

A Beginner's Field Guide to Minimal Surfaces

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1 What is a Minimal Surface?

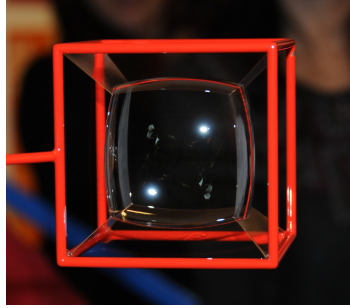


Figure 1: Soap film on a cubical frame [13].

A minimal surface is one which *locally minimizes the area of a surface given a boundary*. The first minimal surfaces which were explicitly defined were the plane, the catenoid, and the helicoid, see Figures 2 and 3. Because of this minimizing property, minimal surfaces can often be spotted in nature, the most well known examples being soap films on wire frames. A minimal surface is defined quantitatively by the following:

Definition 1.1. A minimal surface $\mathcal{M}$ is such that it's mean curvature H is equivalently zero on $\mathcal{M}$.

1.1 Properties of Minimal Surfaces

Let $S \subset \mathbb{R}^3$ be a surface so that at each point $p \in S$ there is a neighborhood $U_p \subset S$ so that U_p is the graph of a map $f = f(x, y)$. We have that the mean curvature, H , of S satisfies

$$H = \frac{f_{xx}(1 + f_y^2) - 2f_x f_y f_{xy} + f_{yy}(1 + f_x^2)}{2(1 + f_x^2 + f_y^2)^{3/2}} \quad (1)$$

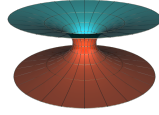


Figure 2: The first nontrivial minimal surface found was a catenoid by Euler in 1741. This is the unique minimal surface of revolution, aside from the plane. Source: [18].

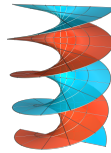


Figure 3: In 1776 Meusnier discovered the helicoid. This is the unique minimal ruled surface, aside from the plane. Source: [18].

(see Problem 2.2.1 in [2]). If S is a minimal surface, setting $H = 0$ gives us Lagrange's equation, which was found by Lagrange in 1762:

$$f_{xx}(1 + f_y^2) - 2f_x f_y f_{xy} + f_{yy}(1 + f_x^2) = 0. \quad (2)$$

We can express equation 2 in terms of the divergence of f :

$$\operatorname{div} \left(\frac{\nabla f}{\sqrt{1 + |\nabla f|^2}} \right) = 0. \quad (3)$$

Using this partial differential equation we can equivalently say that a surface $\mathcal{M} \subset \mathbb{R}^3$ is minimal if and only if for each $p \in \mathcal{M}$, there exists a neighborhood U_p and a function f mapping to $U_p \subset \mathcal{M}$ satisfying 3. If $|\nabla f|$ is very small, then the solution f to 3 looks like a solution to the Laplace equation,

$$\Delta f = 0. \quad (4)$$

The above remark shows that minimal surfaces locally inherit properties of harmonic functions. This is used to prove the following theorem.

Theorem 1.1. *There are no compact minimal surfaces without boundary*

So any closed genus k surface is not minimal. We mentioned how minimal surfaces *locally* minimize area. What about globally?

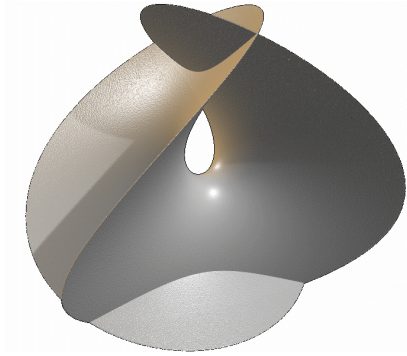


Figure 4: The Enneper surface in $\mathbb{R}^3$. Source: [19]

Theorem 1.2 (Bernstein's Theorem). *If the function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ satisfies 3 on all of $\mathbb{R}^2$, i.e., it globally minimizes area on $\mathbb{R}^2$, then f must be of the form $f(x, y) = ax + by + c$.*

For a proof of Bernstein's theorem, see [9]. We end this section with a statement highlighting the unique properties of the catenoid and the helicoid.

Theorem 1.3. *If $\mathcal{M}$ is a **surface of revolution** that is also minimal, then $\mathcal{M}$ is contained in a plane or a catenoid (Euler). If $\mathcal{N}$ is a **ruled surface** that is also minimal, then $\mathcal{N}$ is contained in a plane or a helicoid (Catalan).*

2 Topics of Interest

2.1 Plateau's Problem

Let $\mathcal{C}$ be a collection of surfaces. In its original and most simple form, Plateau's problem seeks a surface $S \in \mathcal{C}$ which minimizes an "area" function $F : \mathcal{C} \rightarrow \mathbb{R}$, given certain boundary constraints. This problem was first posed by Lagrange in the 1760's. Lebesgue was the first who referred to it as "Plateau's Problem". Plateau is a Belgian physicist and mathematician from the 1800's who is known for his physical studies of soap films on wire frames. One conclusion of his many experiments is that any simple closed curve, i.e., a Jordan curve, bounds at least one soap film. The existence of such a minimal surface for any Jordan curve in $\mathbb{R}^3$ was proved rigorously in the 1930's by Douglas and Rado. During the 1960's, Plateau's problem was extended to higher dimensional surfaces. Much research on Plateau's problem has been generated in the past century with many profound results, for more on the development of minimal surface theory see [4].

Theorem 2.1 (Meeks and Rosenberg). *Every simply connected, properly embedded minimal surface in $\mathbb{R}^3$ is either a plane or a helicoid.*

The above theorem is central to classical minimal surface theory, and may be surprising given we've only thus far discussed surfaces which are fairly nice: the plane, helicoid, and catenoid. But these alone are not an accurate representation of minimal surfaces. For example, the Enneper surface intersects itself in $\mathbb{R}^3$. See [4] for further discussion on pathological soap films with carefully crafted boundaries.

We end this section with a few open problems (as far as is known by the author) in minimal surface theory.

Problem 2.1 (Meeks, 1978). Is every compact, minimal, connected surface bounded by two convex planar curves topologically an annulus?

Conjecture 2.1 (The Convex Curve conjecture, Meeks). Two convex Jordan curves in parallel planes cannot bound a compact minimal surface of positive genus.

Conjecture 2.2 (Liouville Conjecture, Meeks). If $\mathcal{M}$ is a connected, complete, properly embedded minimal surface, and $h : \mathcal{M} \rightarrow \mathbb{R}$ is a positive harmonic function, then h is constant.

2.2 The Double-Bubble Conjecture

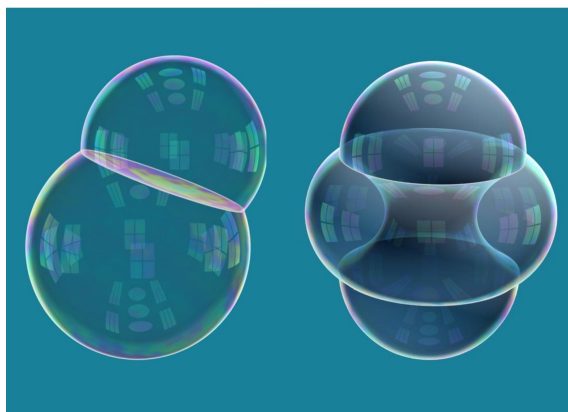


Figure 5: Left: the standard double bubble. Right: a nonstandard double bubble (images by John Sullivan, [16].)

Standard soap bubbles are not minimal surfaces as defined in 1.1, as spheres have a nonzero mean curvature. The pressure inside a soap bubble is higher than outside the bubble, and is given by a product of the mean curvature and the surface tension. But the shape a soap bubble forms takes the minimal surface area required to enclose a certain volume. It is a topic of interest as to how bubbles will connect in a stable way. This is of direct interest in biology, see [8].

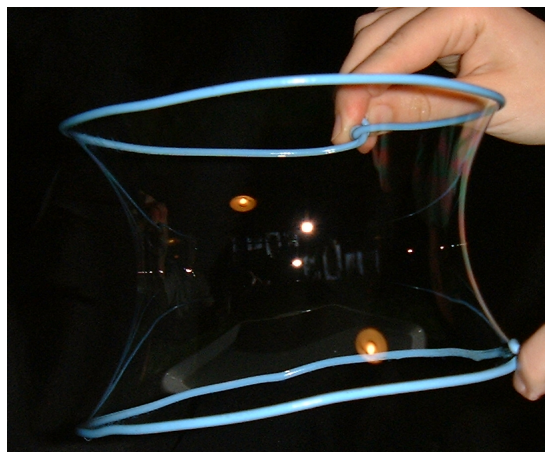


Figure 6: Is every compact, minimal, connected surface bounded by two convex planar curves topologically an annulus? The above surface created by the soap film with boundary composed of two circles is a catenoid. Source: [17].

Theorem 2.2 (Hutchings, Morgan, Ritoré, Ros, 2000). *In $\mathbb{R}^3$, the unique perimeter-minimizing double bubble enclosing and separating regions R_1 and R_2 of prescribed volumes v_1 and v_2 is a standard double bubble, which consists of three spherical caps meeting along a common circle at 120-degree angles. (For equal volumes, the middle cap is a flat disc.)*

The corresponding claim in $\mathbb{R}^2$ was proved in 1990 by the Williams College SMALL REU Geometry group, see [10]. The proof of the double bubble conjecture, [6], was done using stability arguments. For an overview of the proof with history, see [7]. The triple bubble conjecture has been resolved in $\mathbb{R}^2$ [11] but is still open in $\mathbb{R}^3$.

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