Meaning in Mathematics

A Folkloric Account

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Overview

$\mathbb{R}$ is Dedekind complete.

$\mathbb{R}$ is well orderable.

Claim: these two sentences have different meanings.

Problem: Most accounts of meaning say that every mathematical theorem has the same meaning.

Overview

- The "direct" route: The key ideas and definitions of the folkloric account.
- The "indirect" route: How the folkloric account is an obvious tweak to the possible worlds account.

The Folkloric Account

(a) The meaning of a sentence is given by its truth conditions.

(b) Model theory is the proper tool for investigating and characterizing the truth conditions of mathematical statements.

($\star$) The meaning of a mathematical sentence is given by the set of models which make it true.

Models

We have a language $\mathcal{L} = (c_0, \ldots, c_i, P_0, \ldots P_i, f_0, \ldots f_i)$.

A model $\mathcal{M}$ consists of a domain of objects $|\mathcal{M}|$ and interpretations for the symbols in L $(c_0^{\mathcal{M}}, \ldots, c_i^{\mathcal{M}}, P_0^{\mathcal{M}}, \ldots P_i^{\mathcal{M}}, f_0^{\mathcal{M}}, \ldots f_i^{\mathcal{M}})$.

We read $\mathcal{M} \models \phi$ as $\mathcal{M}$ satisfies ϕ or ϕ is true in $\mathcal{M}$. It is defined as follows:

$\mathcal{M} \models P_i t_0 \ldots t_{n-1}$ if and only if $\langle t_0^{\mathcal{M}}, \ldots, t_{n-1}^{\mathcal{M}} \rangle \in P_i^{\mathcal{M}}$

$\mathcal{M} \models \neg\phi$ if and only if $\mathcal{M} \not\models \phi$

$\mathcal{M} \models \phi \to \psi$ if and only if $\mathcal{M} \not\models \phi$ or $\mathcal{M} \models \psi$

$\mathcal{M} \models \exists x \phi(x)$ if and only if there is some $a \in M$ such that $\mathcal{M} \models \phi(a \mapsto x)$ where $\phi(a \mapsto x)$ means that a has been substituted for every free instance of x in ϕ.

The Folkloric Account

($\star$) The meaning of a mathematical sentence is given by the set of models which make it true

Key Definition

The *content* of a sentence ϕ, relative to a set of models $\mathcal{C}$, is the set of such models which make it true, i.e., $[\![\phi]\!]_{\mathcal{C}} = \{\mathcal{M} \in \mathcal{C} : \mathcal{M} \models \phi\}$.

Content is just one aspect but it is an important one and amenable to formal analysis.

We have relativized the notion of content to handle the practice of mathematicians using different theories and model-classes.

Examples

(1) $\mathbb{R}$ is Dedekind complete.

(2) $\mathbb{R}$ is well-orderable.

Consider OF the theory of ordered fields, containing field axioms and order axioms.

There is a model $\mathcal{M} \in mod(OF)$ such that $\mathcal{M} \models (1)$ and $\mathcal{M} \not\models (2)$

Hence, $[\![(1)]\!]_{OF} \neq [\![(2)]\!]_{OF}$, i.e., not synonymous (in OF).

(3) $\mathbb{R}$ is Cauchy complete

For all $\mathcal{M} \in mod(OF)$, $\mathcal{M} \models (1) \leftrightarrow (3)$.

Hence, $[\![(1)]\!]_{OF} = [\![(3)]\!]_{OF}$, i.e., synonymous.

Examples

Let $\mathbb{K}$ be an arbitrary ordered field.

(4) $\mathbb{K}$ is Dedekind complete.

(5) $\mathbb{K}$ is Cauchy complete

There is a model $\mathcal{M} \in mod(OF)$ such that $\mathcal{M} \models (4)$ and $\mathcal{M} \not\models (5)$. Hence, $[\![(4)]\!]_{OF} \neq [\![(5)]\!]_{OF}$, i.e., not synonymous in OF.

Consider the subclass of $mod(OF)$ where all the ordered fields are Archimedian, i.e. every field is such that $\forall xy \exists n(nx > y)$, call it $\mathcal{AOF}$.

For all $\mathcal{M} \in \mathcal{AOF}$, $\mathcal{M} \models (4) \leftrightarrow (5)$. Hence, $[\![(4)]\!]_{\mathcal{AOF}} = [\![(5)]\!]_{\mathcal{AOF}}$ i.e., they are synonymous in $\mathcal{AOF}$.

Wins and Worries

Win: Doesn't trivialize in "either direction" some theorems are synonymous others are not.

Worry: I can just pick and choose model-classes to force judgements.

Win: The account captures a variety of mathematical situations.

Win: The account identifies the "content difference maker."

The “indirect route”: possible worlds account

Possible Worlds Account

(i) The meaning of a sentence is given by its truth conditions.

(ii) Truth conditions are given by alternative ways that the world could have been.

(#) The meaning of a sentence is represented by the set of worlds which make it true.

How do we know what is true in these different possible worlds?

Represent them with a model and use the model!

Possible Worlds Account

Mathematics isn't worldly, mathematics is necessarily true, so...

... every mathematical theorem is true is every possible world, so they're all synonymous. Every false mathematical statement is true at no possible world, so they're all synonymous too.

Lewis: "[The necessary truth of mathematics] is nothing but the systematic expression of my naive pre-philosophical opinion that physics could be different, but not logic or mathematics."

Just don't do this

Conclusion

- The folkloric account can be motivated both on its own grounds and as an development of the possible worlds account for mathematics specifically.
- The relativity of the folkloric account is a benefit and allows it to capture mathematical practice.
- The folkloric account is a methodology not an off-the-shelf solution.
- Further directions include developing an account of selecting model-classes (formal and philosophical) and comparing judgements across model-classes.

Questions?