The general Calderón problem on Riemannian surfaces and inverse problems for minimal surfaces

Tony Liimatainen
University of Helsinki

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Based on joint works with C. Carstea, M. Lassas and L. Tzou,
https://arxiv.org/abs/2310.14268

Introduction

The minimal surface equation in the Euclidean space.

Several ways to define minimal surfaces. A minimal surface embedded in $\mathbb{R}^{n+1}$ is given as a graph of a function $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ if u satisfies the minimal surface equation

$$\begin{cases} \nabla \cdot \left(\frac{\nabla u}{(1+|\nabla u|^2)^{1/2}} \right) = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases}$$

- Quasilinear elliptic. If you linearize at $u = 0$, the principal term is Laplacian.
- A minimal surface has vanishing mean curvature. That is, trace of the tensor $(X, Y) \mapsto \langle \nabla_X N, Y \rangle$ vanishes, X, Y tangential and N the normal of the surface.

More generally, a minimal surface embedded in an $(n+1)$-dimensional Riemannian manifold $(M, \overline{g})$ can be defined to be an n-dimensional submanifold whose mean curvature vanishes.

But what are minimal surfaces?

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Let great Jim Simons (April 25, 1938 – May 10, 2024) explain: Click for video [1]

[1] Jim Simons: "A Short Story of My Life and Mathematics (2022)" `https://tinyurl.com/2adepcdk`

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The class of minimal surfaces even in Euclidean space $\mathbb{R}^3$ is quite rich.

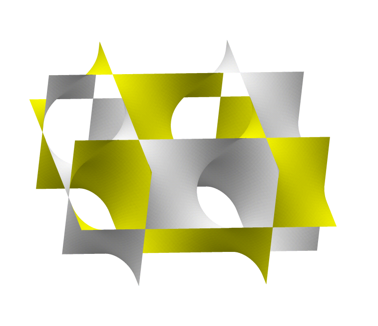

Figure 1: Schwarz D Surface

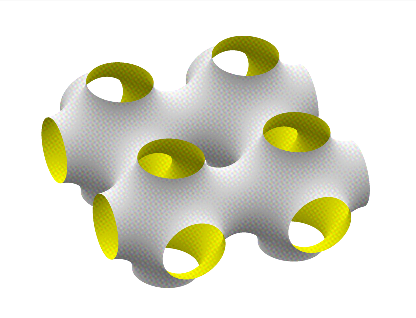

Figure 2: Schwarz P Surface

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Notation and Fermi coordinates

We set up notation. We only consider dimension of the surface $n = 2$. We work in Fermi coordinates relative to a 2D surface Ω embedded in a 3D manifold M. In Fermi coordinates, the 3D metric of $(M, \overline{g})$ reads

$$\overline{g}(s,x) = ds^2 + \sum_{k,l=1}^{2} g_{kl}(s,x)dx^k dx^l, \quad x \in \Omega,\ s \in I \subset \mathbb{R}.$$

- Fermi coordinates always exist. Using the coordinates, we may view M locally as $I \times \Omega$.
- Metric is of the same form as in boundary normal coordinates.
- The 2D metric g on Ω is given by $g_{kl}(s,x)|_{s=0} = g_{kl}(0,x)$, $x \in \Omega$.
- If u is a function over Ω, we write $g_u(x) = g(u(x), x)$.

As mean curvature depends not only on the metric on the minimal surface, but also on the surrounding "ambient" 3D metric $\overline{g}$, the minimal surface equation will depend on $\overline{g}$.

The equation for an embedded minimal surface.

If a minimal surface embedded in *ambient space* $(M, \overline{g})$ is given as a graph of $u : \Omega \to \mathbb{R}$ over 2D surface $\Omega \subset M$, then u satisfies in Fermi coordinates

$$-\frac{1}{\mathrm{Det}(g_u)^{1/2}} \nabla \cdot \left(g_u^{-1} \frac{\mathrm{Det}(g_u)^{1/2}}{\sqrt{1 + |\nabla u|_{g_u}^2}} \right) \nabla u + F(u, \nabla u) = 0 \text{ in } \Omega,$$

where

$$F(u, \nabla u) = \frac{1}{2} \frac{1}{(1 + |\nabla u|_{g_u}^2)^{1/2}} (\partial_s g_u^{-1})(\nabla u, \nabla u) + \frac{1}{2}(1 + |\nabla u|_{g_u}^2)^{1/2} \mathrm{Tr}(g_u^{-1} \partial_s g_u).$$

Here ∇ and $\cdot$ are the Euclidean ones.

- Describes a minimal surface by the graph $x \mapsto (x, u(x)) \in M$, $x \in \Omega$.
- Quite complicated, which results in quite long computations.
- The equation is invariant under isometries of the ambient space $(M, \overline{g})$.
- Not conformally invariant even in 2D $\Rightarrow$ Hope for recovery up to an isometry in the inverse problem, not just conformal mapping.

Areas of minimal surfaces determine the DN map

From now on we consider minimal surfaces given as graphs over Ω, which itself is a minimal surface (Σ, g), i.e. $\Omega = \Sigma$. Dirichlet-to-Neumann (DN) map Λ of the minimal surface equation on Σ given by the usual assignment

$$\Lambda(f) = N \cdot \nabla u^f|_{\partial\Sigma},$$

where u^f solves the minimal surface eq. with boundary value $f \in C^\infty(\partial\Sigma)$ small.

- Local solvability near a given minimal surface follows from the usual implicit function theorem/contraction method.

DN map of the minimal surface equation is determined by areas of minimal surfaces.

- An exercise in calculus of variations. Minimal surfaces minimize
$$u \mapsto \text{Area}(\text{graph}(u)) = \int_\Sigma \sqrt{1 + |\nabla u(x)|^2_{g_u(x)}}\, \det\left(g_u(x)\right)^{1/2} dx^1 \wedge dx^2.$$
- Purely geometric data.

The inverse problem for minimal surfaces. Two formulations.

Let us formulate the inverse problem for minimal surfaces in two equivalent ways. In both of them, we assume that a 2D minimal surface (Σ, g) is embedded in a 3D Riemannian manifold $(M, \overline{g})$.

DN map formulation:

Assume that the Dirichlet-to-Neumann map $\Lambda : f \mapsto N \cdot \nabla u^f$ of the minimal surface equation is known. Determine Σ and the induced 2D metric g on it.

Area data formulation (purely geometric data):

Assume that the area of a minimal surface Σ and areas of its minimal surface perturbations are known. Same task, determine (Σ, g).

- In both cases the determination will at best be possible up to an isometry.
- Will additionally obtain information about the 3D ambient metric $\overline{g}$.

Main results

The main result for the inverse problem of minimal surfaces

Theorem (C. Carstea, M. Lassas, T. L, L. Tzou 2023/2024)

Let $(\Sigma_1, g_1) \subset (M_1, \overline{g}_1)$ and $(\Sigma_2, g_2) \subset (M_2, \overline{g}_2)$ be $2D$ minimal surfaces embedded in 3D Riemannian manifolds, with a mutual boundary $\partial\Sigma$. (Assume boundary determination.)

If the DN maps of the associated minimal surface equations satisfy $\Lambda_{g_1} f = \Lambda_{g_2} f$, for all $f \in C^\infty(\partial\Sigma)$ sufficiently small, then there is an isometry $F : \Sigma_1 \to \Sigma_2$,

$$F^* g_2 = g_1, \quad F|_{\partial\Sigma} = Id.$$

In addition $F^*\eta_2 = \eta_1$, where η_1, η_2 are the 2nd fundamental forms of (Σ_1, g_1) and (Σ_2, g_2).

- General result. No additional assumptions.
- The recovery is up to diffeomorphism, not just to conformal mapping as often in 2D cases.
- Also the second fundamental form recovered, which depends on the 3D metric.
- We work only in 2D because we need to solve an anisotropic Calderón problem at one stage. Many of the arguments work in all dimensions however.

The general Calderón problem on Riemannian surfaces. The other main result.

As said, to solve the inverse problem for minimal surfaces, one must solve the anisotropic Calderón problem on Riemannian surfaces, where both the metric and the potential are unknown:

Theorem (T. L and L. Tzou 2024)

Let (Σ_1, g_1) and (Σ_2, g_2) be Riemannian surfaces with mutual boundary $\partial\Sigma$. If the DN maps of the Shrödinger equations

$$(\Delta_{g_1} + q_1)v_1 = 0 \text{ on } \Sigma_1 \quad \text{and} \quad (\Delta_{g_2} + q_2)v_2 = 0 \text{ on } \Sigma_2$$

agree on $\partial\Sigma$, then there is a conformal mapping $J : \Sigma_1 \to \Sigma_2$ and $c(x) > 0$:

$$g_1 = cJ^*g_2 \quad \text{and} \quad q_1 = c^{-1}J^*q_2.$$

- General result. No additional assumptions.
- Recovery up to conformal mapping is due to a conformal gauge symmetry.
- Generalizes earlier results by Imanuvilov-Uhlmann-Yamamoto 2012, Guillarmou-Tzou 2011 and Lassas-Uhlmann 2001, and others.

Why inverse problems for minimal surfaces? Generalized boundary rigidity.

In the generalized boundary rigidity problem the aim is to construct a manifold from the areas of minimal surfaces.

- The usual boundary rigidity asks if a manifold is determined by lengths of minimal geodesics between pairs of points on its boundary. Minimal geodesics are 1D minimal surfaces.

The first mathematical paper on generalized boundary rigidity is by S. Alexakis, T. Balehowsky & A. Nachman (2020) “How to determine a 3 dimensional manifold from the areas of its minimal surfaces”.

- In generalized boundary rigidity the task is two-fold: By knowing areas minimal surfaces (1) recover minimal surfaces and (2) find a way to glue them together.
- The above paper considered both (1) and (2).
- In this talk we only consider (1). We introduce a higher order linearization method for the problem.

Motivation from AdS/CFT duality in physics (1)

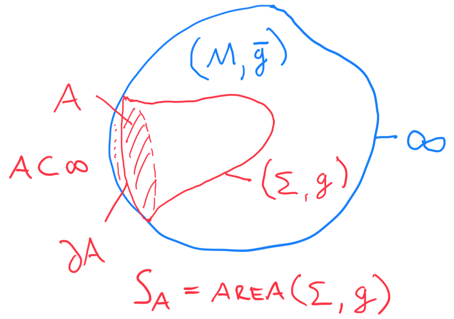
An AdS/CFT duality conjecture in physics by Ryu and Takayanagi states that "entanglement entropies" of a quantum field theory living on the boundary are equivalent to areas of related minimal surfaces (2006, thousands of citations).

- Entanglement entropy S_A is the experienced entropy (state of disorder) of a physical system for an observer who has only access to a subregion A of a larger space.
- Physicists ask: "Is a spacetime determined by entanglement entropies of a quantum field theory (QFT) living on the asymptotic boundary?"
  - Equivalently by the duality: 'Is a spacetime determined by areas of minimal surfaces anchored on the asymptotic boundary?"
  - Conformal field theories (CFTs) are special type of QFTs.
- Physicists give examples where the answer to the above question is yes. Especially these are examples where the generalized boundary rigidity problem is solvable.

Motivation from AdS/CFT duality in physics (2)

The real physical situation is in the noncompact setting.

- The typical physical setting is in asymptotically hyperbolic Riemannian manifolds, such as time slices of Anti de Sitter (AdS) space.
  - Areas of minimal surfaces become infinite $\leftrightarrow$ QFTs have infinite degrees of freedom.
- Considered by conformal compactification, in which case the metric blows up on the boundary:



In this talk everything is compact and nothing blows up on the boundary.

Other earlier results

Inverse problems for the minimal surface equation:

- C. Carstea, Lassas, T. L, L. Oksanen (2022), determination of a minimal surface (Σ, g) embedded in $\Sigma \times \mathbb{R}$, "toy model".
- J. Nurminen (2022, 2023), results for conformally Euclidean metric in $\mathbb{R}^n$.

Physics papers about the construction of a Riemannian manifold from areas of minimal surfaces in context of AdS/CFT duality.

- S. Bilson, N. Bao, CJ. Cao, S. Fischetti, C. Keeler, V. Hubeny, N. Jokela, A. Pönni...

Recent advances in inverse problem for nonlinear equations in general:

- Kurylev, Lassas & Uhlmann (2018), inverse problem for $\square_g u(x,t) + q(x,t)u^2(x,t) = 0$.
- A. Feizmohammadi & L. Oksanen and Lassas, T.L, Y-H. Lin & M. Salo (2019), inverse problems for $\Delta_g u + qu^m = 0$, $m \geq 2$.
- Other recent results for nonlinear *elliptic* by K. Krupchyk, T. Zhou, Y. Kian, R-Y. Lai, H. Liu, L. Tzou, S. Lu, B. Harrach, T. Tyni, L. Potenciano-Machado...

Proof of main theorem

How to recover an embedded minimal surface from the DN map (1)

The recovery is based on the higher order linearization method: Consider $f_j \in C^\infty(\partial\Sigma)$, $j = 1, 2, 3, 4$ and denote by $u = u_{\varepsilon_1 f_1 + \cdots + \varepsilon_4 f_4}$ the solution to the minimal surface equation with boundary data $\varepsilon_1 f_1 + \cdots + \varepsilon_4 f_4$. Denote $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_4)$.

By taking the derivative $\partial_{\varepsilon_j}|_{\varepsilon=0}$ of the solution $u_{\varepsilon_1 f_1 + \cdots + \varepsilon_4 f_4}$, we see that the function

$$v^j := \frac{\partial}{\partial \varepsilon_j}\Big|_{\varepsilon=0} u_{\varepsilon_1 f_1 + \cdots + \varepsilon_4 f_4}$$

solves the first linearized equation

$$\Delta_g v + qv = 0 \text{ on } \Sigma, \tag{1}$$

where both g and q are *unknown*. The DN map of (1) is known from the DN map of the minimal surface equation by differentiation. Thus we can determine (Σ, g) and q up to a conformal mapping using the announced result about the general Calderón problem.

- In fact while q is unknown, it is also $-|\eta|^2 - \mathrm{Ric}(N, N)$. Thus, will obtain information about ambient 3D space.

Still need to recover the conformal factor.

How to recover an embedded minimal surface from the DN map (2)

Let's proceed to find the conformal factor and 2nd fundamental form $\eta(X,Y) = \langle \nabla_X N, Y\rangle_{\overline{g}}$.

- The conformal factor will be found only from the third linearization. We must first solve the second linearized problem to find (a conformal multiple) of the 2×2 matrix field η.

The function $w^{jk} := \frac{\partial^2}{\partial\varepsilon_j\partial\varepsilon_k}\Big|_{\varepsilon=0} u_{\varepsilon_1 f_1+\cdots+\varepsilon_4 f_4}$ satisfies the second linearized equation

$$(\Delta_g + q)w^{jk} = \text{terms of the form } \eta(\nabla v^j, \nabla v^k) + \text{lower order terms.}$$

- The lower order terms are terms contain at most one gradient of a linearized solution v^j. They will be negligible in the "first layer" of the asymptotic analysis to follow.

Since we know the Cauchy data of w^{jk}, from the DN map of the minimal surface equation, it follows that the integral

$$\int_\Sigma v^1\eta(\nabla v^2,\nabla v^3)dV + \int_\Sigma v^2\eta(\nabla v^1,\nabla v^3)dV + \int_\Sigma v^3\eta(\nabla v^1,\nabla v^2)dV$$

$$+ \text{lower order terms}$$

is known.

How to recover an embedded minimal surface from the DN map (3)

We recover the matrix field η (up to a conformal factor) next from the integral quantity above. This is done by choosing special CGO solutions for the linearized equation $(\Delta_g + q)v = 0$.

To recover η, we use as functions v^k, solving $(\Delta_g + q)v^k = 0$, the CGOs constructed by C. Guillarmou and L. Tzou (2011, GAFA) of the form

$$e^{\Phi/h}(a + r_h),$$

where $\Phi = \phi + i\psi$ is a holomorphic Morse function, h small, a is a holomorphic function and r_h is a correction term given by

$$r_h = -\overline{\partial}_\psi^{-1} \sum_{j=0}^{\infty} T_h^j \overline{\partial}_\psi^{*-1}(qa),$$

where $\overline{\partial}_\psi^{-1}$ is defined (modulo localization) by $\overline{\partial}_\psi^{-1} f = \overline{\partial}^{-1}(e^{-2i\psi/h} f)$, where $\overline{\partial}^{-1}$ is the Cauchy-Riemann operator that solves $\overline{\partial}^{-1}\overline{\partial} = \mathrm{Id}$.

- The form of T_h is not important, but it is important to note that the h-dependence of r_h is quite complicated and especially not polynomial: $r_h \neq h a_1 + h^2 a_2 + \cdots + O_{H^k}(h^R)$.

How to recover an embedded minimal surface from the DN map (4)

Plugging in the CGOs $v^k = e^{\Phi_k/h}(a_k + r_k)$, $k = 1, 2, 3$, to the known integral quantity

$$\int_\Sigma v^1 \eta(\nabla v^2, \nabla v^3)\, dV + \int_\Sigma v^2 \eta(\nabla v^1, \nabla v^3)\, dV + \int_\Sigma v^3 \eta(\nabla v^1, \nabla v^2)\, dV + \cdots,$$

yields the term

$$I_{\text{leading}} = \int_\Sigma \hat{v}^1 \eta(\nabla \hat{v}^2, \nabla \hat{v}^3)\, dV + \int_\Sigma \hat{v}^2 \eta(\nabla \hat{v}^1, \nabla \hat{v}^3)\, dV + \int_\Sigma \hat{v}^3 \eta(\nabla \hat{v}^1, \nabla \hat{v}^2)\, dV,$$

where $\hat{v}^k = e^{\Phi_k/h} a_k$, and another term

$$I_{\text{other}} = \text{ other terms that contain integrals of products of } e^{\Phi_k/h} r_k \text{ and their gradients.}$$

We wish to use stationary phase for I_{leading} to recover η and consider I_{other} as negligible.

Stationary phase yields $I_{\text{leading}} = c_0 + O(h) = O(1)$, $c_0 \neq 0$, as $h \to 0$ while the usual L^p estimates $\|r_k\|_{L^p}, \|\nabla r_k\|_{L^p} = O(h^{1/p})$ yield $I_{\text{other}} \sim O(h^{-1})$. Thus

$$I_{\text{other}} \sim O(h^{-1}) > O(1) = I_{\text{leading}}, \quad h \to 0$$

by using L^p estimates. A problem.

A solution: CGOs with phases without critical points

The CGOs we have used so far have phases that have (non-degenerate) critical points. Next we employ the fact that our known integral terms, such as

$$\int_\Sigma v^1 \eta(\nabla v^2, \nabla v^3)\, dV$$

consider products of 3 solutions. For the 3 CGO solutions v_1, v_2, v_3 we choose phases

$$\Phi_1(z) = -z + z^2 + O(|z|^3), \quad \Phi_2(z) = z + z^2 + O(|z|^3), \quad \Phi_3(z) = 2\overline{z}^2 + O(|z|^3).$$

With these phases, the corresponding correction terms r_1 and r_2 have the better decay:

$$\|r_1\|_{L^p}, \|r_2\|_{L^p}, \|\nabla r_1\|_{L^p}, \|\nabla r_2\|_{L^p} = O(h)$$

instead of $O(h^{1/p})$. The reason why such solutions work in our case is that the product of these 3 CGOs have a phase

$$\Phi_1(z) + \Phi_2(z) + \Phi_3(z) = 2z^2 - 2\overline{z}^2,$$

which is still of the stationary phase form.

End of proof

By using the CGOs with phases without critical points, we obtain improved the estimate $I_{\text{other}} = o(1)$. Thus $I_{\text{other}} < O(1) = I_{\text{leading}}$ and we are able to recover a conformal multiple of the second fundamental form η.

- We continue to "lower layers" in the asymptotic analysis in $h \to 0$ to recover the remaining quantities appearing in the second linearization.
- From the third order linearization we finally recover the conformal factor by similar methods as we used for the second order linearization.
- We have recovered the embedded minimal surface (Σ, g) up to an isometry.

Key points from the proof:

1. One has to work down with the linearizations to find the conformal factor.
2. Because the L^p estimates for the correction terms of the standard CGOs were not enough, we introduced new CGOs whose correction terms had sufficient decay.

Back to mathematics of AdS/CFT duality

The geometrical situation of AdS/CFT is in the noncompact setting, where the Riemannian manifold is typically asymptotically hyperbolic.

- The conformal metric blows up when approaching the boundary in a specific way.
- Notable contributions to study geometry and scattering in this setting by R. Graham, M. Zworski, S.Y. A. Chang, J. Lee, S. Alexakis, R. Mazzeo, R. Melrose, M. Anderson...

To approach the generalized boundary rigidity in the above noncompact settings one needs to understand (for starters):

- Renormalized volumes of minimal surfaces embedded in asymptotically hyperbolic spaces.
- Scattering map on embedded minimal surfaces in the above case, a la Graham-Zworski "Scattering matrix in conformal geometry (2001)".
- These are geometric and forward problems.

Summary

1. Recovery of a *general* 2D minimal surface from the DN map of minimal surface equation.
   - Equivalently, a minimal surface is determined by areas of nearby minimal surfaces.
   - Also recovered second fundamental form; information about the ambient 3D space.
2. Announced a solution to the *general* Calderón problem on Riemannian surfaces where both the metric and the potential are unknown.
   - The solution was used in solving the inverse problem minimal surfaces.
3. Introduction of the higher order linearization method to inverse problems with rigidity type data, and to the AdS/CFT correspondence in physics.
   - Higher order variations of areas of minimal surfaces.
   - Entanglement entropies of CFT determine the DN map of minimal surface equation.
4. CGOs with phases without critical points in 2D.
   - Useful for inverse problems of other nonlinear equations in 2D as well.

Slides are available at `https://www.mv.helsinki.fi/home/tjliimat/`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