with the required properties. Since $E_r \leq E_{r+1} \lesssim F$ and

$$E_r = E_1 + (E_2 - E_1) + \dots + (E_r - E_{r-1}) \sim F_1 + F_2 + \dots + F_r \leq F,$$

we have $E_{r+1} - E_r \lesssim F - (F_1 + F_2 + \dots + F_r)$, by the finiteness of R . Hence there is a subprojection F_{r+1} of $F - (F_1 + F_2 + \dots + F_r)$, such that $E_{r+1} - E_r \sim F_{r+1}$; and the inductive construction of the sequence (F_k) proceeds. Finally,

$$E = E_1 + \sum_2^{\infty} (E_k - E_{k-1}) \sim \sum_1^{\infty} F_k \leq F.$$

If R is a von Neumann algebra with unitary group U and predual R_* , we can associate with U in U an isometric linear mapping L_U , from R_* onto R_* , by defining

$$(L_U \omega)(A) = \omega(U^*AU) \quad (\omega \in R_*, A \in R, U \in U).$$

When $\omega \in R_*$, we shall denote by Q_ω the norm closed convex hull in R_* of the set $K_\omega = \{L_U \omega : U \in U\}$.

Lemma 5.2. If (E_n) is an orthogonal sequence of projections in a finite von Neumann algebra R , and $\omega \in R_*$, then $\tau(E_n) \rightarrow 0$, uniformly for τ in Q_ω .

Proof. It suffices to prove that $\tau(E_n) \rightarrow 0$ uniformly for τ in K_ω . We suppose this is not the case, and in due course obtain a contradiction. There is a positive δ , a subsequence (F_n) of (E_n) , and a sequence (τ_n) in K_ω , such that $|\tau_n(F_n)| \geq \delta$ for each n . With U_n a unitary in R such that $\tau_n = L_{U_n} \omega$, and G_n the projection $U_n^* F_n U_n$, we have

$$(5.2.1) \quad G_n \in R, \quad G_n \sim F_n, \quad |\omega(G_n)| \geq \delta.$$

We define

$$\begin{aligned} P_{m,n} &= V\{G_n, G_{n+1}, \dots, G_m\} \quad (1 \leq n \leq m), \\ P_n &= V\{G_j : j \geq n\} = V\{P_{m,n} : m \geq n\}, \\ P &= \bigwedge \{P_n : n \geq 1\}. \end{aligned}$$

Then $P_{n,n} \leq P_{n+1,n} \leq P_{n+2,n} \leq \dots$, and we assert that

$$(5.2.2) \quad P_{m,n} \lesssim F_n + F_{n+1} + \dots + F_m \quad (m \geq n).$$

Since $P_{n,n} = G_n \sim F_n$, (5.2.2) is satisfied when $m = n$. If $r \geq n$, and (5.2.2) has been proved when $m = r$, we have

$$\begin{aligned} P_{r,n} &\lesssim F_n + F_{n+1} + \dots + F_r \\ G_{r+1} \vee^P P_{r,n} - P_{r,n} &\sim G_{r+1} - G_{r+1} \wedge^P P_{r,n} \leq G_{r+1} \sim F_{r+1}; \end{aligned}$$

and addition yields

$$G_{r+1} \vee^P P_{r,n} \lesssim F_n + F_{n+1} + \dots + F_{r+1}.$$

Since $G_{r+1} \vee^P P_{r,n} = P_{r+1,n}$, (5.2.2) holds also when $m = r+1$.

From (5.2.2), $P_{m,n} \lesssim \sum_n^\infty F_j$ ($m \geq n$). By Lemma 5.1, $P_n \lesssim \sum_n^\infty F_j$. Finiteness of R now entails $I - \sum_n^\infty F_j \lesssim I - P_n \leq I - P$; and a further application of Lemma 5.1 gives

$$I = \bigvee \{I - \sum_n^\infty F_j : n \geq 1\} \lesssim I - P.$$

Thus $P = 0$. Since $P_1 \geq P_2 \geq P_3 \geq \dots$, $P_n \geq G_n$ and $\bigwedge \{P_n : n \geq 1\} = 0$, it follows that both (P_n) and (G_n) converge strongly (hence, also, ultraweakly) to 0. Since $\omega \in R_*$, $\omega(G_n) \rightarrow 0$, contradicting (5.2.1).

Lemma 5.3. If R is a finite von Neumann algebra and $\omega \in R_*$, then Q_ω is $\sigma(R_*, R)$ compact.

Proof. Since $\sigma(R_*, R)$ is the weak topology on the Banach space R_* , the norm closed convex set Q_ω is also $\sigma(R_*, R)$ closed. It is $\sigma(R_*, R)$ relatively compact in R_* , by Lemma 5.2 and Theorem 4.7.

Theorem 5.4. Suppose that R is a finite von Neumann algebra with center C and unitary group U .

(i) Each ω in C_* extends uniquely to an element τ of R_*^* such that $\tau(U^*AU) = \tau(A)$ whenever $A \in R$ and $U \in U$; moreover, $\tau \in R_*$, $\|\tau\| = \|\omega\|$, and τ is positive if ω is positive.

(ii) There is a unique norm continuous linear mapping T from R into C such that $T(A) = T(U^*AU)$ and $T(C) = C$ whenever $A \in R$, $U \in U$, $C \in C$. Moreover, T is norm decreasing and ultraweakly continuous, $T(CA) = CT(A)$ whenever $A \in R$ and $C \in C$, and $T(A)$ is positive whenever A is positive.

Proof. It is convenient to leave aside for the moment the questions of existence, and to deal first with uniqueness and some of the additional stated properties of τ and T . With ω a given element of C_* , suppose that $\tau \in R_*^*$, $\tau|_C = \omega$, and $\tau(U^*AU) = \tau(A)$ whenever $A \in R$ and $U \in U$. Given A in R , choose an element C in $\kappa_R(A) \cap C$; so that $\|C\| \leq \|A\|$, and C is positive if A is positive. It is clear that τ is constant on $\kappa_R(A)$, so

$$\tau(A) = \tau(C) = \omega(C), \quad |\tau(A)| \leq \|\omega\| \|C\| \leq \|\omega\| \|A\|,$$

and $\tau(A) \geq 0$ if ω and A are positive. This shows that τ is uniquely determined, that $\|\tau\| \leq \|\omega\|$ (whence $\|\tau\| = \|\omega\|$, since $\omega = \tau|C$), and that τ is positive if ω is positive. A similar argument shows that T , if it exists, is unique, norm decreasing and positive. Furthermore, if C_0 is a unitary element of C , the norm continuous linear mapping $T_0 : R \rightarrow C$, defined by $T_0(A) = C_0^*T(C_0A)$, satisfies $T_0(U^*AU) = T_0(A)$ and $T_0(C) = C$ whenever $A \in R$, $U \in U$, $C \in C$. By the uniqueness just established, $T_0 = T$; so $T(C_0A) = C_0T(A)$. This has been proved with A in R and C_0 unitary in C ; by linearity, it remains valid for all C_0 in C .

We turn now to the questions of existence and ultraweak continuity. With ω in C_* , there exists ρ in R_* such that $\rho|C = \omega$, by Theorem 2.4 (iv). We now use the notation introduced before the statement of Lemma 5.2. The isometric linear mappings $L_U : R_* \rightarrow R_*$ are $\sigma(R_*, R)$ continuous, and leave invariant the $\sigma(R_*, R)$ compact convex set Q_ρ in R_* . Since $\sigma(R_*, R)$ is the weak topology on the Banach space R_* , the Ryll-Nardzewski fixed point theorem [12, 10] asserts the existence of an element τ of Q_ρ such that $L_U\tau = \tau$ for each U in U . Hence $\tau \in R_*$ and

$$\tau(U^*AU) = (L_U\tau)(A) = \tau(A) \quad (A \in R, U \in U).$$

With C in $\mathcal{C}$,

$$(L_U \rho)(C) = \rho(U^*CU) = \rho(C) = \omega(C)$$

for each U in $\mathcal{U}$. Since τ is the norm limit of a sequence of convex combinations of $L_U \rho$'s, we have $\tau(C) = \omega(C)$; so $\omega = \tau|_{\mathcal{C}}$. This proves the existence and ultraweak continuity of τ , and so completes the proof of part (i) of the theorem.

For each ω in $\mathcal{C}_*$, we denote by $S\omega$ the unique element τ of $\mathcal{R}_*$ occurring in (i). Thus S is a bounded linear operator from $\mathcal{C}_*$ into $\mathcal{R}_*$. Since $\mathcal{R}$ and $\mathcal{C}$ can be identified with the Banach dual spaces of $\mathcal{R}_*$ and $\mathcal{C}_*$ respectively, the adjoint operator S^* corresponds to a bounded linear operator T from $\mathcal{R}$ into $\mathcal{C}$, such that $\omega(T(A)) = (S\omega)(A)$ whenever $\omega \in \mathcal{C}_*$ and $A \in \mathcal{R}$. Since the adjoint operator S^* is weak* continuous, T is ultraweakly continuous. From the properties of $S\omega (= \tau)$ stated in part (i), we have

$$\omega(T(U^*AU)) = (S\omega)(U^*AU) = (S\omega)(A) = \omega(T(A)),$$

$$\omega(T(C)) = (S\omega)(C) = \omega(C)$$

whenever $\omega \in \mathcal{C}_*$, $A \in \mathcal{R}$, $U \in \mathcal{U}$ and $C \in \mathcal{C}$. Thus $T(U^*AU) = T(A)$ and $T(C) = C$. This completes the proof of the theorem.

The mapping T described in Theorem 5.4 (ii) is called the center-valued trace on $\mathcal{R}$.

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