Linearized Calderón problem on cylindrical Riemannian manifolds and exponentially accurate solutions

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Outline

Introduction

Anisotropic Calderón problem

If (M, g) is a Riemannian manifold with boundary, $n \geq 3$, the anisotropic Calderón problem amounts to determining (M, g) from the knowledge of the DN-map up to a diffeomorphism.

- Let $w = w_f$ solve the Riemannian Laplace equation

$$\Delta_g w = 0 \text{ in } M, \qquad w|_{\partial M} = f.$$

- The DN map is defined as

$$\Lambda : C^\infty(\partial M) \to C^\infty(\partial M), \quad \Lambda(f) = \partial_\nu w|_{\partial M}$$

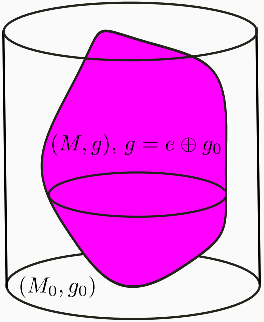
We study today a variant of the problem of determination of an unknown potential q on a known (M, g) from the DN map of the Shrödinger equation

$$\Delta_g u + qu = 0 \text{ in } M, \qquad u|_{\partial M} = f.$$

Cylindrical manifolds

Determination of q is open on a general Riemannian manifold (M, g), but considerable progress has been made in the class of transversally anisotropic (TA) manifolds.

- Transversally anisotropic manifolds are cylindrical Riemannian manifolds, $M \subset \mathbb{R} \times M_0$. Here (M_0, g_0) is called the transversal manifold.



- These manifolds have an analogue of complex geometric optics solutions, which I will explain.

Earlier results

Results about determining the potential q on TA manifolds include:

- DS Ferreira, Kenig, Salo and Uhlmann (2009) initiated the study on TA manifolds. Uniqueness of the unknown q if (M_0, g_0) is simple. Roughly speaking manifold is simple if its points are connected by unique geodesics.
- DS Ferreira, Kurylev, Lassas, Salo (2014): Uniqueness of q if (M_0, g_0) has invertible geodesic ray transformation. It is not completely known when it is possible to invert the geodesic ray transformation on a general Riemannian manifold. There are also counterexamples.
- DS Ferreira, Kurylev, Lassas, L, Salo (2018) studied singularities of q under a different geometric conditions on (M_0, g_0). No assumption on the invertibility of geodesic ray transformation/simple manifold. Uses solutions corresponding to intersecting geodesics, which I will explain.

The linearized problem of determining q from the DN map of $\Delta_g u + qu = 0$ amounts to showing that if

$$\int_M f u_1 u_2 \, dV = 0,$$

for all u_j with $\Delta_g u_j = 0$ in M, then $f \equiv 0$.

Definition (The geometric condition)

Let $\gamma_1, \gamma_2 : [a,b] \to M_0$ be geodesics that intersect the boundary transversally.

- A vector $(z_0, \xi_0) \in TM_0$ is said to be generated by the pair (γ_1, γ_2) of geodesics if γ_1 and γ_2 intersect at z_0 and

$$\dot\gamma_1(0) + \dot\gamma_2(0) = \xi_0.$$

- A pair of (γ_1, γ_2) of geodesics is admissible if they generate (z_0, ξ_0) and only intersect at z_0.

The result

Let $f = f(x_1, x')$ with $(x_1, x') \in \mathbb{R} \times M_0$ and let $\hat{f}(\lambda, \cdot\,)$ denote the partial Fourier transformation of f in the x_1 variable at a frequency $\lambda \in \mathbb{R}$.

Theorem (KK, L, MS 2020)
Let (M, g) be a transversally anisotropic manifold with (M_0, g_0) analytic. Assume that $f \in L^\infty(M)$ satisfies

$$\int_M f u_1 u_2 \, dV_g = 0, \tag{1}$$

for all $u_j \in L^2(M)$ with $\Delta_g u_j = 0$ in M^{int}. Let $(z_0, \xi_0) \in TM_0^{\mathrm{int}}$ be generated by an admissible pair of geodesics. Then for any $\lambda \in \mathbb{R}$, one has

$$(z_0, \xi_0) \notin WF_a(\hat{f}(\lambda, \cdot\,)) \subset TM_0^{\mathrm{int}} \setminus \{0\}.$$

Corollary (KK, L, MS 2020)
If each tangent vector $(z_0, \xi_0) \in TM_0^{\mathrm{int}}$ of the transversal manifold is generated by an admissible pair of geodesics, then the linearized problem is solvable, $f \equiv 0$.

Gaussian beams with exponentially small error

Sketch of the proof of the main theorem

The proof consists of 2 major observations. We denote $(x_1, x') \in \mathbb{R} \times M_0$.

1. Due to the geometry of $M \subset \mathbb{R} \times M_0$, we are able to construct an analogue of complex geometric optics (CGOs) with an exponentially small correction term.
   - Here a CGO on a cylindrical manifold solves $\Delta u = 0$ and is of the form $u = e^{sx_1}(v_s(x') + r)$, where $r = r(x_1, x')$ a correction term and s large.
   - As (M_0, g_0) is analytic we are able to show that r can be taken to be exponentially small in s.
2. We observed in our earlier work that the identity $\int_M f u_1 u_2 \, dV_g = 0$ for certain CGOs leads to an FBI-type transform of the unknown f.
   - Since the error term r is exponentially small, the FBI-transform allows to study analytic wave front set of f.
   - Under the assumptions of the theorem about admissible geodesics, it follows that $\hat{f}(\lambda, \cdot) \in L^\infty(M_0)$ is analytic. Thus $\hat{f}(\lambda, \cdot)$ can be recovered from boundary determination. This recovers f.

CGOs and Gaussian beams

The proof of our new results crucially depends on properties of Gaussian beams. As said a CGO on a transversally anisotropic manifolds reads:

$$u = e^{-sx_1}(v(x') + r).$$

Here we take $v = v_s$ to be a Gaussian beam, which is an approximate eigenfunction on (M_0, g_0) of energy s^2. Here $s = \tau + i\lambda$, τ large, and r is a correction term.

- Compare with CGOs in $\mathbb{R}^n$: $u = e^{-\tau\eta\cdot x}(e^{i\tau\xi\cdot x} + r)$, $\eta \perp \xi$, where $e^{i\tau\xi\cdot x}$ is an eigenfunction of energy τ^2 on a hyperplane orthogonal to η.

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A Gaussian beam v is a special approximate eigenfunction on (M_0, g_0) which concentrates near a given geodesic $\gamma(t)$. If (t, y) are coordinates near γ with y coordinates transversal to γ, then

$$v \sim e^{s\varphi}(a_0 + a_{-1}/\tau + \cdots), \quad s = \tau + i\lambda$$

where

$$\varphi \approx it - a|y|^2, \quad a > 0.$$

Exponentially accurate Gaussian beams on the transversal manifold

Theorem (KK, L, MS (2020))
Let (M_0, g_0) be analytic compact Riemannian manifold with boundary. Let γ be a non-tangential geodesic, and let $\lambda \in \mathbb{R}$ and $s = \tau + i\lambda$. There is a family of functions $v(x;s)$ on M_0 and $c > 0$ such that $\operatorname{supp}(v(\,\cdot\,;s))$ is supported near γ and

$$\|(-\Delta_g - s^2)v\|_{L^2(M_0)} = \mathcal{O}(e^{-c\tau})$$

as $\tau \to \infty$. Let $p \in$ graph(γ) and let $t_1 < \cdots < t_{N_p}$ be the times when $\gamma(t_l) = p$. Near p, we have $v| = v^{(1)} + \cdots + v^{(N_p)}$, where each $v^{(l)}$ has the form

$$v^{(l)}(x;s) = e^{is\varphi(x)}a(x;\tau),$$

$$\varphi(\gamma(t)) = t, \quad \nabla\varphi(\gamma(t)) = \dot{\gamma}(t), \quad \mathit{Im}\,(\nabla^2\varphi(\gamma(t))) \geq 0, \quad \mathit{Im}\,(\nabla^2\varphi)|_{\dot{\gamma}(t)^\perp} > 0,$$

where $\varphi = \varphi^{(l)}$ analytic and $a^{(l)}$ an elliptic classical analytic symbol (will explain later).

Constructing the Gaussian beams

Gaussian beams $v = v_s$, which are locally sums of functions of the form

$$v(x; s) = e^{is\varphi(x)} a(x; \tau),$$

are constructed in two steps. We have (set $s = \tau$ here for simplicity)

$$e^{-i\tau\varphi}(-\Delta_g - \tau^2)e^{i\tau\varphi} a = \tau^2[\langle d\varphi, d\varphi\rangle_g - 1]a - i\tau[2\langle d\varphi, da\rangle_g + (\Delta_g\varphi)a] - \Delta_g a.$$

1. First, find (an analytic) φ solving the eikonal equation $\langle d\varphi, d\varphi\rangle_g = 1$.
2. After finding φ, insert φ to the transport equation,

$$i\tau(L + \Delta\varphi)a + \Delta a = 0,$$

where L is the vector field $2\nabla\varphi \cdot \nabla$, to find

$$a = a_0 + a_1/\tau + a_2/\tau^2 + \cdots .$$

Constructing a solution to the eikonal equation

Often eikonal equation is solved only approximatively by a Taylor expansion argument that e.g. leads to a Riccati equation. We instead use the machinery of Lagrangian manifolds. We view $\langle d\varphi, d\varphi\rangle_g = 1$ as the Hamilton-Jacobi equation for

$$p(x,\xi) := |\xi|_g^2 - 1 = g(x)\xi\cdot\xi - 1$$

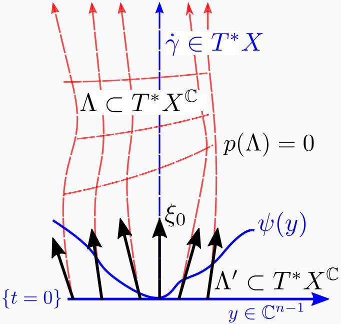
given as

$$\begin{cases} p(x, d\varphi(x)) = 0, \quad x = (t,y), \\ \varphi(0,y) = \psi(y), \\ d\varphi(0,0) = \xi_0. \end{cases}$$

- A nonlinear PDE for φ in the coordinates $x = (t,y)$ on a neighborhood of γ.
- The set $\{(0,y)\}$ is the initial data surface of dimension $\dim(M_0) - 1$. We specify initial conditions by choosing ξ_0 and $\psi(y)$.
- A solution by constructing a Lagrangian submanifold of T^*M_0.

Solving the Hamilton-Jacobi equation. Slide 1

- Let (t, y) coordinates on a neighborhood of the geodesic γ, $t \in \mathbb{R}$ and $y \in \mathbb{R}^{n-1}$. The tangent bundle is locally parametrized by (t, y, τ, η). We define the **black arrows**. (X here is M_0.)
- We set $\xi_0 := \dot\gamma(0)$ and thus $p(\xi_0) = 0$. By the implicit function theorem there is $\tau = \lambda(y, \eta)$ such that $p(0, y, \lambda(y, \eta), \eta) = 0$.
- To implement the initial condition $\varphi(0, y) = \psi(y)$, we also choose $\eta = \psi'_y(y)$.



Solving the Hamilton-Jacobi equation. Slide 2

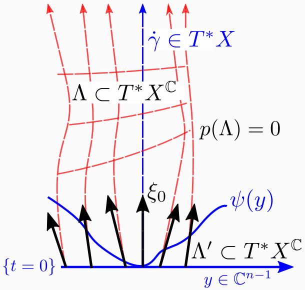
- Let us define the **black arrows** as the set

$$\Lambda' := \{(0, y, \tau, \eta) : \eta = \psi'_y(y), \tau = \lambda(y, \eta), y \in \mathsf{neigh}(0, \mathbb{C}^{n-1})\} \subset \mathbb{C}^{2n}.$$

- Let H_p be the Hamiltonian vector field $TX \to TTX$ of $p = p(x, \xi) = |\xi|_g^2 - 1$. The integral curves of the flow of H_p are geodesics (considered as curves in TX).
- Since $p(\Lambda') = 0$ and the flow of H_p preserves p, we have that

$$\Lambda := \mathsf{Flow}_{H_p}(\Lambda') \text{ satisfies } p(\Lambda) = 0.$$

Red curves are $\Lambda \subset TX$.



Solving the Hamilton-Jacobi equation. Slide 3

- Now Λ by constriction is a (complex) Lagrangian submanifold of $T^*X^{\mathbb{C}}$. Thus it can be parametrized by a function $\varphi : U \to \mathbb{C}$, U open in X, as

$$\Lambda|_U = \{(x, d\varphi(x)) : x \in U\} \subset T^*X^{\mathbb{C}}.$$

- Since $p(\Lambda) = 0$, we have $p(x, d\varphi(x)) = \langle d\varphi, d\varphi \rangle_g - 1 = 0$. Thus the parametrizing function φ solves the eikonal equation!

This is a local construction and it must be patched to have a global solution on a neighborhood of the original geodesic γ.

- To have a global solution, we choose the initial data $\psi(y)$ so that $\text{Im}(\text{Hess}_y \psi) > 0$.
- With this choice, we may use the theory of *positive Lagrangians* to have a global solution.
  - Analogous to choosing initial data with positive imaginary part for the Riccati equation and showing that this condition is preserved by the solution.

Finding the amplitude of the Gaussian beams

- Now that we have found phase function φ for the Gaussian beam ansatz $e^{is\varphi(x)}a(x;\tau)$, we insert φ to the transport equation

$$i\tau(L+\Delta\varphi)a+\Delta a=0,$$

where $La = 2\langle d\varphi, da\rangle_g = 2\nabla\varphi\cdot\nabla a$.

- We look for a classical analytic symbol $a(x;\tau)=\sum_{k=0}^{N}\tau^{-k}a_k(x)$: A symbol a is classical analytic if

$$|a_k(x)|\leq C^{k+1}k^k, \quad a_k \text{ analytic.} \tag{2}$$

- By choosing flowout coordinates such that L corresponds to ∂_r, we may integrate the transport equation.
- To show that the integration produces a classical analytic symbol a near γ, need to use a method of nested neighborhoods. Goes back to Sjöstrand and is non-trivial.

Completing the Gaussian beam construction

Now that we have both the phase and amplitude functions, we may study how large is the error for $e^{is\varphi(x)}a(x;\tau)$ being a true eigenfunction of Δ of energy s^2. (Again $s=\tau$ for simplicity.)

- We let N depend on τ. That is $N=N(\tau)\to\infty$ as $\tau\to\infty$.
- Then by our construction, we have

$$e^{-i\tau\varphi}(-\Delta_g-\tau^2)e^{i\tau\varphi}\left(\sum_{j=0}^{N}\tau^{-j}a_j\right)=\tau^{-N}\Delta a_N.$$

- Since a is classical analytic, we have $|\Delta a_N|\le C\sup_x|a_N(x)|$ by Cauchy estimates. Also, $|a_N(x)|\le C^{N+1}N^N$ and RHS is bounded by $\tau^{-N}C^{N+1}N^N$.
- Set $N(\tau)=\frac{\tau}{eC}$ to finally have by direct substitution

$$|\tau^{-N}\Delta a_N|\le Ce^{-\frac{\tau}{eC}}.$$

Proof of the main theorem

Choosing Gaussian beams

Recall, $s = \tau + i\lambda$. For large τ and fixed λ, the Gaussian beam looks like

$$v \approx e^{i\tau t - a\tau|y|^2}, \quad a > 0.$$

- Concentrates strongly near the geodesic γ (good), but oscillates along γ (bad).

If we plug in the CGO solutions $e^{sx_1}(v(x') + r)$ in the linearized problem, we get

$$\int_{M_0} \hat{f}(2\lambda, x')v_1(x')v_2(x')\,dx' = \mathcal{O}(e^{-c\tau}), \quad c > 0.$$

We are free to choose geodesics γ and η corresponding to Gaussian beams v_1 and v_2.

Remark:

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Remark:

- If we would choose $\eta(t) = \gamma(-t)$, the oscillations would cancel. We get by $\tau \to \infty$

$$0 = \int_\gamma \hat{f}(2\lambda, \gamma(t))e^{2\lambda t} dt.$$

- Reduces to inverting a geodesic ray transformation as observed by DS Ferreira et al.

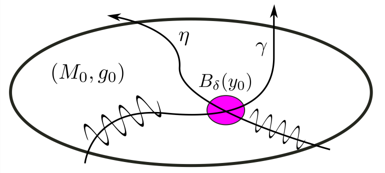
Intersecting Gaussian beams

We instead choose Gaussian beams corresponding to intersecting geodesics γ and η yielding:

$$v_1(x')v_2(x') \approx e^{is\xi_0 \cdot x - as|x|^2}$$

where

$$\xi_0 = \dot{\gamma}(0) + \dot{\eta}(0), \quad s = \tau + i\lambda, \quad a > 0.$$



- Provides excellent concentration to the intersection point, but oscillation still kills information by non-stationary phase.

FBI-transformation

The FBI-transformation of a function f on $\mathbb{R}^n$ is given by

$$(\mathsf{FBI}_b f)(\xi) = \int_{\mathbb{R}^n} f(x) e^{i\xi\cdot x - b|x|^2} dx, \quad b > 0, \quad \xi \in \mathbb{R}^n.$$

- Originally used by Bros and Iagolnitzer to study the real analytic wave front set.
- The kernel of the FBI-transformation looks almost the same as $v_1(x')v_2(x') \approx e^{is\xi_0\cdot x - as|x|^2}$, if we choose $s\xi_0 = \xi$ and set $as = b$.

Fact: Let $\xi_0 \in \mathbb{R}^n$. Assume that there is a neighborhood $U \subset \mathbb{R}^n \times \mathbb{R}^n$ of ξ_0 such that for all $\xi \in U$ the FBI-transformation of a function f

$$(\mathsf{FBI}_b f)(\xi) = \mathcal{O}(e^{-C_1|\xi|}).$$

Then $\xi_0 \notin WF_a(f)$.

End of the proof of the main theorem

Recall that $(x_1, x') \in \mathbb{R} \times M_0$. Combining things, we have

$$\int_{M_0} \hat{f}(2\lambda, x') e^{is\xi_0 \cdot x' - as|x'|^2} A \, dx' = \int_{M_0} \hat{f}(2\lambda, x') v_1(x') v_2(x') \, dx' = \mathcal{O}(e^{-c\tau}), \qquad (3)$$

where $A(x') > 0$. This holds in a neighborhood $U \subset TM$ of $\xi_0 \in T_{z_0}M$.

- It follows that $\xi_0 \notin WF_a(\hat{f}(2\lambda, \cdot\,))$.
- If any $\xi_0 \in TM_0$ is generated by an admissible pair of geodesics then

  $$WF_a(\hat{f}(2\lambda, \cdot\,)) = \emptyset$$

  and thus $\hat{f}(2\lambda, \cdot\,)$ is analytic in M_0.
- Recall that in the uniqueness proof in the linearized Calderón problem $f = q_1 - q_2$.
- By boundary determination Taylor series of f vanishes on ∂M_0. Thus $\hat{f}(2\lambda, \cdot\,) \equiv 0$ for all $\lambda \in \mathbb{R}$.
- By inverting the Fourier transformation in the flat x_1-direction, $f \equiv 0$. Thus $q_1 = q_2$. QED

Summary

Summary

- A new result on the linearized Calderón problem on transversally anisotropic (cylindrical) manifolds.
  - The known geometry is assumed analytic, while the unknown $q \in L^\infty$.
- The proof was based on new method, which uses: exponentially accurate quasimodes corresponding to intersecting geodesics and the FBI transformation.
  - The method does not involve inverting an integral transformation.
  - The phase function of a quasimode by a Lagrangian submanifold argument.
  - All vectors of TM_0 need to be generated by an admissible pair of geodesics (didn't discuss much). In a corresponding inverse problem for a nonlinear Shrödinger equation, the admissibility condition can be removed.
- The proof is not completely satisfactory in the sense that even though $q \in L^\infty$, the reconstruction is by analytic continuation from the boundary.
  - Stability?
  - Different ways to use exponentially accurate solutions?

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Kiitos!