NOTES ON FILLABLE SURGERIES ON POSITIVE TORUS KNOTS

The purpose of this note is to provide a proof of the following theorem.

Theorem 1 (Etnyre-Min-Tosun-Varvarezos, [3]). *Let r be any negative rational number with continued fraction expansion as $r = [-d_1, -d_2, \cdots, -d_k]$ where $d_1 \geq 1, d_i \geq 2, 1 < i \leq k$. Then, up to isotopy, there are exactly $(pq - p - q + d_1)(d_2 - 1) \cdots (d_k - 1)$ symplectically fillable contact structures on $S^3_r(T_{p,q})$.*

Remark 2. This result when $r = -\frac{1}{n}$ and $(p, q) = (2, 3)$ was proved in [9, Theorem 1.9]

The following preliminary result, which was proved by the third author and Mark while working on [10], will play a key role during our proof.

Lemma 3. *If ξ be a fillable contact structure on $Y(-1; r_1, r_2, r_3)$ with $r_1 + r_2 + r_3 < 1$, then $t(\xi) < 0$.*

Remark 4. Let $0 < p^* < q$ denote the unique integer satisfying $pp^* \equiv 1 \pmod q$, and similarly $0 < q^* < p$ unique integer satisfying $qq^* \equiv 1 \pmod p$. Finally, let $r = \frac{a}{b}$ be a negative rational number. Using normalized invariants we can write $Y = S^3_{\frac{a}{b}}(T_{p,q})$ in the theorem, as a small Seifert fibered space, as $Y(-1; \frac{p-q^*}{p}, \frac{q-p^*}{q}, \frac{b}{bpq-a})$, and from which we observe easily that it satisfies the hypothesis on Seifert invariants in Lemma 3.

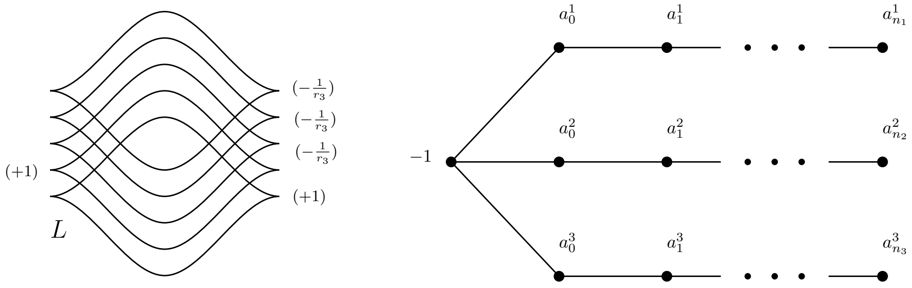

FIGURE 1. On the left, contact structures with maximal twisting number zero on $Y(-1; r_1, r_2, r_3)$. On the left, the plumbing of disk bundles according to star-shaped graph with boundary $Y(-1; r_1, r_2, r_3)$.

Proof. Consider the Seifert fibered space $Y = Y(-1; r_1, r_2, r_3)$, and let $e(Y) := r_1 + r_2 + r_3 - 1$. Let ξ be a tight contact structure on Y with $tw(\xi) = 0$. In [8, Proposition 6.1], it was proved that any such structure is given by one of the contact surgery diagram in Figure 1. We claim that contact structures described by these contact surgery diagrams, under the assumption of $r_1 + r_2 + r_3 < 1$, all have vanishing Ozváth-Szabó contact invariant c^+. In particular, we will derive $c^+(\xi) = 0$, and hence ξ is not filled by any symplectic manifold. To prove our claim, we start with setting some notation.

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For each of the rational number r_1, r_2, r_3 in $(0,1)$ we write

$$-\frac{1}{r_i} = [a_0^i, a_1^i, \cdots, a_{n_i}^i] = a_0^i - \cfrac{1}{a_1^i - \cfrac{1}{\ddots - \cfrac{1}{a_{n_i}^i}}}$$

for some uniquely determined integers $a_j^i \leq -2$.

Let $W = W(-1; r_1, r_2, r_3)$ denote the 4-manifold obtained by plumbing disk bundles according to a star-shaped graph as in Figure 1. It is a classical result, which can be found in [11, Theorem 5.2], that

$$b_2^+(W) = \begin{cases} 1 & \text{if } e(Y) > 0 \\ 0, & \text{if } e(Y) < 0 \end{cases} \tag{1}$$

Let (Y', ξ') denote contact manifold given by the contact surgery diagram in Figure 1 with the Legendrian knot L (the bottom component) is ignored–technically we are considering a family of contact structures here but which one we choose will not matter for our argument below. We make two observations. First contact structure ξ' for any choice of r_1, r_2, r_3 is Stein fillable. To see this note that the single contact (+1) surgery in the diagram results a Stein fillable contact manifold $(S^1 \times S^2, \xi_{std})$ as the Legendrian unknot bounds a regular Lagrangian equatorial disk in (B^4, w_{std}). Second, by using elementary Kirby calculus, one can easily see that the 3-manifold Y' is a small Seifert fibered space with $e_0 = 0$ – see discussion below and Figure 2 where the drawing in solid curves represents $-Y'$. Any such manifold is an L-space by a result of Lisca-Stipsicz [8, Theorem 1.1].

Finally, let $-X$ be the cobordism from (Y', ξ') to (Y, ξ) which is induced by the contact (+1) surgery on L in Figure 1. In a moment we will be interested the map induced by this cobordism, and its effect on the contact class. So, for usual orientation reasons, we switch the orientation and consider X as a cobordism from $-Y'$ to $-Y$. We prove now that the cobordism X has $b_2^+ = 1$ under the assumption $r_1 + r_2 + r_3 < 1$. To see this, we first convert the contact framings in Figure 1 to smooth framings, and after series of Kirby moves we obtain Figure 2 where the dotted curve is the knot L that is carried along. More precisely, we first switch the orientation of smooth manifold that underlies the contact surgery diagram, which amounts to changing the signs of the framings, then perform a blow up, followed by blow down of the $+1$ curve, Finally, applying three obvious negative Rolfsen twists gives $-Y'$, and the 2-handle attached along the dotted curve describes the cobordim X. This description shows immediately that X embeds in $\widehat{W} = W(-2; 1-r_1, 1-r_2, 1-r_3) \# \overline{\mathbb{C}P^2}$, and that $\partial \widehat{W} = Y(-2; 1-r_1, 1-r_2, 1-r_3) = -Y$. Since $e(-Y) = -e(Y) = 1 - (r_1 + r_2 + r_3) > 0$ whenever $r_1 + r_2 + r_3 < 1$, we conclude, by Equation 1, that $W(-2; 1-r_1, 1-r_2, 1-r_3)$ has $b_2^+ = 1$, and hence so does $\widehat{W}$. Finally, by simple Novikov formula we obtain then that $b_2^+(W) = 1$.

We now return to implications of these on maps induced on relevant Heegaard Floer groups. Since $b_2^+(X) > 0$, the induced map $F^\infty_{X,\mathfrak{s}} : HF^\infty(-Y', \mathfrak{s}_{|_{Y'}}) \to HF^\infty(-Y, \mathfrak{s}_{|_Y})$ is zero for all spin^c structures [12, Lemma 8.2]. Moreover since Y' is an L-space, the map $\pi : HF^\infty(-Y', \mathfrak{s}_{|_{Y'}}) \to HF^+(-Y', \mathfrak{s}_{|_{Y'}})$ is surjective. In other words $F^+_{X,\mathfrak{s}} \circ \pi = 0$. Using the surjectivity of π again, we obtain $F^+_{X,\mathfrak{s}} = 0$.

Finally, returning back to contact geometry, recall that (Y, ξ) is the result of a contact $(+1)$ surgery on (Y', ξ') for some ξ'. Hence, by [4, Lemma 2.11] there exists a (unique) spin^c structure $\mathfrak{t}$ on X such that $F^+_{X,\mathfrak{t}}(c^+(Y', \xi')) = c^+(Y, \xi)$. Since the map $F^+_{X,\mathfrak{t}} = 0$, we obtain that $c^+(Y, \xi) = 0$, which proves the initial claim above, and finishes the proof.

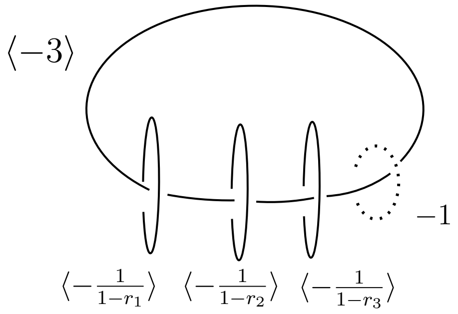

FIGURE 2. The cobordism X which is given as a 2-handle attachment along the dotted curve in $-Y'$.

□

Remark 5. We can also describe the surgered manifold $S^3_{\frac{a}{b}}(T_{p,q})$ by using the (unnormalized) Seifert invariants $(0, \frac{p}{q^*}, \frac{q}{q-p^*}, \frac{bpq-a}{b})$. From which the order of $H_1(S^3_{\frac{a}{b}}(T_{p,q}))$ is

$$q^*q(bpq-a) + (q-p^*)p(bpq-a) + pq(-b) = \pm a \text{ and hence } p^*p + q^*q - pq = 1 \tag{2}$$

All in place we proceed with the proof of the main theorem. Recall, we take $r = \frac{a}{b}$ with $a < 0$ and $\frac{a}{b} = [-d_1, -d_2, \cdots, -d_k]$ where $d_1 \geq 1$ and $d_i \geq 2$, $1 \leq i \leq k$.

Proof of Theorem 1. The proof will be completed in two steps:

0.1. **Lower Bound.** In this step we prove, by drawing explicit contact surgery diagrams, that there are at least $(pq - p - q + d_1)(d_2 + 1) \cdots (d_k + 1)$ many pairwise non-isotopic Stein fillable contact structures on $S^3_r(T_{p,q})$.

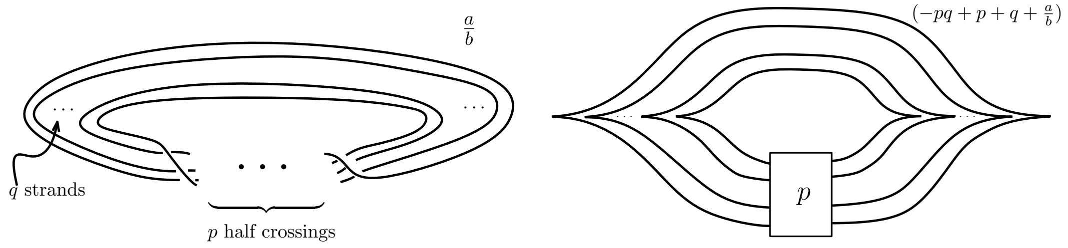

FIGURE 3. On the left is the $\frac{a}{b}$ surgery on the positive (p, q)–torus knot. On the right is contact $(-pq + p + q + \frac{a}{b})$ surgery on the Legendrian positive (p, q)–torus knot with $\mathrm{tb} = pq - p - q$.

The contact surgery diagram in Figure 3 can be converted, by using an algorithm due to Ding-Geiges-Stipsicz [1], to a contact surgery on a Legendrian link obtained as follows. First, note that the contact surgery coefficient $r' = (-pq + p + q + \frac{a}{b})$ has continued fraction expansion of $[-(pq-p-q) - d_1, -d_2, \cdots, -d_k]$. Starting with this input, we form the Legendrian link $L = L_1 \cup L_2 \cup \cdots L_k$ in the Ding-Geiges-Stipsicz algorithm as follows.

(1) L_1 is the max-tb Legendrian positive (p,q)–torus knot K that has been stabilized $(pq - p - q + d_1 - 1)$ times.
(2) L_i is the push off of L_{i-1}, $i = 2, \cdots, k$, that has been stabilized $(d_k - 2)$ times.

Then contact (r') surgery on K can be described as contact surgery on the Legendrian link L where the contact surgery coefficients are all -1 (that is we perform Legendrian surgery on L). Call this contact structure $\xi_{r'}$.

Now since Legendrian surgery preserves fillability, contact structure $\xi_{r'}$ is Stein fillable. On the other hand there is actually an ambiguity in the definition of contact structure $\xi_{r'}$. This results from the choice of stabilizations that are performed on each L_i above. But this will work to our advantage. Indeed, the choices of stabilizations yield $(pq - p - q + d_1)(d_2 - 1) \cdots (d_k - 1)$ many possible contact structures, and each of which is Stein fillable by the previous argument. Now, one data that keeps track of this different choices is the rotation vector: $\mathbf{rot}(L) = (\mathrm{rot}(L_1), \mathrm{rot}(L_2), \cdots, \mathrm{rot}(L_k))$. By a result of Eliashberg [2] and Gompf [5, Proposition 2.3] the first Chern class of the resulting Stein structure is captured by this rotation vector, and moreover by a result of Lisca-Matic [7] different rotation vectors yield different (i.e. non-isotopic) contact structures on the boundary. Indeed by a result of Plamenevskaya [13] even the corresponding contact classes are different. So, this completes the claim that there are at least $(pq-p-q+d_1)(d_2+1)\cdots(d_k+1)$ many non-isotopic Stein fillable contact structures on $S^3_r(T_{p,q})$ when r is negative.

0.2. **Upper Bound.** We now prove that there are at most $(pq-p-q+d_1)(d_2+1)\cdots(d_k+1)$ many pairwise non-isotopic fillable structures on $S^3_r(T_{p,q})$. More precisely, we will list all possible tight structures with negative maximal twisting and note that this number matches with $(pq-p-q+d_1)(d_2+1)\cdots(d_k+1)$.

This part of proof will follow some classic convex surface theory arguments. For the rest of the argument below Y will denote $S^3_r(T_{p,q})$.

If Y is equipped with a contact structure, we can isotope each singular fiber F_i to be Legendrian with twisting number $n_i < 0$, and take V_i to be its standard neighborhood with slope$(\Gamma_{\partial V_i}) = \frac{1}{n_i}$ – in particular with appropriate slope conventions, slope $\frac{1}{n_i}$ on ∂V_i corresponds to the vector $(n_i, 1)^T$.

After a small isotopy in the neighborhood V_i, we can make the ruling curves on $-\partial(M \setminus V_i)$ to have infinite slope; that is $(0,1)^T$ curves, and in short we will call such curves vertical. When measured with respect to $\partial(V_i)$, that is after applying A_i^{-1} to curves $(0,1)^T$, they become $\frac{p}{q}$, $\frac{q}{p}$ and $-\frac{bpq-a}{u}$. This information together with the Twist Number Lemma will help us to increase the twisting numbers n_i. To be able to find bypasses, and increase the twisting numbers we will often need to consider annuli between $-\partial(Y \setminus V_i)$ and $-\partial(Y \setminus V_j)$, $i \neq j = 1, 2, 3$, with boundary along vertical Legendrian curves. Such an annulus will be called vertical and can always be made convex.

When measured in $-\partial(Y \setminus V_i)$ slope$(\Gamma_{\partial V_i}) = \frac{1}{n_i}$ become

$$s_1 = \frac{q^* n_1 + p^* - q}{pn_1 - q}, \; s_2 = \frac{(p^* - q)n_2 + q^*}{qn_2 - p} \text{ and } s_3 = -\frac{bn_3 + v}{(bpq - a)n_3 + u}. \tag{3}$$

Let $\mathcal{A}$ be a vertical annulus between V_1 and V_2. Note that since $t(\xi) < 0$. Moreover $\partial\mathcal{A}$ intersects $\Gamma_{\partial(Y\setminus V_1)}$ and $\Gamma_{\partial(Y\setminus V_2)}$ exactly $2(pn_1 - q)$ and $2(qn_2 - p)$ times, respectively. In what follows we prove that by finding enough bypasses we can increase the twisting numbers (and hence thicken V_1 and V_2) up to $n_1 = n_2 = -1$ and prove that thickening beyond this will result all contact structures must have zero twisting. There are two cases.

Case 1: If $pn_1 - q \neq qn_2 - p$, the dividing set of $\mathcal{A}$, by the Imbalance Principle, has at least one boundary parallel arc, which bounds a bypass disk, on ∂V_1 and/or on ∂V_2 side. By attaching this bypass, we can increase the twisting numbers k_1, k_2 incrementally by the Twist Number Lemma [6]. As long as we remain under Case 1, we can continue this process and increase twisting numbers n_1, n_2 up to -1 because of our choice of ruling slopes and the Twist Number Lemma.

Case 2: If $pn_1 - q = qn_2 - p$ and the annulus $\mathcal{A}$ has no boundary parallel arcs, then we cut along $\mathcal{A}$ and round the corners by using the Edge Rounding Lemma [6, Lemma 3.11]. To this end, observe first that a neighborhood of $Y \setminus (V_1 \cup V_2 \cup \mathcal{A})$ is a solid torus with four edges. By rounding these edges we obtain a smooth torus $\partial(Y \setminus V_1 \cup V_2 \cup \mathcal{A})$ that is isotopic to a neighborhood of F_3, and identify this torus with $\mathbb{R}^2/\mathbb{Z}^2$ in the same way as $\partial(M \setminus V_3)$. We can use the Edge Rounding Lemma to compute the slope of the convex boundary: Each rounding between $-\partial(Y \setminus V_1)$ or $-\partial(Y \setminus V_2)$ and $\mathcal{A}$ changes the slope by an amount of $-\frac{1}{2(2(pn_1-q))}$. Since there are four edges to round we compute the slope on the $\partial(Y \setminus V_1 \cup V_2 \cup \mathcal{A})$ as

$$s(\Gamma_{\partial(Y\setminus V_1 \cup V_2 \cup \mathcal{A})}) = \frac{q^* n_1 + p^* - q}{pn_1 - q} + \frac{(p^* - q)n_2 + q^*}{qn_2 - p} - \frac{1}{pn_1 - q} = \frac{n_1 + 1 - q}{pqn_1 - q^2}.$$

In the calculation above we used the assumption that $\frac{pn_1 + p - q}{q} = n_2$ and the fact that $pp^* + qq^* - pq = 1$. Now, when measured in ∂V_3 we get.

$$s_{n_1} = A_3^{-1}(pqn_1 - q^2, -n_1 - 1 + q)^T = \frac{an_1 + (bpq - a)(q - 1) - bq^2}{(u - vpq)n_1 + vq^2 + u - uq}.$$

Note that $s_{n_1} < 0$ for all $n_1 < 0$ as we can easily show that its numerator is always positive and the denominator is always negative. Moreover, as a rational function, s_{n_1} is increasing as $n_1 \to -\infty$. So,

$$s_{n_1} = \frac{an_1 + (bpq - a)(q - 1) - bq^2}{(u - vpq)n_1 + vq^2 + u - uq} < \frac{a}{u - vpq}.$$

In particular we replace V_3 with a neighborhood $V_3' \subset V_3$ having slope $\frac{a}{u - vpq}$. When measured in $-\partial(Y \setminus V_3)$ we get a torus, after some simplifications and using the fact that $-(bpq - a)v + bu = 1$, parallel to $-\partial(\Sigma \times S^1)$ having slope.

$$s(\Gamma_{-\partial(Y\setminus V_3')}) = A_3(u - vpq, a)^T = -\frac{1}{pq}$$

We shall use this information to find thickenings of V_1 and V_2 with slopes corresponding to $n_1 = -1$ and $n_2 = -1$, respectively. Consider a vertical annulus $\mathcal{A}_{13}$ between V_1 and V_3. One can easily check that $|pn_1 - q| > pq$ whenever $n_1 < -q + 1$. So, there are bypasses on the V_1 side, attaching those we can increase, by using the Twist Number Lemma, the twisting number n_1 up to $n_1 = -q+1$ and get slope $s_1 = \frac{qq^*-p^*-q^*+q}{pq-p+q}$. Similarly, consider a vertical annulus $\mathcal{A}_{23}$ between V_2 and V_3, and let m be the smallest natural number such that $qm > p$. Then $|qn_2 - p| > pq$ whenever $n_2 < -p + m$. So we can increase the twisting number n_2 up to $n_2 = -p + m$ and get slope $s_2 = -\frac{pq-pp^*+p^*m-qm+q^*}{pq+p-qm}$. Finally consider a vertical annulus $\mathcal{A}_{12}$ between V_1 and V_2. Now as long as $pq - p + q \neq pq + p - qm$ there will be bypasses on V_1 or V_2 side depending on which one is bigger. So we search for n_1, n_2 such that

$$pn_1 - q = qn_2 - p$$

We easily find that $n_1 = -qn - 1$ and $n_2 = -pn - 1$, $n \geq 0$ characterizes all the tuples (n_1, n_2) satisfying above. Since we already know that $n_1 \geq -q + 1$ and $n_2 = -p + m$, it must be that $n_1 = n_2 = -1$. So we obtain that

$$n_1 = -1, s_1 = -\frac{q + q^* - p^*}{p+q} \text{ and } n_2 = -1, s_2 = -\frac{q + q^* - p^*}{p+q}$$

At this point there are two more possibilities:

Case 2a. There are no more bypasses on any vertical annulus between V_1 and V_2. As above we can cut along any vertical annulus between these neighborhoods, and round the corners by using the Edge Rounding Lemma. The resulting torus is isotopic to $\partial(Y \setminus V_3)$ with slope

$$s(\Gamma_{\partial(Y \setminus V_3)}) = \frac{1}{p+q}$$

When measured in ∂V_3 this slope becomes.

$$s = A_3^{-1}(p+q, -1)^T = \frac{b(pq-p-q) - a}{v(p+q) - u} = [-d_k, -d_{k-1}, \cdots, -(pq-p-q) - d_1].$$

By the classification of tight contact structures on the solid tori [6, Theorem 2.3], there are exactly $(d_k + 1)(d_{k-1} + 1) \cdots (pq - p - q + d_1)$ tight contact structures. On the other hand, each of V_1 and V_2 carries a unique tight contact structures as they are standard neighborhoods (i.e. $s(\Gamma_{\partial V_2} = s(\Gamma_{\partial V_3}) = -1)$. Similarly on $\Sigma \times S^1 = Y \setminus (V_1 \cup V_2 \cup V_3)$ there is a unique tight contact structure. This is because we can reconstruct Y as $(Y \setminus V_3) \cup V_3$ where $Y \setminus V_3$ is made of $V_1 \cup V_2 \cup N(\mathcal{A})$. Since V_1, V_2 each carries a unique tight contact structure and as the tight structure in $N(\mathcal{A})$ is uniquely determined by the dividing set of $\mathcal{A}$, we conclude that $Y \setminus V_3$ carries a unique tight structure relative to ots boundary. So all possible tight structures are resulting from V_3. Therefore, under the assumption above, we obtain total of $(d_k + 1)(d_{k-1} + 1) \cdots (pq - p - q + d_1)$ tight contact structures on Y.

Case 2b. There are bypasses between on some vertical annulus between V_1 and V_2. In this case there must be bypasses on both sides. Attaching those we can increase twisting n_1, n_2 to be $n_1 = n_2 = 0$ which yield slopes $s_1 = \frac{q-p^*}{q}$ and $s_2 = -\frac{q^*}{p}$. Since $p > q$, a vertical annulus between V_1 and V_2 will have more bypasses on V_2 side. We argue that this bypasses will not a result a torus with slope $\frac{x}{q}$ for some x. To see this, note that $s_1 = \frac{q-p^*}{q}$ and $-s_2 = \frac{q^*}{p}$ form a basis of $\mathbb{Z}^2$ as

$\frac{q-p^*}{q} \cdot \frac{q^*}{p} = pq - pp^* - qq^* = -1$. In particular, there is an edge from $\frac{q-p^*}{q}$ and $\frac{q^*}{p}$. Note also that $\frac{q^*}{p} > \frac{q-p^*}{q}$. See Figure 4.

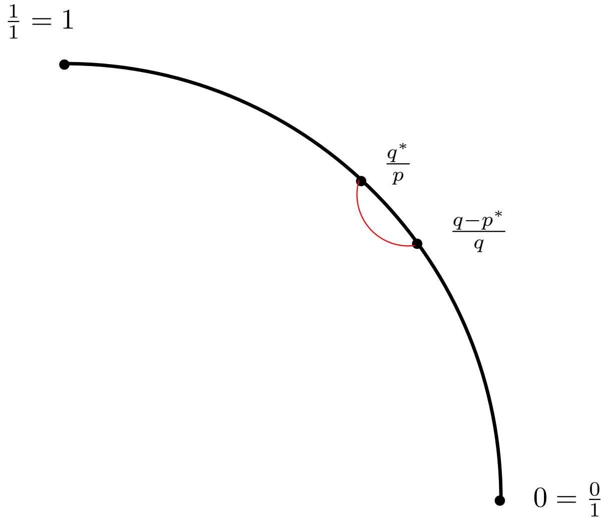

FIGURE 4. The piece of Farey graph, where its vertices are integer valued tuples and are in $1-1$ corresponds with the fractions, and two vertices are connected by an edge if and only if they form a basis for $\mathbb{Z}^2$.

A bypass attachment on ∂V_2 will result a new convex torus, and its slope can be tracked as follows (see [6, Lemma 3.15]): The new slope is a point on the Farey graph which is closest to ∞ (ruling slope along which the bypass is attached) and with an edge to $-\frac{q^*}{p}$. In other words, this new torus will have slope $-\frac{x}{y}$ such that $-\frac{x}{y} < -\frac{q^*}{p}$ and that the edge from $-\frac{x}{y}$ and $-\frac{q^*}{p}$ is longest possible. Reflecting this around the x-axis we claim that there must also be an edge from $\frac{x}{y}$ to $\frac{q-p^*}{q}$. One way to see this last claim is as follows: Observe that the edge from $\frac{q^*}{p}$ to $\frac{x}{y}$, as it suppose to be the longest possible, must be shielded by another edge starting from $\frac{x}{y}$. And this edge we claim must land at $\frac{q-p^*}{q}$ as any other point on the Farey graph between $\frac{q^*}{p}$ and where the edge from $\frac{x}{y}$ landed, when considered as a fraction, must have denominator strictly bigger than p. But $-\frac{q-p^*}{q}$ is in that interval and $p > q$ by our convention. So it must be that the edge starting from $\frac{x}{y}$ lands at $\frac{q-p^*}{q}$. Moreover, since p and q are relatively prime, $y \neq p$ (as otherwise we get $2q = p$). See Figure 5. Continuing this way we obtain that bypasses on V_2 side (and possibly then more bypasses on V_1 side) yield slopes $s_1 = 0$ and $s_2 = -1$. At this point there are two sub-cases:

Case 2ba. There are more bypasses between $V_1$and V_2. Attaching those we get $s_1 = \infty$ (and hence $s_2 = s_3 = \infty$). Now a (vertical) regular Legendrian fiber on ∂V_1 will have $tw(L) = 0$. Hence $t(\xi) = 0$. By Lemma 3 the contact structure ξ cannot be fillable (though it is possible that ξ is tight).

Case2bb. There are no more bypasses between V_1 and V_2. Similar to Case 2 and Case 2a, we can cut along a vertical annulus between these neighborhoods, and round the corners by using

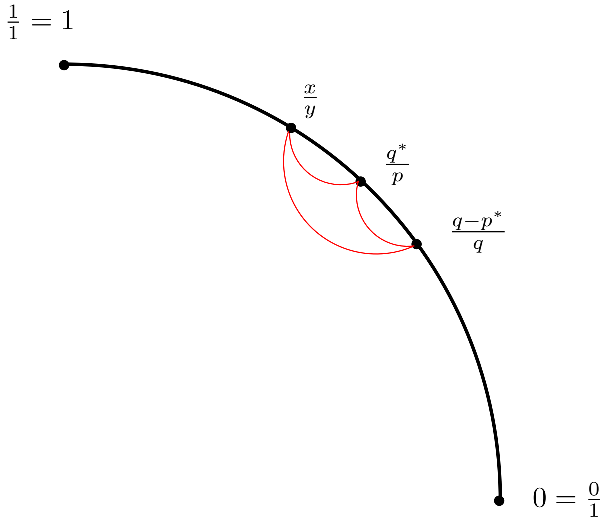

FIGURE 5. The piece of Farey graph, where its vertices are integer valued tuples and are in $1-1$ corresponds with the fractions, and two vertices are connected by an edge iff they form a basis for $\mathbb{Z}^2$.

the Edge Rounding Lemma. The resulting torus is isotopic to $\partial(Y \setminus V_3)$ with slope $s(\Gamma_{\partial(Y\setminus V_3)}) = 0-1+1=0$. When measured in ∂V_3 this slope becomes

$$s = A_3^{-1}(1,0)^T = -\frac{b}{v}.$$

Now since $-\frac{b}{v}+\frac{bpq-a}{u} = -\frac{1}{uv} < 0$, we find that there exists a neighborhood $V_3' \subset V_3$ with slope $s(\Gamma_{\partial V_3'}) = -\frac{bpq-a}{u}$. When measured $-\partial(Y\setminus V_3)$ this slope becomes

$$s(\Gamma_{-\partial(Y\setminus V_3)}) = A_3(-u, bpq-a)^T = (0,1)^T = \infty.$$

which implies again that there exists a regular Legendrian fiber L with $tw(L)=0$, and hence $t(\xi)=0$. By Lemma 3, we obtain that ξ is not fillable. Therefore, our count under *Case 2a* gives that there are at most $(pq-p-q+d_1)(d_2-1)\cdots(d_k-1)$ fillable contact structures on Y, which happens to be the minimum number of fillable contact structures on Y obtained in the lower bound part above. This completes the proof.

□

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