Rationality and Reverse Mathematics of Nash Equilibria

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Game Theory

A *n-player continuous game* is a tuple $G = (A_1, \dots, A_n, u_1, \dots, u_n)$ where each A_i is a compact metric space denoting the i^{th} agent's set of pure strategies (i.e. an agent's possible actions) and each $u_i : A_i \rightarrow \mathbb{R}$ is a continuous function corresponding to the i^{th} agent's utility (pay-off) function.

A *Nash equilibrium* on G is a strategy set $\langle x_1, \dots, x_n \rangle$ such that for all $1 \leq i \leq n$, for all $y \in A_i$

$$u_i(x_1, \dots, x_i, \dots, x_n) \geq u_i(x_1, \dots, y, \dots, x_n)$$

Roughly speaking, a Nash equilibrium is optimal because unilaterally changing one's strategy doesn't increase one's utility.

Theorem (Glicksberg 1952): Every continuous game has a Nash equilibrium

Rationality

1. Rational agents are assumed to be utility maximizers.
2. Nash equilibria are traditionally interpreted as rationally recommended actions since they maximize utility.
3. A rationally recommended action needs to be possible for the agent to perform ("should implies can")
4. By the Church-Turing Thesis, computability is an upper bound on abilities of finite agents.

Therefore, rationally recommending equilibria implicitly requires that they are computable.

Continuous games can have non-computable Nash equilibria, which even ideal finite agents can't "find". This presents a problem for the traditional interpretation of equilibria as "rationally recommended" actions.

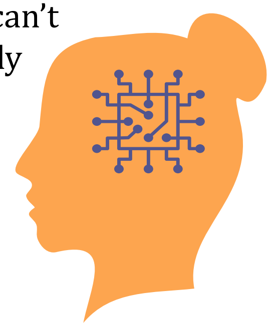
Theorem (AM): The existence of Nash equilibria for continuous games is equivalent to Weak Kőnig's Lemma.

Reverse Mathematics

Reverse mathematics investigates which axioms are necessary rather than merely sufficient for a given theorem. It demonstrates this by showing that the axioms are equivalent to the theorem in a weak base theory, which is strong enough to prove interesting equivalences without proving the statements directly. Such equivalences allow us to **quantify the strength of mathematical resources needed for a given theorem**.

Result

The existence of Nash equilibria for continuous games 'reverses to' Weak Kőnig's Lemma, (i.e., they are provably equivalent in the weak base theory). This improves on previous non-computability theorems for equilibria. Since Weak König's Lemma is not computably true, this equivalence shows that there are continuous games with non-computable Nash equilibria. Hence, although these equilibria exist, agents can't calculate them so they can't be rationally recommended.



"There is no point in prescribing a particular substantively rational solution if there exists no procedure for finding that solution with an acceptable amount of computing effort." - [Simon 1976]

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References can be found by scanning this QR Code