Lebesgue's Theorem Concerning The Differentiability of Monotone Functions

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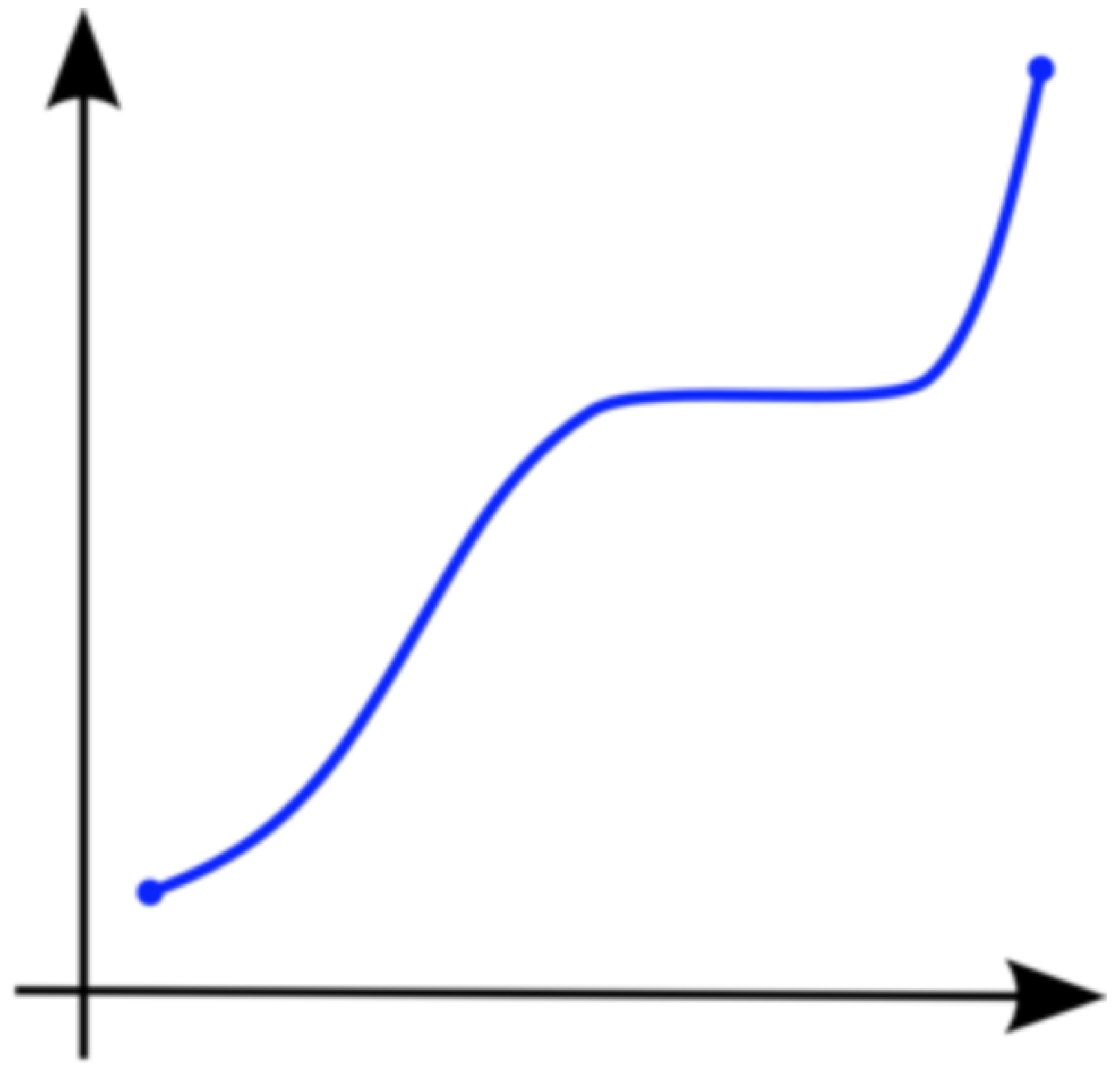
Importance

In the Royden book, we learn that mathematicians Frigyes Riesz and Béla Szőkefalvi-Nagy remark that Lebesgue's Theorem is "one of the most important in real variable theory."

Interestingly, in 1872 Karl Weierstrass presented mathematics with his continuous function on an open interval which failed to be differentiable at any point. Further pathology was revealed and there followed a period of uncertainty regarding the spread of pathology in mathematical analysis.

Lebesgue's Theorem, which was published in 1904, and its consequences, helped restore confidence in the harmony of mathematics analysis.



A Monotonic Function.

Precursor Definitions

These definitions are vital to understanding the proof.

- **Upper and Lower Derivative**

$$\overline{D}f(x) = \lim_{h\to 0}\left[\sup_{0<t\le h}\frac{f(x+t)-f(x)}{t}\right] \quad (1) \qquad \underline{D}f(x) = \lim_{h\to 0}\left[\inf_{0<t\le h}\frac{f(x+t)-f(x)}{t}\right] \quad (2)$$

- **Vitali Covering** A collection $\mathcal{F}$ of closed, bounded, nondegenerate (more than one point) intervals is said to cover a set E in the sense of Vitali provided for each point $x \in E$ and $\varepsilon > 0$, there is an interval $I \in \mathcal{F}$ that contains x and has $\ell(x) < \varepsilon$.
- **Vitali Covering Lemma** Let E be a set of finite outer measure and $\mathcal{F}$ a collection of closed, bounded intervals that covers E in the sense of Vitali. Then, for each $\varepsilon > 0$, there is a finite disjoint subcollection $\{I_k\}_{k=1}^{n}$ of $\mathcal{F}$ for which

$$m^*\left[E \sim \bigcup_{k=1}^{n} I_k\right] < \varepsilon.$$

- **Lemma 3** Let f be an increasing function on the closed, bounded interval $[a, b]$. Then, for $\alpha > 0$,

$$m^*\{x \in (a,b) | \overline{D}f(x) \ge \alpha\} \le \frac{1}{\alpha}\cdot[f(b) - f(a)]$$

The Main Results

Lebesgue proved that a monotone function must be differentiable almost everywhere

Outline of the Proof

Lebesgue's Theorem *If the function f is monotone on the open interval (a,b), then it is differentiable almost everywhere on (a,b).*

Outline of Proof

1. We assume that f is increasing, (a, b) is bounded, or that (a, b) is the union of an ascending sequence of open, bounded intervals. Continuity of Lebesgue measure is used.

2. We define the following

$$E_{\alpha,\beta} = \{x \in (a,b) \mid \overline{D}f(x) > \alpha > \beta > \underline{D}f(x)\}$$

Where α and β are rational numbers.
It is sufficient to conclude our proof by showing $E_{\alpha,\beta}$ has outer measure zero.

3. For $\varepsilon > 0$, let $E = E_{\alpha,\beta}$.
Let $\mathcal{F}$ be the collection of closed bounded intervals in an open set $\mathcal{O}$ and show that $\mathcal{F}$ is a Vitali covering of E.

4. Vitali Covering Lemma is used throughout the next few steps. First to show there is a finite disjoint subcollection $\{[c_k, d_k]\}_{k=1}^{n}$ of $\mathcal{F}$ where

$$m^*\left[E \sim \bigcup_{k=1}^{n}[c_k, d_k]\right] < \varepsilon$$

5. Then for

$$E \subseteq \mathcal{O} \subseteq (a,b) \text{ and } m(\mathcal{O}) < m^*(E) + \varepsilon$$

The union of disjoint intervals and preceding results yields

$$\sum_{k=1}^{n}[f(d_k) - f(c_k)] < \beta\left[\sum_{k=1}^{n}(d_k - c_k)\right] \le \beta\cdot m(\mathcal{O}) \le \beta\cdot[m^*(E) + \varepsilon]$$

6. We use Lemma 3 restricted to $[c_k, d_k], 1 \le k \le n$

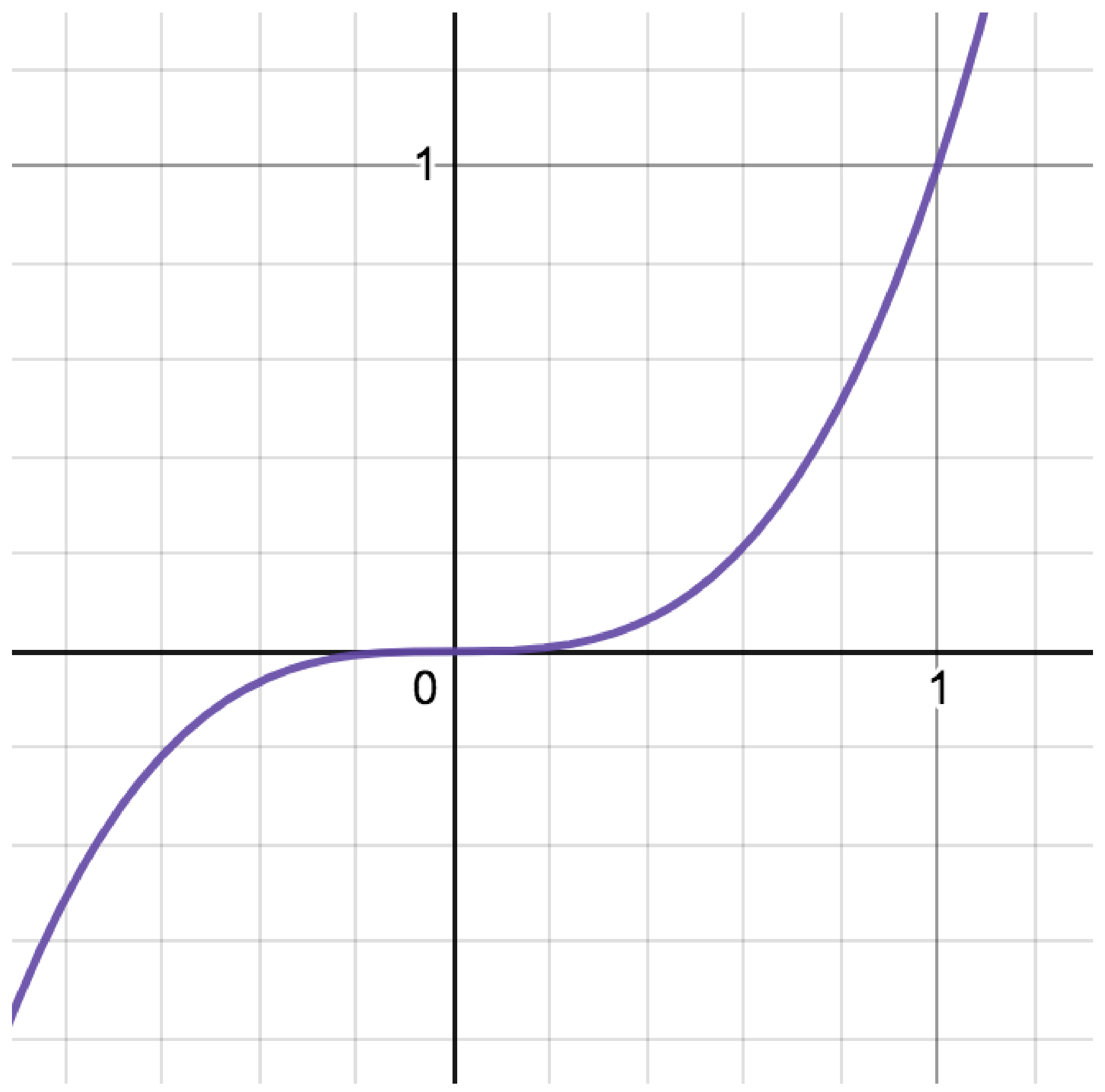
$$m^*(E \cap (c_k, d_k)) \le \frac{1}{\alpha}[f(d_k) - f(c_k)].$$

7. Combination of above steps we show

$$m^*(E) \le \frac{\beta}{\alpha}m^*(E + m^*(E) + \frac{1}{\alpha}\cdot\varepsilon + \varepsilon \text{ for all } \varepsilon > 0$$

And $m^*(E) = 0$.

Applications



This trivial example of the graph of x^3 is monotonic increasing on any open interval, therefore is differentiable almost everywhere on that interval.

The Royden book tells us that one of the best applications of Lebesgue's Theorem is the case where E is a set of measure zero contained in the open interval (a, b), there is an increasing function on (a, b) that fails to be differentiable at each point in E.

Also, Lebesgue's Theorem is used to derive the following Corollary:
Corollary 4 Let f be an increasing function on the closed, bounded interval, $[a, b]$. Then f' is integrable over $[a, b]$.

References

Nelson, G. S. (2015). ***A User-Friendly Introduction to Lebesgue Measure and Integration*****. American Mathematical Society.**

Royden, H. L., & Fitzpatrick, P. (2010). ***Real Analysis*****. Pearson Education.**

Acknowledgements

Professor Gavosto

Math 151C Class