Tricolorability and Other Knot Invariants
UCI Math Circle

1 Introduction

1.1 Suggested Materials

While not entirely necessary, having rubber bands and string available to the class may help with visualizing knot diagrams and how we can "wiggle," "twist," and "stretch" equivalent knots into each other. For the exercises about tricolorability and knot coloring, it can be immensely helpful to have a set of colored pencils, crayons, or pens available to help in understanding these types of invariants.

1.2 Acknowledgments

This lesson plan was made possible by heavily referencing course notes from UCSD's Math 190 taught by Professor Justin Roberts and his "Knots Knotes." Images used in this lesson have either been taken from Wikimedia Commons (public domain), drawn in Inkscape, or taken from Aribi's 2013 paper "The L2-Alexander invariant detects the unknot."

2 Directions for the Instructor

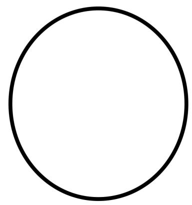
Start by working through the background information portion of the student worksheet together as a class. Explain to the students that a mathematical knot is not too different from a knot they would see in their everyday life, but that we just required the ends be glued together. When discussing knot diagrams and knot equivalence, demonstrate to the class that you can twist and pull a rubber band into all kinds of shapes without actually changing the fact that it is the same knot. Explain that a knot does not have a single diagram. For the unknot diagram examples it may help to demonstrate how we could "untangle" the diagrams so that they look like the standard circle diagram of the unknot. This can be done by either twisting a rubber band into the shape described in the diagram and then untwisting it, or by drawing a series of steps on the blackboard.

When discussing the concept of an invariant, begin by elaborating on the "is blue" backpack invariant described in the background information. Make sure the class understands that this invariant can only be used to detect when backpacks are different and that we cannot use it to conclude that a given blue backpack is the one we are searching for. Ask the class if they can think of another example of an invariant and ask what it can be used to detect and what it fails to detect. Explain to the class that in theory, we could take any two knots and try to untangle one into the other (similar to how we did for the unknot examples,) but that this process can be extremely difficult, if not impossible for us to do completely. Use this as a starting point to explain the significance of invariants as a way of detecting similarities and differences among knots, and how we can use these to enrich our understanding of knots.

It may be very helpful for the students to have pens or colored pencils available in three different colors. In the worksheet and solutions the colors used are red, green, and blue. Of course, these choices of color are arbitrary and explain to the class that they can use their own set of colors, if desired. Explain that they must use the same three colors throughout each problem and they may need to relabel some of the diagrams in the worksheet and solutions to fit their color scheme.

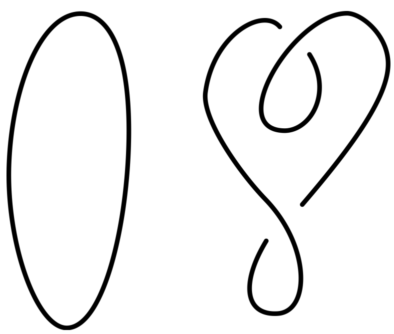
3 Background Information and Worksheet

We all have some type of experience with knots, whether it be our shoe laces or a tangled ball of yarn, knots appear all over the physical world. A mathematical knot is not too different than one you would see in your everyday life, either. A knot, as defined in mathematics, is a closed loop of string in 3-dimensional space. Notice that your shoe laces and a tangled ball of yarn are not quite mathematical knots, their ends aren't (usually) glued together, but a rubber band is a good example. In fact, the rubber band is an example of what we call the unknot (or the trivial knot.) The unknot is the most simple knot that exists, and it's not difficult to see why either. Below you will find a diagram of the unknot.



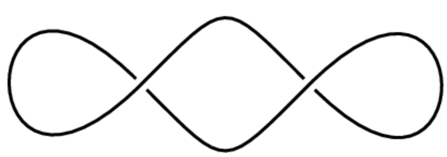
A diagram of the unknot.

A knot diagram is a way of representing a knot when projected into 2-dimensional space. It's important to understand that the same knot may be represented by dozens and dozens of different diagrams. To see why, let's look at some possible diagrams for the unknot.



Two different diagrams for the unknot.

Observe that in the diagram on the right hand side, we may take the top and bottom loops and untangle them so that the diagram looks identical to the one on the left hand side. It may be really helpful to have a rubber band or hair tie nearby, as you can easily twist the bands into configurations shown in the diagrams and see for yourself that the left and right hand sides are really equivalent! Let's look at one more example for the unknot. Try to think about how you could untwist this one to make it look like the the first diagram we saw of the unknot.



Yet another diagram of the unknot.

If you were given two different diagrams for unknown knots, how could you tell whether or not they represented the same thing? One way of doing this is to take one of the knots and wiggle, stretch, pull, and twist it until it looked like the other one. One thing we cannot do is cut the knot and glue the ends back together. If we can make the two knots look identical without cutting and re-gluing them, then they must be the same. But what if we try and try but still cannot get the two knots to look identical? It can be very difficult for us to try every possible combination of twists and pulls and it's very likely that we missed at least one. How could we know that it wasn't the combination we missed that makes the knots look identical? In this case we haven't actually proven anything; we can't say that the two knots are different, but we can't say they are the same either. We only know that what we have tried so far hasn't worked. This is where the concept of an invariant comes in.

An invariant is a property that all equivalent things must share. For example, if you know your backpack is blue and you're in a classroom full of different colored backpacks, you can quickly determine which backpack is *not* yours by getting rid of all the backpacks that aren't blue. In this case "is blue" is the invariant and any backpack that is yours, that is, any backpack equivalent to your own must have the property "is blue." Notice that this invariant *cannot* tell us when two backpacks are the same. Of course, there could be many blue backpacks in the classroom and if the only information we know about your backpack is that it's blue, then we can't select yours from the pile of all equally blue backpacks. We are now ready to learn about a knot invariant called "tricolorability."

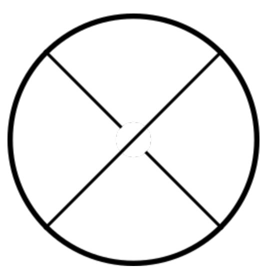
Problem Set

1. Tricolorability is a knot invariant similar to the "is blue" backpack invariant described above. A knot is tricolorable if we can color each arc in a diagram of it using three colors which satisfy the following rules:

   (i) At least two colors must be used to color the diagram,

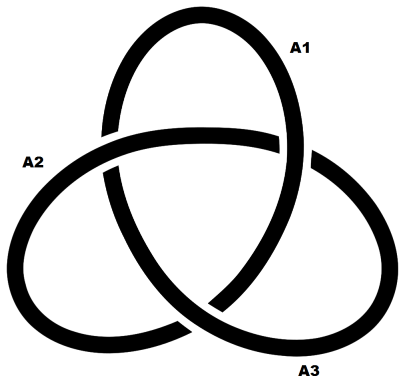
   (ii) At each crossing in the diagram, either all arcs are different colors or all arcs are the same color.

   (Don't worry if this doesn't make sense just yet, we'll do an example soon!) A "crossing" in a knot diagram is the part where one string goes over and one goes over. If we zoom in on a diagram, it looks a little something like this.

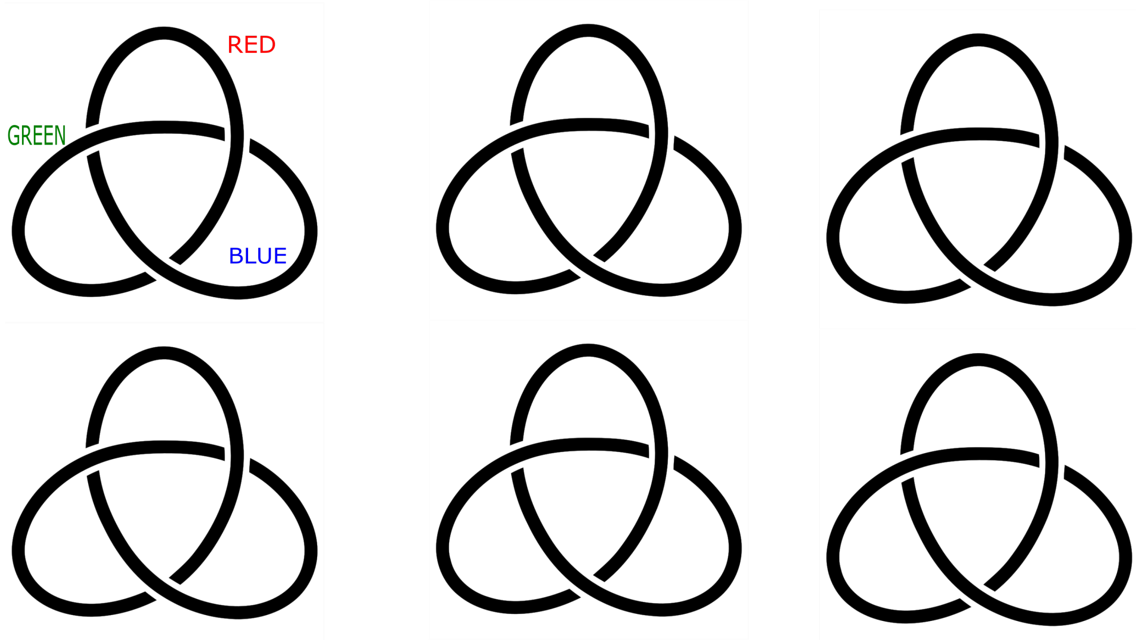
   An example of a crossing.

   Notice that a crossing has three component or "arcs." The diagonal line that goes from left to right is the "over" arc and there are two "under" arcs on its left and right sides. The coloring rules tell us that at each crossing all three arcs must be all different colors or all the same and that over the entire diagram we must use at least two colors. The diagram below is for a knot known as the trefoil. Let's show that it is tricolorable as follows. Color the arc labeled A1 with red, if you have colored pencils/pens/crayons, or write "red" above it otherwise. Color the arc labeled A2 with Green, or write "Green" over it, and finally color the arc labeled A3 with blue or write "Blue" over it.



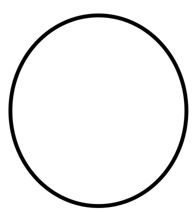
Explain why this this knot is tricolorable (explain how it satisfies all the rules of tricoloring.)

2. Look at the trefoil you colored in above. Is this the only possible way to color it so that the rules of tricolorability are satisfied? Try to produce different, yet valid, tricolorings below by coloring arcs similarly to how we did in the first problem. How many did you find? The coloring from above is included below.



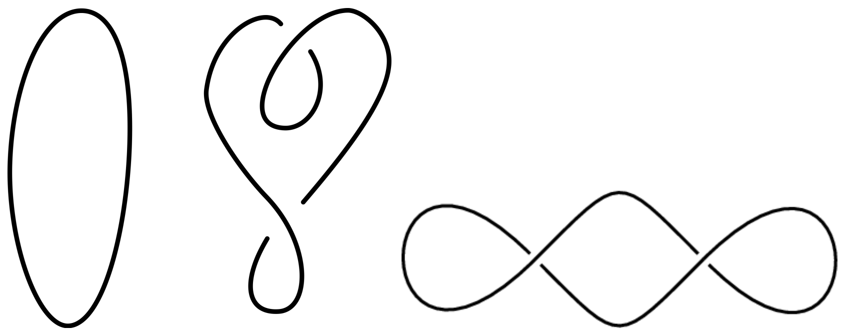
3. Give a mathematical argument for how you could find the number of tricolorings of the trefoil without having to draw every possibility. Hint: think about what rules we must satisfy for tricolorings and how the trefoil has three arcs. When we color the first arc, we have three choices for color. How many choices for color do we have when we color the second arc? When we color the third?

4. Is the unknot tricolorable? If it is, show that you can color the diagram below so that the above rules for tricoloring are satisfied. If not, explain what rule fails.

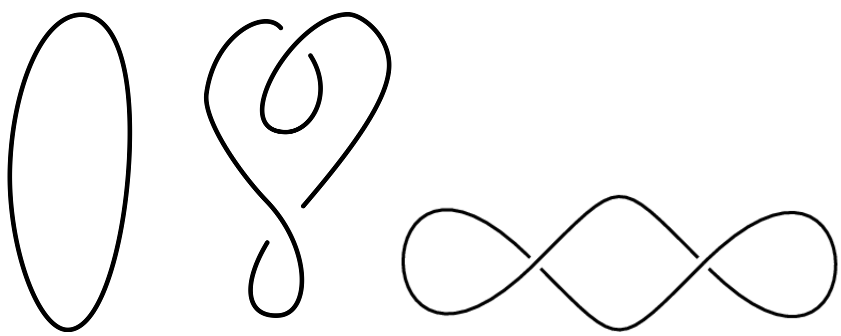


5. We showed earlier that many different diagrams can represent the same knot. Based on your answer to the previous question, are these diagrams tricolorable? Don't try coloring in

the diagrams just yet, remember that these diagrams are equivalent to the unknot and that tricolorability is a knot invariant. That is, if two knots are equivalent, they should either both be tricolorable, or both not be tricolorable.

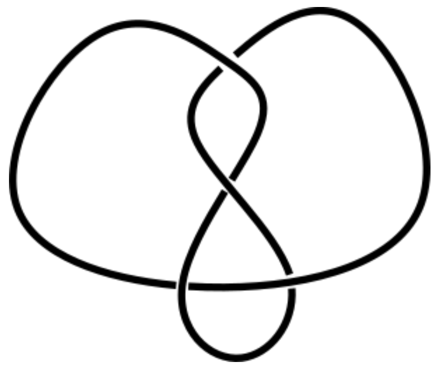


6. If you didn't know the diagrams below were equivalent to the unknot, show how you would determine whether or not they were tricolorable by coloring them.



7. We showed earlier that the trefoil was tricolorable. You should have found above that the unknot is not tricolorable. Can we use this information to determine whether or not the trefoil is equivalent to the unknot? Be sure to explain why or why not.

8. Determine whether the figure-eight knot (a diagram for it is pictured below) is tricolorable.

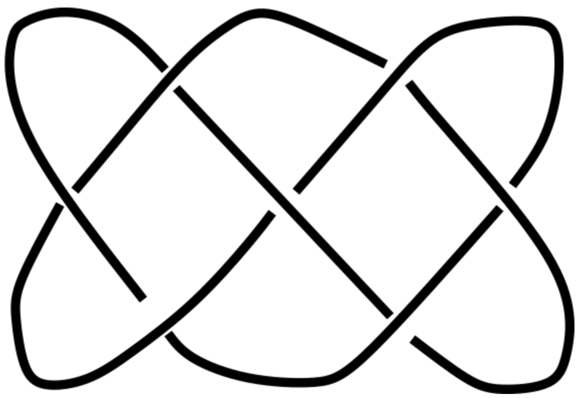


9. Use your answer to Question 8 to answer the following questions about the figure-eight knot.

   (a) What can we say about the relationship between the unknot and the figure-eight knot? Can we conclude that they are different? The same? Or do we not have enough information with just the tricolorability invariant?

   (b) What can we say about the relationship between the trefoil and the figure-eight knot? Can we conclude that they are different? The same? Or do we not have enough information with just the tricolorability invariant?

10. Determine whether the knot corresponding to the diagram below is tricolorable. (This is called the 7_4 knot.)



11. Use your answer from Question 10 to answer the following questions about the 7_4 knot.

    (a) What can we say about the relationship between the 7_4 knot and the unknot? Can we conclude that they are different? The same? Or do we not have enough information with just the tricolorability invariant?

    (b) What can we say about the relationship between the 7_4 knot and the trefoil? Can we conclude that they are different? The same? Or do we not have enough information with just the tricolorability invariant?

So far we've been working with the tricolorability invariant. This is an example of a "true-false" invariant. That is, for any given knot we have either that it is true the knot is tricolorable, or it is false. We will now look at a more general invariant called the 3-coloring number. The rules for the 3-coloring invariant are almost identical to the rules for tricolorability, except we no longer require that at least two colors be used, using just one color is sufficient. To summarize, the number of 3-colorings of a knot is the number of ways we can color its arcs with a maximum of three colors such that at each crossing either all arcs are the same color, or all are different.

12. Since the number of 3-colorings is an invariant, if two knots are equivalent would we expect them to have the same number of three colorings? If two knots have different numbers of 3-colorings what can we conclude about them? If we don't know whether two knots are equivalent, but we do know they have the same number of 3-colorings, can we conclude that they

are equivalent?

13. How many 3-colorings are there for the unknot?

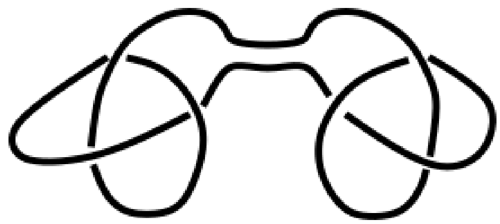
14. How many 3-colorings are there for trefoil? Hint: your solution to problem 2 is all the possible tricolorings of the trefoil. Tricolorings are valid 3-colorings, so think about what rule is different for 3-colorings to find the remaining possibilities.

15. If a knot is not tricolorable, how many 3-colorings does it have? Explain your reasoning.

16. Using your answer to Question 15 and Question 8, how many 3-colorings does the figure-eight knot have?

4 Challenge/Take-Home Problem

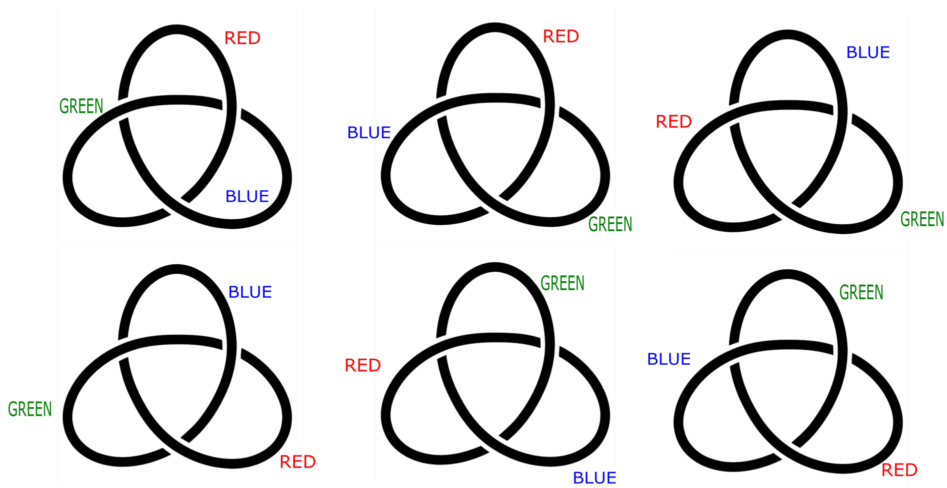
The diagram below is for a knot made up of two trefoils connected together (connecting knots like this is called their "knot sum" or "connect sum.") How many 3-colorings does this knot have? Hint: Start by trying to make a valid 3-coloring and notice how many choices of color you have for each arc. Use a counting argument similar to Problem 3 on the worksheet.



5 Solutions to Worksheet

1. Notice that we have used three colors over the diagram, so the first rule of tricolorability is satisfied. There are three crossings to examine. Starting with the top-left crossing and working counter-clockwise we see that the colors at each crossing are given by red-green-blue, green-blue-red, and blue-red-green, so the second rule of tricolorability is satisfied.

2. Here are all valid tricolorings for the colors Red-Green-Blue. Not matter what three colors you used, there should be 6 valid tricolorings for the trefoil.



3. Pick one arc on the trefoil to color. We have three choices for the color we can place on that arc. Pick another arc on the trefoil. If we wanted to color that arc the same color as the first, then notice that the third arc must also be the same color. But this violates the rule that at least two colors must be used. Then for the second arc, we have only 2 choices for color. For the third arc, we only have one choice of color, since the other two arcs completely determine what it must be. We see that $3 \times 2 \times 1 = 6$, which gives us all possible tricolorings of the trefoil.

4. The diagram given in this problem has only one arc, which means we can only place one color on the diagram. It automatically fails the first rule of tricolorability (that we must place at least two colors on the diagram) and we conclude that the unknot is not tricolorable.

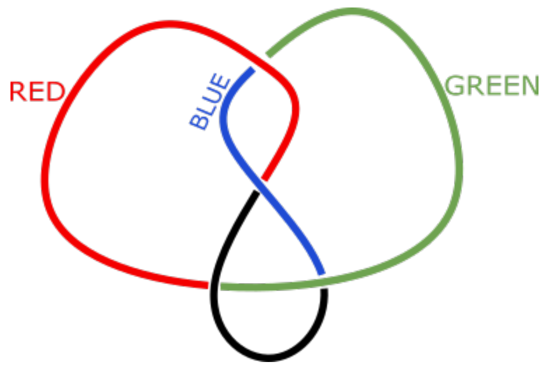
5. Since tricolorability is an invariant, we know that if two knots are the same then they must either be both tricolorable or both be not tricolorable. We argued earlier that all three diagrams represent the unknot, and since the unknot is not tricolorable, the three diagrams in this problem must also be not tricolorable.

6. Notice that in the left-most diagram there is just one arc, so we can only place one color on the diagram. It automatically fails the first condition for tricolorability. The other two diagrams have only two arcs and when we try to place more than one color on the diagrams the second tricolorability rule will always fail.

7. So far we have been able to show that the unknot is not tricolorable, but the trefoil IS tricolorable. We know that since tricolorability is a knot invariant, if two knots are equivalent they

must either both be tricolorable or both be not tricolorable. We can conclude that the trefoil and the unknot are different knots. It turns out that trying to prove the unknot and trefoil are different knots without using tricolorability is extremely difficult and gives us one example of how tricolorability can be a very powerful tool.

8. We claim that the figure-eight knot is NOT tricolorable. Notice what happens when we try to give it a valid three coloring.

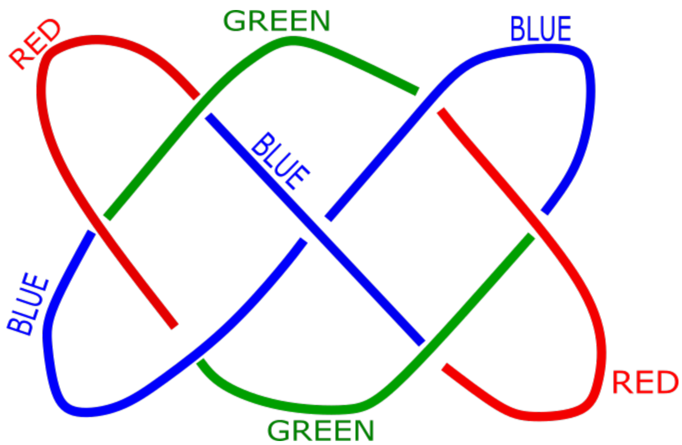


Suppose we pick red as our first color and we place it on the arc labeled "red" above. Then we suppose we pick a color different from red, green, and place it on the arc labeled "green" above. Then to satisfy the tricoloring rules, we must make the arc labeled "blue" blue. But then there is no way to color the last arc. Notice that no matter what color we put, there will be a crossing that doesn't have either all colors the same or all colors different. A similar problem arises if we pick red and then red again to color the first two arcs. To satisfy the tricoloring rules, every single arc must be red. But then only one color was used and we haven't satisfied the tricoloring rules.

9. (a) While both the unknot and the figure-eight knot are not tricolorable, we cannot conclude anything about their equivalence. If the only thing I know about your backpack is that it isn't blue, I still can't select yours from a pile of not-blue backpacks.

   (b) Since the trefoil is tricolorable but the figure-eight knot is not, we can conclude that they must be different, since any two equivalent knots must either both be tricolorable or both be not tricolorable.

10. To determine whether or not the 7_4 knot is tricolorable, we start coloring arcs and see if we run into any problems. Here's one way of tricoloring the knot. Since we have found at least one valid tricoloring, we may conclude that the 7_4 knot is tricolorable.

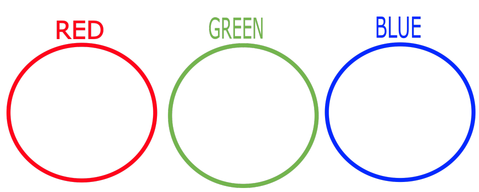


11. (a) Since tricolorability is a knot invariant, if two knots are equivalent they both must be tricolorable. We know that the unknot is not tricolorable but have shown that the 7_4 is tricolorable. We can conclude that the 7_4 knot is not equivalent to the unknot.

    (b) We know that both the 7_4 knot and the trefoil are tricolorable. Tricolorability does not provide us enough information to conclude that these knots are the same or different. This is similar to the "is blue" backpack invariant described earlier. If the only thing I know about your backpack is that it's blue, I can't select it from a pile of blue backpacks.

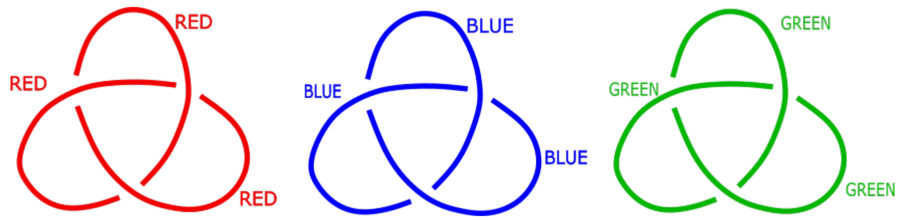
12. Since the number of 3-colorings is a knot invariant, if two knots are equivalent they must have the same 3-coloring number. If we are two given knots and each one has a different number of 3-colorings, then we can conclude that the knots are different. However, if we are given two unknown knots and we show that both have the same number of three colorings, we can't say anything about the relationship between the two knots. This is just like the example with the blue backpack. If the only thing I know about your backpack is that it's blue, I can't select yours from a pile of blue backpacks.

13. Using the first diagram shown of the unknot (the one that looks like a circle), we see that there is just one arc. Then, for each of the three colors there is just one way place it on the unknot's diagram. This gives 3 total 3-colorings. They are shown below.



14. There are 9 total 3-colorings of the trefoil. Notice that we have 2 "degrees of freedom" when picking colors. When we pick an arc to color it, was have 3 choices for color. When we pick a second arc to color, we still have 3 choices for color. However, we only have 1 choice for color for the last arc, because we must satisfy the 3-coloring rules. Once we have determined 2 colors there is only one possibility for the third. We see that $3 \times 3 \times 1 = 9$. The 9 total

colorings are seen as the 6 colorings shown in Question 2 plus the additional “trivial” colorings shown below.

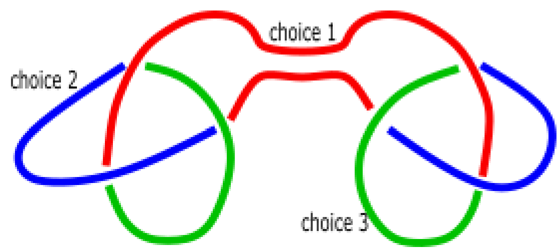


15. If a knot is not tricolorable, then we know that there does not exist a way of placing at least 2 colors on its diagram such that each crossing either has all different colors or all the same color. Then, if we take just 1 color and place it on every arc that is a valid 3-coloring. Since we have 3 choices for color this gives a total of 3 3-colorings.

16. In Question 8 we found that the figure-eight knot is not tricolorable. In Question 15 we saw that if a knot is not tricolorable then it has 3 total 3-colorings. Hence, the figure-eight knot has 3 total 3-colorings.

6 Solution to Challenge/Take-Home Problem

Let's use the following diagram to help us understand this problem.



We try to build a valid 3-coloring and keep track of how many choices we have at each arc. Look at the arc labeled "choice 1." We have three possible colors to pick. Look at the arc labeled "choice 2." We also have three possible colors to pick for this arc. Notice that for the arcs between choice 2 and choice 3 we have no choice of color because the colors for choices 1 and 2 completely determine them. Notice for the arc labeled "choice 3", however, we have three choices of color (to see this, try it out for yourself and notice that no matter what color you pick you can still get a valid 3-coloring.) Once we have picked a color for choice 3, the remaining arc's color is determined and we have a valid three coloring. We had three choices to make, each with three possible colors to pick from, so we see there are $3 \times 3 \times 3 = 27$ possible 3-colorings of this knot.