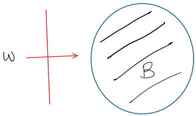
The fixed angle inverse scattering problem

Rakesh*
University of Delaware

November 21, 2024

*Joint work with Lauri Oksanen and Mikko Salo
Supported by NSF DMS 1615616, 1908391

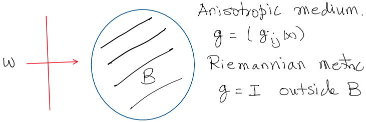
The Riemannian metric problem



Anisotropic medium.
$g = (g_{ij}(x))$
Riemannian metric
$g = I$ outside B

-

The Riemannian metric problem

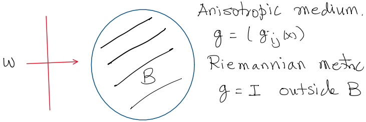


-

- ω a unit vector in $\mathbb{R}^n$, $U_\omega(x,t)$ the solution of the IVP

$$(\partial_t^2 - \Delta_g)U_\omega = 0, \qquad \text{on } \mathbb{R}^n \times \mathbb{R},$$
$$U_\omega(x,t) = H(t - x\cdot\omega) \qquad \text{on } \mathbb{R}^n \times (-\infty,-1).$$

The Riemannian metric problem

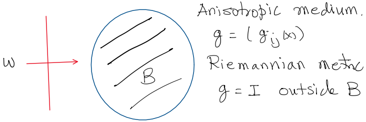


-
- ω a unit vector in $\mathbb{R}^n$, $U_\omega(x,t)$ the solution of the IVP

$$\begin{aligned} (\partial_t^2 - \Delta_g)U_\omega &= 0, && \text{on } \mathbb{R}^n \times \mathbb{R}, \\ U_\omega(x,t) &= H(t - x\cdot\omega) && \text{on } \mathbb{R}^n \times (-\infty,-1). \end{aligned}$$

- Here $\Delta_g = |g|^{-1/2}\partial_i\left(|g|^{1/2}g^{ij}\partial_j\right)$. Here $(g^{ij}) = (g_{ij})^{-1}$ and $|g| = \det g$.

The Riemannian metric problem

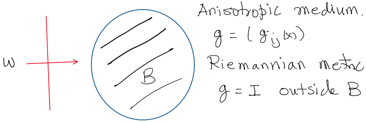


-
- ω a unit vector in $\mathbb{R}^n$, $U_\omega(x,t)$ the solution of the IVP

$$\begin{aligned} (\partial_t^2 - \Delta_g)U_\omega &= 0, && \text{on } \mathbb{R}^n \times \mathbb{R}, \\ U_\omega(x,t) &= H(t - x\cdot\omega) && \text{on } \mathbb{R}^n \times (-\infty,-1). \end{aligned}$$

- Here $\Delta_g = |g|^{-1/2}\partial_i\left(|g|^{1/2}g^{ij}\partial_j\right)$. Here $(g^{ij}) = (g_{ij})^{-1}$ and $|g| = \det g$.
- Similar problem for $\rho(x)\partial_t^2 - \Delta$ where $1/\sqrt{\rho}$ is medium velocity (a positive smooth function) with $\rho(x) = 1$ outside B.

The Riemannian metric problem



-
- ω a unit vector in $\mathbb{R}^n$, $U_\omega(x,t)$ the solution of the IVP

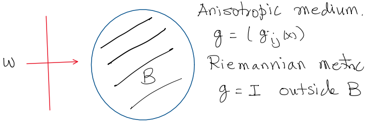
$$
\begin{aligned}
(\partial_t^2 - \Delta_g) U_\omega &= 0, \qquad &&\text{on } \mathbb{R}^n \times \mathbb{R}, \\
U_\omega(x,t) &= H(t - x \cdot \omega) \qquad &&\text{on } \mathbb{R}^n \times (-\infty, -1).
\end{aligned}
$$

- Here $\Delta_g = |g|^{-1/2} \partial_i \left(|g|^{1/2} g^{ij} \partial_j \right)$. Here $(g^{ij}) = (g_{ij})^{-1}$ and $|g| = \det g$.
- Similar problem for $\rho(x)\partial_t^2 - \Delta$ where $1/\sqrt{\rho}$ is medium velocity (a positive smooth function) with $\rho(x) = 1$ outside B.
- $\Omega = \{e_i\} \cup \{(e_i + e_j)/\sqrt{2}\}_{i \neq j}$; $n(n+1)/2$ incoming wave dir.

$$
\mathcal{F}_T : g \to [U_\omega|_{\partial B \times (-\infty, T)}]_{\omega \in \Omega} \qquad\qquad \text{(forward map)}.
$$

Question: Is $\mathcal{F}_T$ injective?

The Riemannian metric problem



-

- ω a unit vector in $\mathbb{R}^n$, $U_\omega(x,t)$ the solution of the IVP

$$\begin{aligned} (\partial_t^2 - \Delta_g)U_\omega &= 0, \qquad \text{on } \mathbb{R}^n \times \mathbb{R}, \\ U_\omega(x,t) &= H(t - x\cdot\omega) \qquad \text{on } \mathbb{R}^n \times (-\infty,-1). \end{aligned}$$

- Here $\Delta_g = |g|^{-1/2}\partial_i\left(|g|^{1/2}g^{ij}\partial_j\right)$. Here $(g^{ij}) = (g_{ij})^{-1}$ and $|g| = \det g$.
- Similar problem for $\rho(x)\partial_t^2 - \Delta$ where $1/\sqrt{\rho}$ is medium velocity (a positive smooth function) with $\rho(x) = 1$ outside B.
- $\Omega = \{e_i\} \cup \{(e_i + e_j)/\sqrt{2}\}_{i\neq j}$; $n(n+1)/2$ incoming wave dir.

$$\mathcal{F}_T : g \to [U_\omega|_{\partial B\times(-\infty,T)}]_{\omega\in\Omega} \qquad \text{(forward map)}.$$

Question: Is $\mathcal{F}_T$ injective?

- $\mathcal{F}_T$ nonlinear. Formally determined problem. If $\psi : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism with $\psi(x) = x$ outside B then $\mathcal{F}(g) = \mathcal{F}(\psi^*(g))$.

The results

Theorem [Oksanen, R, Salo 2024]. Suppose g is a Riemannian metric on $\mathbb{R}^n$ with $g = g_{Eucl}$ outside B. If $\mathcal{F}_T(g) = \mathcal{F}_T(g_{Eucl})$ and T large enough, there is a diffeomorphism $\psi : \mathbb{R}^n \to \mathbb{R}^n$ with $\psi(x) = x$ outside B such that $\psi^*(g) = g_{Eucl}$.

The results

Theorem [Oksanen, R, Salo 2024]. Suppose g is a Riemannian metric on $\mathbb{R}^n$ with $g = g_{Eucl}$ outside B. If $\mathcal{F}_T(g) = \mathcal{F}_T(g_{Eucl})$ and T large enough, there is a diffeomorphism $\psi : \mathbb{R}^n \to \mathbb{R}^n$ with $\psi(x) = x$ outside B such that $\psi^*(g) = g_{Eucl}$.

- Similar results for $\rho\partial_t^2 - \Delta$ for positive function $\rho(x)$ with $\rho = 1$ outside B. Need only one direction ω.

The results

Theorem [Oksanen, R, Salo 2024]. Suppose g is a Riemannian metric on $\mathbb{R}^n$ with $g = g_{Eucl}$ outside B. If $\mathcal{F}_T(g) = \mathcal{F}_T(g_{Eucl})$ and T large enough, there is a diffeomorphism $\psi : \mathbb{R}^n \to \mathbb{R}^n$ with $\psi(x) = x$ outside B such that $\psi^*(g) = g_{Eucl}$.

- Similar results for $\rho\partial_t^2 - \Delta$ for positive function $\rho(x)$ with $\rho = 1$ outside B. Need only one direction ω.
- Similar result for $\Box_h$ where $h(x,t)$ is a Lorentzian metric (details in later slides). Lorentzian case requires considerable more work than the Riemannian case.

The results

Theorem [Oksanen, R, Salo 2024]. Suppose g is a Riemannian metric on $\mathbb{R}^n$ with $g = g_{Eucl}$ outside B. If $\mathcal{F}_T(g) = \mathcal{F}_T(g_{Eucl})$ and T large enough, there is a diffeomorphism $\psi : \mathbb{R}^n \to \mathbb{R}^n$ with $\psi(x) = x$ outside B such that $\psi^*(g) = g_{Eucl}$.

- Similar results for $\rho\partial_t^2 - \Delta$ for positive function $\rho(x)$ with $\rho = 1$ outside B. Need only one direction ω.
- Similar result for $\Box_h$ where $h(x,t)$ is a Lorentzian metric (details in later slides). Lorentzian case requires considerable more work than the Riemannian case.
- Our result a small step towards the injectivity of $\mathcal{F}$. **Does not** assume absence of focal points for boundary normal flow of g (associated with U_ω).

The results

Theorem [Oksanen, R, Salo 2024]. Suppose g is a Riemannian metric on $\mathbb{R}^n$ with $g = g_{Eucl}$ outside B. If $\mathcal{F}_T(g) = \mathcal{F}_T(g_{Eucl})$ and T large enough, there is a diffeomorphism $\psi : \mathbb{R}^n \to \mathbb{R}^n$ with $\psi(x) = x$ outside B such that $\psi^*(g) = g_{Eucl}$.

- Similar results for $\rho\partial_t^2 - \Delta$ for positive function $\rho(x)$ with $\rho = 1$ outside B. Need only one direction ω.
- Similar result for $\Box_h$ where $h(x, t)$ is a Lorentzian metric (details in later slides). Lorentzian case requires considerable more work than the Riemannian case.
- Our result a small step towards the injectivity of $\mathcal{F}$. **Does not** assume absence of focal points for boundary normal flow of g (associated with U_ω).
- Few results for (multi-dim) formally determined inverse problems for hyperbolic operators with **non-constant velocity**.

The results

Theorem [Oksanen, R, Salo 2024]. Suppose g is a Riemannian metric on $\mathbb{R}^n$ with $g = g_{Eucl}$ outside B. If $\mathcal{F}_T(g) = \mathcal{F}_T(g_{Eucl})$ and T large enough, there is a diffeomorphism $\psi : \mathbb{R}^n \to \mathbb{R}^n$ with $\psi(x) = x$ outside B such that $\psi^*(g) = g_{Eucl}$.

- Similar results for $\rho\partial_t^2 - \Delta$ for positive function $\rho(x)$ with $\rho = 1$ outside B. Need only one direction ω.
- Similar result for $\Box_h$ where $h(x, t)$ is a Lorentzian metric (details in later slides). Lorentzian case requires considerable more work than the Riemannian case.
- Our result a small step towards the injectivity of $\mathcal{F}$. **Does not** assume absence of focal points for boundary normal flow of g (associated with U_ω).
- Few results for (multi-dim) formally determined inverse problems for hyperbolic operators with **non-constant velocity**.
- There is no such **partial** result for the anisotropic (elliptic) Impedance Tomography problem (operator Δ_g, data D-N map) if $n \geq 3$, even though that is an overdetermined problem.

Partial history

- For $\rho(z)\partial_t^2 - \partial_z^2$, injectivity, stability etc. of $\mathcal{F}_T$. Gelfand-Levitan, Krein, $\cdots$.

Partial history

- For $\rho(z)\partial_t^2 - \partial_z^2$, injectivity, stability etc. of $\mathcal{F}_T$. Gelfand-Levitan, Krein, $\cdots$.
- For $\rho\partial_t^2 - \Delta$, [Romanov, 2002] showed $\mathcal{F}_T$ is injective if ρ close to 1.
  The assumption guarantees boundary normal geodesic flow (associated with U_ω) has no focal points. Even with this simplifying assumption, result is non-trivial.

Partial history

- For $\rho(z)\partial_t^2 - \partial_z^2$, injectivity, stability etc. of $\mathcal{F}_T$. Gelfand-Levitan, Krein, $\cdots$.
- For $\rho\partial_t^2 - \Delta$, [Romanov, 2002] showed $\mathcal{F}_T$ is injective if ρ close to 1.
  The assumption guarantees boundary normal geodesic flow (associated with U_ω) has no focal points. Even with this simplifying assumption, result is non-trivial.
- For $\rho\partial_t^2 - \Delta$, [Romanov, 2002] showed, with $\omega = e_z$, $\mathcal{F}$ is injective over a small interval z if $\rho(x, y, z)$ is analytic in x, y. Uses ideas from 1-dim case.

Partial history

- For $\rho(z)\partial_t^2 - \partial_z^2$, injectivity, stability etc. of $\mathcal{F}_T$. Gelfand-Levitan, Krein, $\cdots$.
- For $\rho\partial_t^2 - \Delta$, [Romanov, 2002] showed $\mathcal{F}_T$ is injective if ρ close to 1.
  The assumption guarantees boundary normal geodesic flow (associated with U_ω) has no focal points. Even with this simplifying assumption, result is non-trivial.
- For $\rho\partial_t^2 - \Delta$, [Romanov, 2002] showed, with $\omega = e_z$, $\mathcal{F}$ is injective over a small interval z if $\rho(x, y, z)$ is analytic in x, y. Uses ideas from 1-dim case.
- For $\rho\partial_t^2 - \Delta$ (and for $\Box + q$), but with a different type of source

  $$U(\cdot, 0) = f(\cdot), \quad U_t(\cdot, 0) = 0, \qquad \text{on } B \times [0, T]$$

  where f always positive. [Bukhgeim and Klibanov, 1981] proved injectivity if there is a convex function for the Riemannian metric ρI on $\mathbb{R}^n$.

Partial history

- For $\rho(z)\partial_t^2 - \partial_z^2$, injectivity, stability etc. of $\mathcal{F}_T$. Gelfand-Levitan, Krein, $\cdots$.
- For $\rho\partial_t^2 - \Delta$, [Romanov, 2002] showed $\mathcal{F}_T$ is injective if ρ close to 1.
  The assumption guarantees boundary normal geodesic flow (associated with U_ω) has no focal points. Even with this simplifying assumption, result is non-trivial.
- For $\rho\partial_t^2 - \Delta$, [Romanov, 2002] showed, with $\omega = e_z$, $\mathcal{F}$ is injective over a small interval z if $\rho(x, y, z)$ is analytic in x, y. Uses ideas from 1-dim case.
- For $\rho\partial_t^2 - \Delta$ (and for $\Box + q$), but with a different type of source

  $$U(\cdot, 0) = f(\cdot), \quad U_t(\cdot, 0) = 0, \qquad \text{on } B \times [0, T]$$

  where f always positive. [Bukhgeim and Klibanov, 1981] proved injectivity if there is a convex function for the Riemannian metric ρI on $\mathbb{R}^n$.
- For $\partial_t^2 - \Delta_g$, for "overdetermined" problem of recovering g from hyperbolic D-N data, Belishev and Kurylev (BC method) reconstructed (including uniqueness) the manifold and Riemannian metric g;

Partial history

- For $\rho(z)\partial_t^2 - \partial_z^2$, injectivity, stability etc. of $\mathcal{F}_T$. Gelfand-Levitan, Krein, $\cdots$.
- For $\rho\partial_t^2 - \Delta$, [Romanov, 2002] showed $\mathcal{F}_T$ is injective if ρ close to 1.
  The assumption guarantees boundary normal geodesic flow (associated with U_ω) has no focal points. Even with this simplifying assumption, result is non-trivial.
- For $\rho\partial_t^2 - \Delta$, [Romanov, 2002] showed, with $\omega = e_z$, $\mathcal{F}$ is injective over a small interval z if $\rho(x, y, z)$ is analytic in x, y. Uses ideas from 1-dim case.
- For $\rho\partial_t^2 - \Delta$ (and for $\Box + q$), but with a different type of source

  $$U(\cdot, 0) = f(\cdot), \quad U_t(\cdot, 0) = 0, \qquad \text{on } B \times [0, T]$$

  where f always positive. [Bukhgeim and Klibanov, 1981] proved injectivity if there is a convex function for the Riemannian metric ρI on $\mathbb{R}^n$.
- For $\partial_t^2 - \Delta_g$, for "overdetermined" problem of recovering g from hyperbolic D-N data, Belishev and Kurylev (BC method) reconstructed (including uniqueness) the manifold and Riemannian metric g;
  [Stefanov and Uhlmann 2005] proved stability near simple metrics.

Partial history

- For $\rho(z)\partial_t^2 - \partial_z^2$, injectivity, stability etc. of $\mathcal{F}_T$. Gelfand-Levitan, Krein, $\cdots$.
- For $\rho\partial_t^2 - \Delta$, [Romanov, 2002] showed $\mathcal{F}_T$ is injective if ρ close to 1.
  The assumption guarantees boundary normal geodesic flow (associated with U_ω) has no focal points. Even with this simplifying assumption, result is non-trivial.
- For $\rho\partial_t^2 - \Delta$, [Romanov, 2002] showed, with $\omega = e_z$, $\mathcal{F}$ is injective over a small interval z if $\rho(x, y, z)$ is analytic in x, y. Uses ideas from 1-dim case.
- For $\rho\partial_t^2 - \Delta$ (and for $\Box + q$), but with a different type of source

  $$U(\cdot, 0) = f(\cdot), \quad U_t(\cdot, 0) = 0, \qquad \text{on } B \times [0, T]$$

  where f always positive. [Bukhgeim and Klibanov, 1981] proved injectivity if there is a convex function for the Riemannian metric ρI on $\mathbb{R}^n$.
- For $\partial_t^2 - \Delta_g$, for "overdetermined" problem of recovering g from hyperbolic D-N data, Belishev and Kurylev (BC method) reconstructed (including uniqueness) the manifold and Riemannian metric g;
  [Stefanov and Uhlmann 2005] proved stability near simple metrics.
- For formally determined problems for the **constant** velocity operator $\Box + q$, [Rakesh and Salo, 2020] showed $q \to [U_\omega, U_{-\omega}]|_{\partial B \times \mathbb{R}}$ is injective and stable.

Partial history

- For $\rho(z)\partial_t^2 - \partial_z^2$, injectivity, stability etc. of $\mathcal{F}_T$. Gelfand-Levitan, Krein, $\cdots$.
- For $\rho\partial_t^2 - \Delta$, [Romanov, 2002] showed $\mathcal{F}_T$ is injective if ρ close to 1. The assumption guarantees boundary normal geodesic flow (associated with U_ω) has no focal points. Even with this simplifying assumption, result is non-trivial.
- For $\rho\partial_t^2 - \Delta$, [Romanov, 2002] showed, with $\omega = e_z$, $\mathcal{F}$ is injective over a small interval z if $\rho(x, y, z)$ is analytic in x, y. Uses ideas from 1-dim case.
- For $\rho\partial_t^2 - \Delta$ (and for $\square + q$), but with a different type of source

  $$U(\cdot, 0) = f(\cdot), \quad U_t(\cdot, 0) = 0, \qquad \text{on } B \times [0, T]$$

  where f always positive. [Bukhgeim and Klibanov, 1981] proved injectivity if there is a convex function for the Riemannian metric $\rho \boldsymbol{I}$ on $\mathbb{R}^n$.
- For $\partial_t^2 - \Delta_g$, for "overdetermined" problem of recovering g from hyperbolic D-N data, Belishev and Kurylev (BC method) reconstructed (including uniqueness) the manifold and Riemannian metric g;
  [Stefanov and Uhlmann 2005] proved stability near simple metrics.
- For formally determined problems for the **constant** velocity operator $\square + q$, [Rakesh and Salo, 2020] showed $q \to [U_\omega, U_{-\omega}]|_{\partial B \times \mathbb{R}}$ is injective and stable.
- [Merono et al, 2021] Similar result for A recovery problem for operator $\square + A \cdot \nabla$.
  [Ma et al, 2022] Similar result for q problem for operator $\partial_t^2 - \Delta_g + q$.

Result for Lorentzian metric

- Suppose $(\mathbb{R}^{n+1}, h(x,t))$ **time oriented** Lorentzian metric with signature $(-,+,,\cdots,+)$, with following properties.

Result for Lorentzian metric

- Suppose $(\mathbb{R}^{n+1}, h(x,t))$ **time oriented** Lorentzian metric with signature $(-,+,,\cdots,+)$, with following properties.

  (a) Outside $B \times \mathbb{R}$, $h = h_{Min}$ (Minkowski metric) and dt future directed;

Result for Lorentzian metric

- Suppose $(\mathbb{R}^{n+1}, h(x,t))$ **time oriented** Lorentzian metric with signature $(-,+,,\cdots,+)$, with following properties.

  (a) Outside $B \times \mathbb{R}$, $h = h_{Min}$ (Minkowski metric) and dt future directed;

  (b) $[(\mathbb{R}^{n+1}, h)$ globally hyperbolic$]$ There is a smooth surjective function $\tau : \mathbb{R}^{n+1} \to \mathbb{R}$ (temporal function) with $d\tau$ future directed, and every inextendible timelike curve intersects $\tau = c$ exactly once (for each $c \in \mathbb{R}$);

Result for Lorentzian metric

- Suppose $(\mathbb{R}^{n+1}, h(x,t))$ **time oriented** Lorentzian metric with signature $(-,+,,\cdots,+)$, with following properties.

  (a) Outside $B \times \mathbb{R}$, $h = h_{Min}$ (Minkowski metric) and dt future directed;

  (b) [$(\mathbb{R}^{n+1}, h)$ globally hyperbolic] There is a smooth surjective function $\tau : \mathbb{R}^{n+1} \to \mathbb{R}$ (temporal function) with $d\tau$ future directed, and every inextendible timelike curve intersects $\tau = c$ exactly once (for each $c \in \mathbb{R}$);

  (c) (UCP) Let $I = (r, \infty)$; if u smooth on $\overline{B} \times I$ solves

  $$\Box_h u = 0 \text{ on } \overline{B} \times I, \qquad u|_{\partial B \times I} = \partial_\nu u|_{\partial B \times I} = 0,$$

  then $u = 0$ on $\overline{B} \times (r_1, \infty)$ for some $r_1 > r$.

Result for Lorentzian metric

- Suppose $(\mathbb{R}^{n+1}, h(x,t))$ **time oriented** Lorentzian metric with signature $(-,+,,\cdots,+)$, with following properties.

  (a) Outside $B \times \mathbb{R}$, $h = h_{Min}$ (Minkowski metric) and dt future directed;

  (b) $[(\mathbb{R}^{n+1}, h)$ globally hyperbolic$]$ There is a smooth surjective function $\tau : \mathbb{R}^{n+1} \to \mathbb{R}$ (temporal function) with $d\tau$ future directed, and every inextendible timelike curve intersects $\tau = c$ exactly once (for each $c \in \mathbb{R}$);

  (c) (UCP) Let $I = (r, \infty)$; if u smooth on $\overline{B} \times I$ solves

  $$\Box_h u = 0 \ \text{ on } \overline{B} \times I, \qquad u|_{\partial B \times I} = \partial_\nu u|_{\partial B \times I} = 0,$$

  then $u = 0$ on $\overline{B} \times (r_1, \infty)$ for some $r_1 > r$.

- If ω unit vector in $\mathbb{R}^n$, $s \in \mathbb{R}$ (time delay), let $U_{\omega,s}(x,t)$ be solution of IVP

  $$\Box_h U_{\omega,s} = 0 \ \text{ on } \mathbb{R}^n \times \mathbb{R}, \qquad U_{\omega,s}|_{\tau \ll 0} = H(t - s - x\cdot\omega).$$

Result for Lorentzian metric

- Suppose $(\mathbb{R}^{n+1}, h(x,t))$ **time oriented** Lorentzian metric with signature $(-,+,,\cdots,+)$, with following properties.

  (a) Outside $B \times \mathbb{R}$, $h = h_{Min}$ (Minkowski metric) and dt future directed;

  (b) $[(\mathbb{R}^{n+1}, h)$ globally hyperbolic$]$ There is a smooth surjective function $\tau : \mathbb{R}^{n+1} \to \mathbb{R}$ (temporal function) with $d\tau$ future directed, and every inextendible timelike curve intersects $\tau = c$ exactly once (for each $c \in \mathbb{R}$);

  (c) (UCP) Let $I = (r, \infty)$; if u smooth on $\overline{B} \times I$ solves

  $$\Box_h u = 0 \ \text{ on } \overline{B} \times I, \qquad u|_{\partial B \times I} = \partial_\nu u|_{\partial B \times I} = 0,$$

  then $u = 0$ on $\overline{B} \times (r_1, \infty)$ for some $r_1 > r$.

- If ω unit vector in $\mathbb{R}^n$, $s \in \mathbb{R}$ (time delay), let $U_{\omega,s}(x,t)$ be solution of IVP

  $$\Box_h U_{\omega,s} = 0 \ \text{ on } \mathbb{R}^n \times \mathbb{R}, \qquad U_{\omega,s}|_{\tau \ll 0} = H(t - s - x \cdot \omega).$$

- Choose incoming wave directions: $\Omega = \{\pm e_i\}_{i=1}^n \cup \{(e_i + e_j)/\sqrt{2}\}_{i \neq j;\, i,j=1\cdots n}$. Define $\mathcal{F} : h \to [U_{\omega,s}|_{\partial B \times \mathbb{R}}]_{\omega \in \Omega, s \in \mathbb{R}}$.

Result for Lorentzian metric

- Suppose $(\mathbb{R}^{n+1}, h(x,t))$ **time oriented** Lorentzian metric with signature $(-,+,,\cdots,+)$, with following properties.

  (a) Outside $B \times \mathbb{R}$, $h = h_{Min}$ (Minkowski metric) and dt future directed;

  (b) [$(\mathbb{R}^{n+1}, h)$ globally hyperbolic] There is a smooth surjective function $\tau : \mathbb{R}^{n+1} \to \mathbb{R}$ (temporal function) with $d\tau$ future directed, and every inextendible timelike curve intersects $\tau = c$ exactly once (for each $c \in \mathbb{R}$);

  (c) (UCP) Let $I = (r, \infty)$; if u smooth on $\overline{B} \times I$ solves

  $$\Box_h u = 0 \ \text{ on } \overline{B} \times I, \qquad u|_{\partial B \times I} = \partial_\nu u|_{\partial B \times I} = 0,$$

  then $u = 0$ on $\overline{B} \times (r_1, \infty)$ for some $r_1 > r$.

- If ω unit vector in $\mathbb{R}^n$, $s \in \mathbb{R}$ (time delay), let $U_{\omega,s}(x,t)$ be solution of IVP

  $$\Box_h U_{\omega,s} = 0 \ \text{ on } \mathbb{R}^n \times \mathbb{R}, \qquad U_{\omega,s}|_{\tau \ll 0} = H(t - s - x\cdot\omega).$$

- Choose incoming wave directions: $\Omega = \{\pm e_i\}_{i=1}^n \cup \{(e_i + e_j)/\sqrt{2}\}_{i \neq j;\, i,j=1\cdots n}$. Define $\mathcal{F} : h \to [U_{\omega,s}|_{\partial B \times \mathbb{R}}]_{\omega \in \Omega, s \in \mathbb{R}}$.

- **Theorem** [Oksanen, R, Salo (2024)] If $\mathcal{F}(h) = \mathcal{F}(h_{Min})$ then $\psi^*(h) = h_{Min}$ for some diffeomorphism $\psi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ with $\psi(x,t) = (x,t)$ outside $B \times R$.

Result for Lorentzian metric

- Suppose $(\mathbb{R}^{n+1}, h(x,t))$ **time oriented** Lorentzian metric with signature $(-,+,,\cdots,+)$, with following properties.

  (a) Outside $B \times \mathbb{R}$, $h = h_{Min}$ (Minkowski metric) and dt future directed;

  (b) [$(\mathbb{R}^{n+1}, h)$ globally hyperbolic] There is a smooth surjective function $\tau : \mathbb{R}^{n+1} \to \mathbb{R}$ (temporal function) with $d\tau$ future directed, and every inextendible timelike curve intersects $\tau = c$ exactly once (for each $c \in \mathbb{R}$);

  (c) (UCP) Let $I = (r, \infty)$; if u smooth on $\overline{B} \times I$ solves
  $$\Box_h u = 0 \text{ on } \overline{B} \times I, \qquad u|_{\partial B \times I} = \partial_\nu u|_{\partial B \times I} = 0,$$
  then $u = 0$ on $\overline{B} \times (r_1, \infty)$ for some $r_1 > r$.

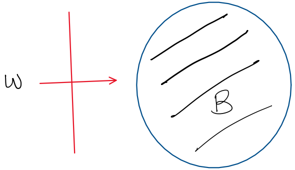
- If ω unit vector in $\mathbb{R}^n$, $s \in \mathbb{R}$ (time delay), let $U_{\omega,s}(x,t)$ be solution of IVP
  $$\Box_h U_{\omega,s} = 0 \text{ on } \mathbb{R}^n \times \mathbb{R}, \qquad U_{\omega,s}|_{\tau \ll 0} = H(t - s - x \cdot \omega).$$

- Choose incoming wave directions: $\Omega = \{\pm e_i\}_{i=1}^n \cup \{(e_i + e_j)/\sqrt{2}\}_{i \neq j;\, i,j=1\cdots n}$. Define $\mathcal{F} : h \to [U_{\omega,s}|_{\partial B \times \mathbb{R}}]_{\omega \in \Omega, s \in \mathbb{R}}$.

- **Theorem** [Oksanen, R, Salo (2024)] If $\mathcal{F}(h) = \mathcal{F}(h_{Min})$ then $\psi^*(h) = h_{Min}$ for some diffeomorphism $\psi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ with $\psi(x,t) = (x,t)$ outside $B \times R$.

- Formally determined problem. Also, h has UCP if $h = -dt^2 + g$ for $t \geq T$, for some Riemannian metric $g(x)$ on $\mathbb{R}^n$.

Proof of Riemannian result: recall the result



Anisotropic medium.
$g = (g_{ij}(x))$
Riemannian metric
$g = I$ outside B

ω a unit vector in $\mathbb{R}^n$, $U_\omega(x,t)$ the solution of the IVP

$$\begin{aligned} (\partial_t^2 - \Delta_g)U_\omega &= 0, && \text{on } \mathbb{R}^n \times \mathbb{R}, \\ U_\omega(x,t) &= H(t - x\cdot\omega) && \text{on } \mathbb{R}^n \times (-\infty,-1). \end{aligned}$$

Incoming wave directions: $\Omega = \{e_i\}_{i=1}^n \cup \{(e_i+e_j)/\sqrt{2}\}_{i\neq j;\, i,j=1,\cdots n}$.

$$\mathcal{F}_T : g \to [U_\omega|_{\partial B\times(-\infty,T)}]_{\omega\in\Omega} \qquad \text{(forward map)}.$$

Theorem If $\mathcal{F}_T(g) = \mathcal{F}_T(g_{Eucl})$ and T large enough, there is a diffeomorphism $\psi : \mathbb{R}^n \to \mathbb{R}^n$ with $\psi(x) = x$ outside B such that $\psi^*(g) = g_{Eucl}$.

Proof: The Lagrangian distribution U_ω

- For Riemannian manifold $(\mathbb{R}^n, g)$, $T^*(\mathbb{R}^n)$ identified with $T(\mathbb{R}^n)$.
- Null bicharacteristics of $\partial_t^2 - \Delta_g$ identified with the time parametrized unit speed geodesics (and their velocities).

Proof: The Lagrangian distribution U_ω

- For Riemannian manifold $(\mathbb{R}^n, g)$, $T^*(\mathbb{R}^n)$ identified with $T(\mathbb{R}^n)$.
- Null bicharacteristics of $\partial_t^2 - \Delta_g$ identified with the time parametrized unit speed geodesics (and their velocities).
- For function $\alpha(x)$ on $\mathbb{R}^n$, 1-form $d\alpha$ identified with vector field $\nabla_g \alpha$. In coordinates $\nabla_g \alpha = g^{-1} \nabla \alpha$.

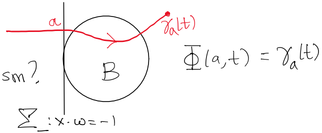
Proof: The Lagrangian distribution U_ω

- For Riemannian manifold $(\mathbb{R}^n, g)$, $T^*(\mathbb{R}^n)$ identified with $T(\mathbb{R}^n)$.
- Null bicharacteristics of $\partial_t^2 - \Delta_g$ identified with the time parametrized unit speed geodesics (and their velocities).
- For function $\alpha(x)$ on $\mathbb{R}^n$, 1-form $d\alpha$ identified with vector field $\nabla_g\alpha$. In coordinates $\nabla_g\alpha = g^{-1}\nabla\alpha$.
- Define $\Sigma_- = \{x \in \mathbb{R}^n : x \cdot \omega = -1\}$. U_ω is Lagrangian distribution with Lagrangian manifold generated by boundary normal flow $\Phi_\omega : \Sigma_- \times \mathbb{R} \to \mathbb{R}^n$

$$\Phi(a, t) = \gamma_a(t), \qquad a \in \Sigma_-, \quad t \in \mathbb{R},$$

where $t \to \gamma_a(t)$ is the geodesic with $\gamma_a(0) = a$, $\dot\gamma(0) = \omega$.

Is Φ a diffeomorphism?

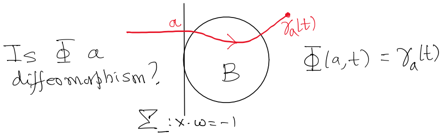


Proof: The Lagrangian distribution U_ω

- For Riemannian manifold $(\mathbb{R}^n, g)$, $T^*(\mathbb{R}^n)$ identified with $T(\mathbb{R}^n)$.
- Null bicharacteristics of $\partial_t^2 - \Delta_g$ identified with the time parametrized unit speed geodesics (and their velocities).
- For function $\alpha(x)$ on $\mathbb{R}^n$, 1-form $d\alpha$ identified with vector field $\nabla_g \alpha$. In coordinates $\nabla_g \alpha = g^{-1} \nabla \alpha$.
- Define $\Sigma_- = \{x \in \mathbb{R}^n : x \cdot \omega = -1\}$. U_ω is Lagrangian distribution with Lagrangian manifold generated by boundary normal flow $\Phi_\omega : \Sigma_- \times \mathbb{R} \to \mathbb{R}^n$

$$\Phi(a,t) = \gamma_a(t), \qquad a \in \Sigma_-, \quad t \in \mathbb{R},$$

where $t \to \gamma_a(t)$ is the geodesic with $\gamma_a(0) = a$, $\dot{\gamma}(0) = \omega$.



- If Φ_ω is a diffeomorphism, then there is a global solution $\alpha_\omega : \mathbb{R}^n \to \mathbb{R}$ of the eikonal equation $\|\nabla_g \alpha_\omega\| = 1$ with $\alpha_\omega(x) = x \cdot \omega$ outside B.
Then U_ω has the nice form

$$U_\omega(x,t) = u_\omega(x,t) H(t - \alpha_\omega(x)), \qquad (x,t) \in \mathbb{R}^n \times \mathbb{R},$$

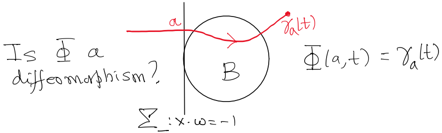
for some smooth function $u_\omega(x,t)$ on $\mathbb{R}^n \times \mathbb{R}$.

Proof: The Lagrangian distribution U_ω

- For Riemannian manifold $(\mathbb{R}^n, g)$, $T^*(\mathbb{R}^n)$ identified with $T(\mathbb{R}^n)$.
- Null bicharacteristics of $\partial_t^2 - \Delta_g$ identified with the time parametrized unit speed geodesics (and their velocities).
- For function $\alpha(x)$ on $\mathbb{R}^n$, 1-form $d\alpha$ identified with vector field $\nabla_g \alpha$. In coordinates $\nabla_g \alpha = g^{-1} \nabla \alpha$.
- Define $\Sigma_- = \{x \in \mathbb{R}^n : x \cdot \omega = -1\}$. U_ω is Lagrangian distribution with Lagrangian manifold generated by boundary normal flow $\Phi_\omega : \Sigma_- \times \mathbb{R} \to \mathbb{R}^n$

$$\Phi(a, t) = \gamma_a(t), \qquad a \in \Sigma_-, \quad t \in \mathbb{R},$$

where $t \to \gamma_a(t)$ is the geodesic with $\gamma_a(0) = a$, $\dot{\gamma}(0) = \omega$.



- If Φ_ω is a diffeomorphism, then there is a global solution $\alpha_\omega : \mathbb{R}^n \to \mathbb{R}$ of the eikonal equation $\|\nabla_g \alpha_\omega\| = 1$ with $\alpha_\omega(x) = x \cdot \omega$ outside B.
Then U_ω has the nice form

$$U_\omega(x, t) = u_\omega(x, t) H(t - \alpha_\omega(x)), \qquad (x, t) \in \mathbb{R}^n \times \mathbb{R},$$

for some smooth function $u_\omega(x, t)$ on $\mathbb{R}^n \times \mathbb{R}$.
- We **do not** assume that the Φ_ω, associated to $(\mathbb{R}^n, g)$ and ω, is a diffeomorphism.

Main steps of the proof

An elementary argument shows that $\mathcal{F}(g) = \mathcal{F}(g_{Eucl})$ implies that for each $\omega \in \Omega$,

$$U_\omega(x,t) = H(t - x\cdot\omega), \qquad \text{for all } (x,t) \text{ outside } B\times\mathbb{R}. \qquad \text{(exterior property)}$$

Proof consists of two parts.

Main steps of the proof

An elementary argument shows that $\mathcal{F}(g) = \mathcal{F}(g_{Eucl})$ implies that for each $\omega \in \Omega$,

$$U_\omega(x,t) = H(t - x\cdot\omega), \qquad \text{for all } (x,t) \text{ outside } B\times\mathbb{R}. \qquad \text{(exterior property)}$$

Proof consists of two parts.

- A geometrical and topological argument which shows that the *exterior property* implies that Φ_ω is a diffeomorphism for each of the directions $\omega \in \Omega$.

Main steps of the proof

An elementary argument shows that $\mathcal{F}(g) = \mathcal{F}(g_{Eucl})$ implies that for each $\omega \in \Omega$,

$$U_\omega(x,t) = H(t - x \cdot \omega), \qquad \text{for all } (x,t) \text{ outside } B \times \mathbb{R}. \qquad \text{(exterior property)}$$

Proof consists of two parts.

- A geometrical and topological argument which shows that the *exterior property* implies that Φ_ω is a diffeomorphism for each of the directions $\omega \in \Omega$.

  Hence there is the global solution α_ω of the eikonal equation $\|\nabla_g \alpha\| = 1$, with $\alpha_\omega(x) = x \cdot \omega$ outside B, and U_ω has the nice form $U_\omega(x,t) = u_\omega(x,t) H(t - \alpha_\omega(x))$.

Main steps of the proof

An elementary argument shows that $\mathcal{F}(g) = \mathcal{F}(g_{Eucl})$ implies that for each $\omega \in \Omega$,

$$U_\omega(x,t) = H(t - x\cdot\omega), \qquad \text{for all } (x,t) \text{ outside } B\times\mathbb{R}. \qquad \text{(exterior property)}$$

Proof consists of two parts.

- A geometrical and topological argument which shows that the *exterior property* implies that Φ_ω is a diffeomorphism for each of the directions $\omega \in \Omega$.

  Hence there is the global solution α_ω of the eikonal equation $\|\nabla_g\alpha\| = 1$, with $\alpha_\omega(x) = x\cdot\omega$ outside B, and U_ω has the nice form $U_\omega(x,t) = u_\omega(x,t)H(t-\alpha_\omega(x))$.
- Define the map $\psi : \mathbb{R}^n \to \mathbb{R}^n$ with

$$\psi(x) = [\alpha_{e_1}(x), \cdots, \alpha_{e_n}(x)], \qquad x \in \mathbb{R}^n.$$

Main steps of the proof

An elementary argument shows that $\mathcal{F}(g) = \mathcal{F}(g_{Eucl})$ implies that for each $\omega \in \Omega$,

$$U_\omega(x,t) = H(t - x\cdot\omega), \qquad \text{for all } (x,t) \text{ outside } B\times\mathbb{R}. \qquad \text{(exterior property)}$$

Proof consists of two parts.

- A geometrical and topological argument which shows that the *exterior property* implies that Φ_ω is a diffeomorphism for each of the directions $\omega \in \Omega$.

  Hence there is the global solution α_ω of the eikonal equation $\|\nabla_g \alpha\| = 1$, with $\alpha_\omega(x) = x\cdot\omega$ outside B, and U_ω has the nice form $U_\omega(x,t) = u_\omega(x,t)H(t-\alpha_\omega(x))$.
- Define the map $\psi : \mathbb{R}^n \to \mathbb{R}^n$ with

$$\psi(x) = [\alpha_{e_1}(x), \cdots, \alpha_{e_n}(x)], \qquad x \in \mathbb{R}^n.$$

  Note that $\psi(x) = x$ outside B. Then, using a PDE argument for the nice form of U_ω, one shows that the *exterior property* implies ψ is a diffeomorphism with ψ' an orthogonal matrix in the $(\mathbb{R}^n, g)$ sense. Hence $g = \psi^*(g_{Eucl})$.

Main steps of the proof

An elementary argument shows that $\mathcal{F}(g) = \mathcal{F}(g_{Eucl})$ implies that for each $\omega \in \Omega$,

$$U_\omega(x,t) = H(t - x\cdot\omega), \qquad \text{for all } (x,t) \text{ outside } B \times \mathbb{R}. \qquad \text{(exterior property)}$$

Proof consists of two parts.

- A geometrical and topological argument which shows that the *exterior property* implies that Φ_ω is a diffeomorphism for each of the directions $\omega \in \Omega$.

  Hence there is the global solution α_ω of the eikonal equation $\|\nabla_g \alpha\| = 1$, with $\alpha_\omega(x) = x\cdot\omega$ outside B, and U_ω has the nice form $U_\omega(x,t) = u_\omega(x,t)H(t - \alpha_\omega(x))$.
- Define the map $\psi : \mathbb{R}^n \to \mathbb{R}^n$ with

  $$\psi(x) = [\alpha_{e_1}(x), \cdots, \alpha_{e_n}(x)], \qquad x \in \mathbb{R}^n.$$

  Note that $\psi(x) = x$ outside B. Then, using a PDE argument for the nice form of U_ω, one shows that the *exterior property* implies ψ is a diffeomorphism with ψ' an orthogonal matrix in the $(\mathbb{R}^n, g)$ sense. Hence $g = \psi^*(g_{Eucl})$.

  Data for the directions $(e_i + e_j)/\sqrt{2}$ is needed to prove the orthogonality of ψ'.

Main steps of the proof

An elementary argument shows that $\mathcal{F}(g) = \mathcal{F}(g_{Eucl})$ implies that for each $\omega \in \Omega$,

$$U_\omega(x,t) = H(t - x\cdot\omega), \qquad \text{for all } (x,t) \text{ outside } B\times\mathbb{R}. \qquad \text{(exterior property)}$$

Proof consists of two parts.

- A geometrical and topological argument which shows that the *exterior property* implies that Φ_ω is a diffeomorphism for each of the directions $\omega \in \Omega$.

  Hence there is the global solution α_ω of the eikonal equation $\|\nabla_g \alpha\| = 1$, with $\alpha_\omega(x) = x\cdot\omega$ outside B, and U_ω has the nice form $U_\omega(x,t) = u_\omega(x,t)H(t - \alpha_\omega(x))$.
- Define the map $\psi : \mathbb{R}^n \to \mathbb{R}^n$ with

$$\psi(x) = [\alpha_{e_1}(x), \cdots, \alpha_{e_n}(x)], \qquad x \in \mathbb{R}^n.$$

  Note that $\psi(x) = x$ outside B. Then, using a PDE argument for the nice form of U_ω, one shows that the *exterior property* implies ψ is a diffeomorphism with ψ' an orthogonal matrix in the $(\mathbb{R}^n, g)$ sense. Hence $g = \psi^*(g_{Eucl})$.

  Data for the directions $(e_i + e_j)/\sqrt{2}$ is needed to prove the orthogonality of ψ'.
- The first part is true ($\Phi_{g,\omega}$ is a diffeomorphism) if $\mathcal{F}(g) = \mathcal{F}(g_1)$ where g_1 is any metric (not necessarily Euclidean) for which $\Phi_{g_1,\omega}$ is a diffeomorphism.

Main steps of the proof

An elementary argument shows that $\mathcal{F}(g) = \mathcal{F}(g_{Eucl})$ implies that for each $\omega \in \Omega$,

$$U_\omega(x,t) = H(t - x\cdot\omega), \qquad \text{for all } (x,t) \text{ outside } B\times\mathbb{R}. \qquad \text{(exterior property)}$$

Proof consists of two parts.

- A geometrical and topological argument which shows that the *exterior property* implies that Φ_ω is a diffeomorphism for each of the directions $\omega \in \Omega$.

  Hence there is the global solution α_ω of the eikonal equation $\|\nabla_g \alpha\| = 1$, with $\alpha_\omega(x) = x\cdot\omega$ outside B, and U_ω has the nice form $U_\omega(x,t) = u_\omega(x,t)H(t-\alpha_\omega(x))$.
- Define the map $\psi : \mathbb{R}^n \to \mathbb{R}^n$ with

  $$\psi(x) = [\alpha_{e_1}(x), \cdots, \alpha_{e_n}(x)], \qquad x \in \mathbb{R}^n.$$

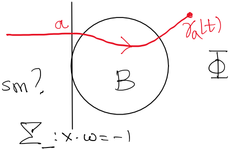
  Note that $\psi(x) = x$ outside B. Then, using a PDE argument for the nice form of U_ω, one shows that the *exterior property* implies ψ is a diffeomorphism with ψ' an orthogonal matrix in the $(\mathbb{R}^n, g)$ sense. Hence $g = \psi^*(g_{Eucl})$.

  Data for the directions $(e_i + e_j)/\sqrt{2}$ is needed to prove the orthogonality of ψ'.
- The first part is true ($\Phi_{g,\omega}$ is a diffeomorphism) if $\mathcal{F}(g) = \mathcal{F}(g_1)$ where g_1 is any metric (not necessarily Euclidean) for which $\Phi_{g_1,\omega}$ is a diffeomorphism.
- The second part works only for $\mathcal{F}(g) = \mathcal{F}(g_{Eucl})$ because for the Euclidean metric $U_{\omega,g_{Eucl}} = 1H(t - x\cdot\omega)$ and we use the fact that $\Delta_g 1 = 0$ for any g.

Proof: The geometrical and topological part

Define $\Sigma_- = \{a \in \mathbb{R}^n : a \cdot \omega = -1\}$.
Define $\Phi : \Sigma_- \times \mathbb{R} \to \mathbb{R}^n$ with $\Phi(a,t) = \gamma_a(t)$.
Here $t \to \gamma_a(t)$ is the geodesic with $\gamma_a(0) = a$, $\dot{\gamma}_a(0) = \omega$.

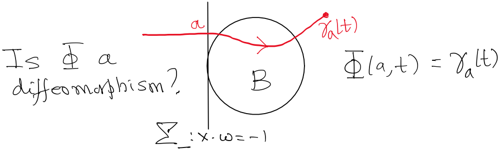
Is Φ a diffeomorphism?



$\Phi(a,t) = \gamma_a(t)$

Proof: The geometrical and topological part

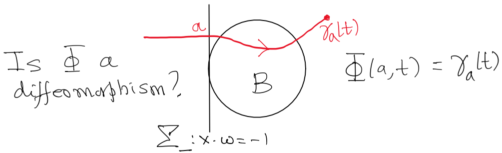
Define $\Sigma_- = \{a \in \mathbb{R}^n : a \cdot \omega = -1\}$.
Define $\Phi : \Sigma_- \times \mathbb{R} \to \mathbb{R}^n$ with $\Phi(a,t) = \gamma_a(t)$.
Here $t \to \gamma_a(t)$ is the geodesic with $\gamma_a(0) = a$, $\dot{\gamma}_a(0) = \omega$.



The exterior property holds so $\mathrm{WF}(U_\omega(x,t)) = \mathrm{WF}(H(t - x\cdot\omega))$ outside $B \times \mathbb{R}$.
So, when $\gamma_a(t)$ is outside B, we have $\dot{\gamma}_a(t) = \omega$ and every point outside B lies on **exactly** one γ_a.

Proof: The geometrical and topological part

Define $\Sigma_- = \{a \in \mathbb{R}^n : a \cdot \omega = -1\}$.
Define $\Phi : \Sigma_- \times \mathbb{R} \to \mathbb{R}^n$ with $\Phi(a, t) = \gamma_a(t)$.
Here $t \to \gamma_a(t)$ is the geodesic with $\gamma_a(0) = a$, $\dot{\gamma}_a(0) = \omega$.

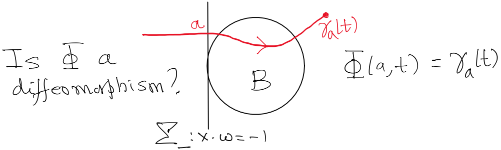


The exterior property holds so $\mathrm{WF}(U_\omega(x,t)) = \mathrm{WF}(H(t - x \cdot \omega))$ outside $B \times \mathbb{R}$.
So, when $\gamma_a(t)$ is outside B, we have $\dot{\gamma}_a(t) = \omega$ and every point outside B lies on **exactly** one γ_a.

This geometric behavior in the exterior is enough to prove that Φ_ω is a diffeomorphism.

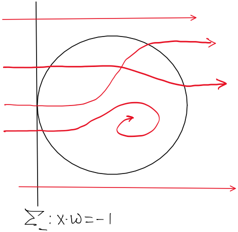
Proof: The geometrical and topological part

Define $\Sigma_- = \{a \in \mathbb{R}^n : a \cdot \omega = -1\}$.
Define $\Phi : \Sigma_- \times \mathbb{R} \to \mathbb{R}^n$ with $\Phi(a,t) = \gamma_a(t)$.
Here $t \to \gamma_a(t)$ is the geodesic with $\gamma_a(0) = a$, $\dot{\gamma}_a(0) = \omega$.



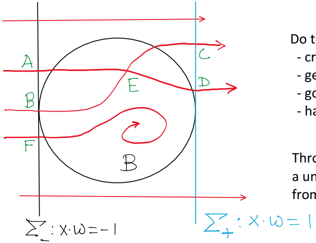
The exterior property holds so $\mathrm{WF}(U_\omega(x,t)) = \mathrm{WF}(H(t - x \cdot \omega))$ outside $B \times \mathbb{R}$. So, when $\gamma_a(t)$ is outside B, we have $\dot{\gamma}_a(t) = \omega$ and every point outside B lies on **exactly** one γ_a.

This geometric behavior in the exterior is enough to prove that Φ_ω is a diffeomorphism.



Do the boundary normal geodesics
- cross each other
- get trapped in B
- go through every point
- have a “caustic”?

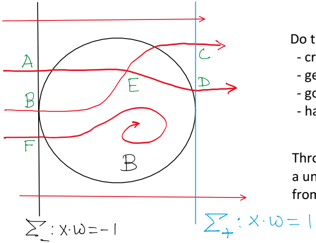
Proof: Details of geometrical and topological part



Do the boundary normal geodesics
- cross each other
- get trapped in B
- go through every point
- have a “caustic”?

Through every point on Σ_+ there is a unique bdry normal geodesic from Σ_-.

Proof: Details of geometrical and topological part

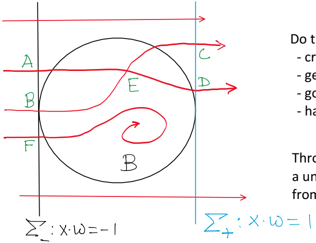


Do the boundary normal geodesics
- cross each other
- get trapped in B
- go through every point
- have a “caustic”?

Through every point on Σ_+ there is a unique bdry normal geodesic from Σ_-.

- Define backward map $f : \Sigma_+ \to \Sigma_-$ with $f(x) = a$ if γ_a goes through $x \in \Sigma_+$.

Proof: Details of geometrical and topological part

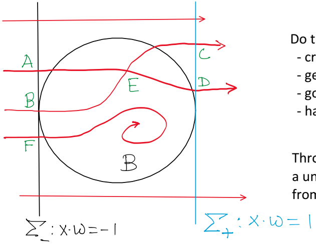


Do the boundary normal geodesics
- cross each other
- get trapped in B
- go through every point
- have a "caustic"?

Through every point on Σ_+ there is a unique bdry normal geodesic from Σ_-.

- Define backward map $f : \Sigma_+ \to \Sigma_-$ with $f(x) = a$ if γ_a goes through $x \in \Sigma_+$.
- Implicit function theorem and transversality imply that f is smooth, $M = \text{range}(f)$ is open in Σ_-, and f^{-1} is smooth.

Proof: Details of geometrical and topological part

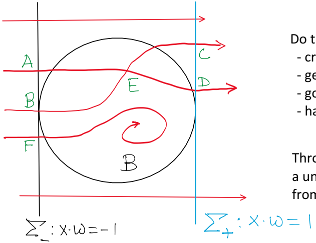


Do the boundary normal geodesics
- cross each other
- get trapped in B
- go through every point
- have a “caustic”?

Through every point on Σ_+ there is a unique bdry normal geodesic from Σ_-.

- Define backward map $f : \Sigma_+ \to \Sigma_-$ with $f(x) = a$ if γ_a goes through $x \in \Sigma_+$.
- Implicit function theorem and transversality imply that f is smooth, $M = \text{range}(f)$ is open in Σ_-, and f^{-1} is smooth.
- So M is diffeomorphic to Σ_+. Note M includes $\Sigma_- \cap \{|a| \geq 1\}$.

Proof: Details of geometrical and topological part

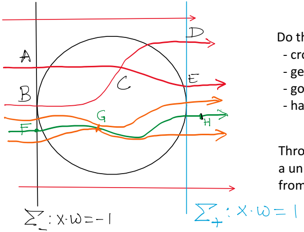


Do the boundary normal geodesics
- cross each other
- get trapped in B
- go through every point
- have a "caustic"?

Through every point on Σ_+ there is a unique bdry normal geodesic from Σ_-.

- Define backward map $f : \Sigma_+ \to \Sigma_-$ with $f(x) = a$ if γ_a goes through $x \in \Sigma_+$.
- Implicit function theorem and transversality imply that f is smooth, $M = \text{range}(f)$ is open in Σ_-, and f^{-1} is smooth.
- So M is diffeomorphic to Σ_+. Note M includes $\Sigma_- \cap \{|a| \geq 1\}$.
- So homology implies $M = \Sigma_-$. Hence no γ_a trapped in B.
- Hence every γ_a crosses Σ_+ at exactly one point, and is shortest path from Σ_- to that point on Σ_+

Proof: Details of geometrical and topological part contd.

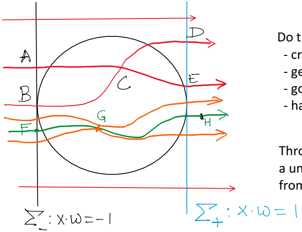


Do the boundary normal geodesics
- cross each other
- get trapped in B
- go through every point
- have a "caustic"?

Through every point on Σ_+ there is a unique bdry normal geodesic from Σ_-.

- We proved every γ_a from Σ_- crosses Σ_+ at exactly one point, and is the shortest path from Σ_- to that point.

Proof: Details of geometrical and topological part contd.

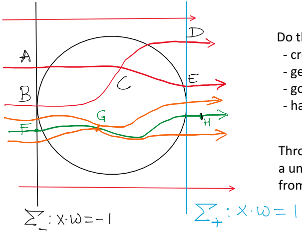


Do the boundary normal geodesics
- cross each other
- get trapped in B
- go through every point
- have a “caustic”?

Through every point on Σ_+ there is a unique bdry normal geodesic from Σ_-.

- We proved every γ_a from Σ_- crosses Σ_+ at exactly one point, and is the shortest path from Σ_- to that point.
- We also know that through every point of Σ_+ there is a γ_a (the bdry normal geodesic) for a unique $a \in \Sigma_-$.
- From this one shows there is a bdry normal geodesic from Σ_- through every point in B! Hence Φ_ω is surjective.

Proof: Details of geometrical and topological part contd.

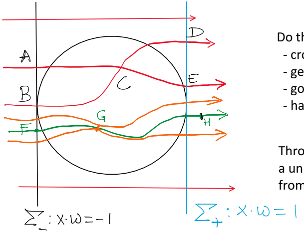


Do the boundary normal geodesics
- cross each other
- get trapped in B
- go through every point
- have a “caustic”?

Through every point on Σ_+ there is a unique bdry normal geodesic from Σ_-.

- We proved every γ_a from Σ_- crosses Σ_+ at exactly one point, and is the shortest path from Σ_- to that point.
- We also know that through every point of Σ_+ there is a γ_a (the bdry normal geodesic) for a unique $a \in \Sigma_-$.
- From this one shows there is a bdry normal geodesic from Σ_- through every point in B! Hence Φ_ω is surjective.
- One also shows that two bdry normal geodesics from Σ_- cannot cross each other! Hence Φ_ω is injective.

Proof: Details of geometrical and topological part contd.

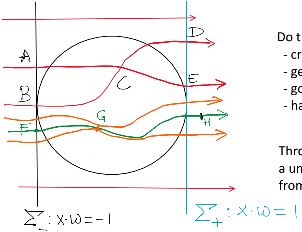


Do the boundary normal geodesics
- cross each other
- get trapped in B
- go through every point
- have a "caustic"?

Through every point on Σ_+ there is a unique bdry normal geodesic from Σ_-.

- We proved every γ_a from Σ_- crosses Σ_+ at exactly one point, and is the shortest path from Σ_- to that point.
- We also know that through every point of Σ_+ there is a γ_a (the bdry normal geodesic) for a unique $a \in \Sigma_-$.
- From this one shows there is a bdry normal geodesic from Σ_- through every point in B! Hence Φ_ω is surjective.
- One also shows that two bdry normal geodesics from Σ_- cannot cross each other! Hence Φ_ω is injective.
- One can show that $\Phi_\omega : \Sigma_- \times \mathbb{R} \to \mathbb{R}^n$ with $\Phi(a, t) = \gamma_a(t)$, is a local diffeomorphism!

Proof: Details of geometrical and topological part contd.

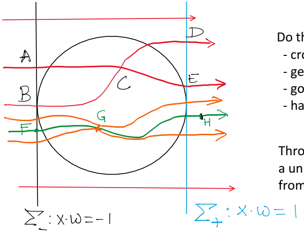


Do the boundary normal geodesics
- cross each other
- get trapped in B
- go through every point
- have a “caustic”?

Through every point on Σ_+ there is a unique bdry normal geodesic from Σ_-.

- We proved every γ_a from Σ_- crosses Σ_+ at exactly one point, and is the shortest path from Σ_- to that point.
- We also know that through every point of Σ_+ there is a γ_a (the bdry normal geodesic) for a unique $a \in \Sigma_-$.
- From this one shows there is a bdry normal geodesic from Σ_- through every point in B! Hence Φ_ω is surjective.
- One also shows that two bdry normal geodesics from Σ_- cannot cross each other! Hence Φ_ω is injective.
- One can show that $\Phi_\omega : \Sigma_- \times \mathbb{R} \to \mathbb{R}^n$ with $\Phi(a, t) = \gamma_a(t)$, is a local diffeomorphism!
- Hence Φ is a diffeomorphism! Completes proof of geometry/topology part for Riemannian case.

Proof: Details of geometrical and topological part contd.



Do the boundary normal geodesics
- cross each other
- get trapped in B
- go through every point
- have a "caustic"?

Through every point on Σ_+ there is a unique bdry normal geodesic from Σ_-.

- We proved every γ_a from Σ_- crosses Σ_+ at exactly one point, and is the shortest path from Σ_- to that point.
- We also know that through every point of Σ_+ there is a γ_a (the bdry normal geodesic) for a unique $a \in \Sigma_-$.
- From this one shows there is a bdry normal geodesic from Σ_- through every point in B! Hence Φ_ω is surjective.
- One also shows that two bdry normal geodesics from Σ_- cannot cross each other! Hence Φ_ω is injective.
- One can show that $\Phi_\omega : \Sigma_- \times \mathbb{R} \to \mathbb{R}^n$ with $\Phi(a,t) = \gamma_a(t)$, is a local diffeomorphism!
- Hence Φ is a diffeomorphism! Completes proof of geometry/topology part for Riemannian case.
- The Lorentzian $\Box_h$ proof more difficult. No shortest distance so more subtle geometric arguments. h inhomogeneous on $B \times \mathbb{R}$ (unbounded) so homology and some geometrical arguments replaced by **Hadamard's theorem: If $f : \mathbb{R}^m \to \mathbb{R}^m$ is a proper local diffeomorphism then f is a diffeomorphism.**