Symplectic Rational Homology Ball Fillings

Smooth World:

Q: What 3 manifolds bound rational homology balls?
i.e, what 3 manifolds are trivial in rational homology cobordism group?

→ Long history: Lots of constructions, lots of obstructions used to define rational blowdowns and construct lots of exotic 4 manifolds, etc.

<u>Note:</u> let $\mathcal{R} \subset \mathbb{Q}$ be the set $p/q > 1$ s.t. $(p,q)=1$, $p=m^2$, and $q, p-q$, or q^* is one of the following

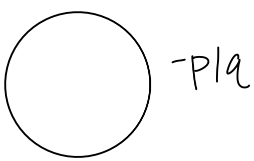
(1) $mh \pm 1$ $(0<h<m, (m,n)=1)$
(2) $mh \pm 1$ $(m,h)=2$
(3) $h(m \pm 1)$ $(h>1$ and $h \mid 2m \pm 1)$
(4) $h(m \pm 1)$ $h \mid m \pm 1$

where $0 < q^* < p$ and $qq^* \equiv 1 \bmod p$

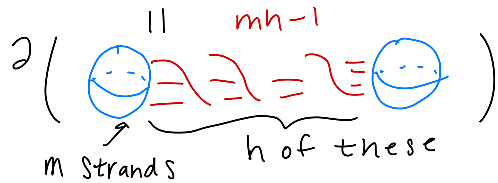
$q \to q^*$ homeomorphic $L(p,q)$
changing $p-q$ and q changes orientation

Lisca 2007

Lens Space $L(p,q)$ bounds a $\mathbb{Q}HB^4 \iff p/q \in \mathcal{R}$



ex: $L(m^2, mh-1)$
$\|$



We are interested in the symp. case

Q: Which contact 3 manifolds are the convex ∂ of symp. $\mathbb{Q}HB^4$?

Lens Spaces:

$(L(p,q),\xi)$ is fillable by a symp. $\mathbb{Q}HB^4$

$\Updownarrow$

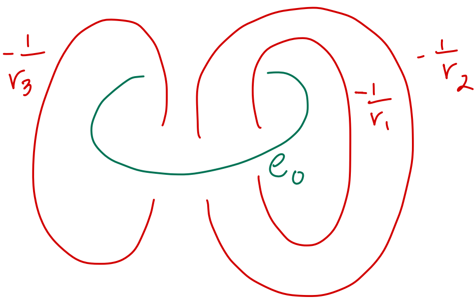
$p=m^2$ and $q=mh-1$ and ξ is the unique universally tight contact structure on $L(p,q)$

Recall: a lens space is a SFS ≤ 2 singular fibers

So, what about SFS with 3 singular fibers
i.e. small SFS

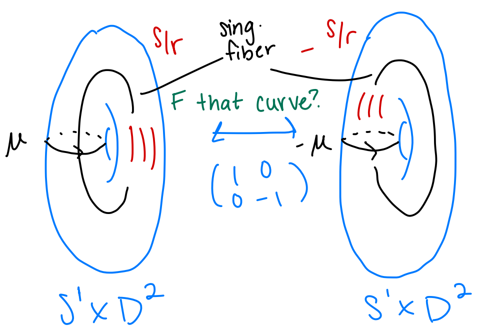
$Y = Y(e_0; r_1, r_2, r_3)$ $r_i \in (0,1)$, $e_0 \in \mathbb{Z}$ $\hookrightarrow$ important!

Y given by surgery diagram:



lets start with a simple construction

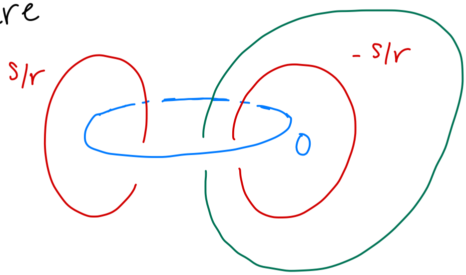
$S^1 \times S^2$ has a lot of SF structures



SF structure to $S^1 \times S^2$

$= S^1 \times S^2$

a surgery picture

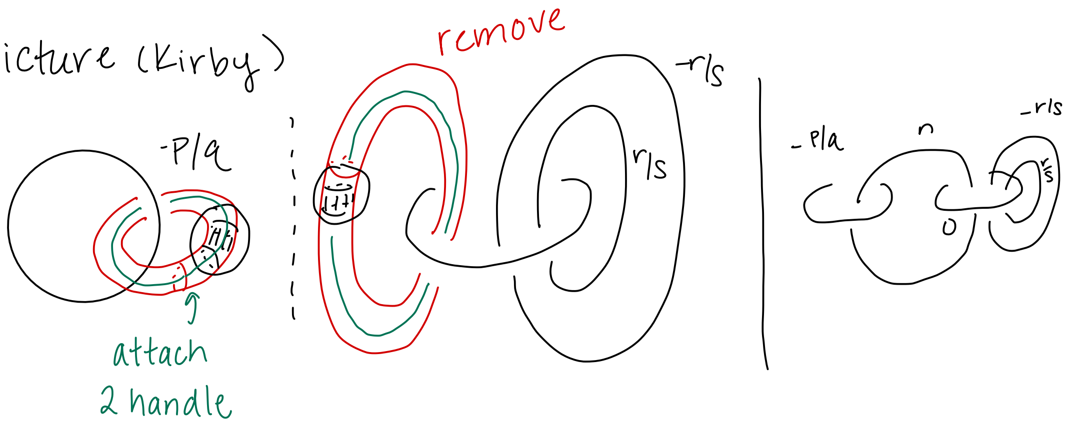


F is a $-s'/s$ curve with
$s'r - r's = 1$
$0 < r' < r$

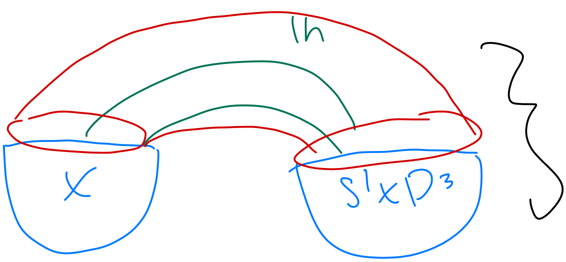
If you do p/q surgery on F, you get a small SFS

<u>Note</u>: if you remove a Heegaard torus from $L(p,q)$ and a nbhd of F above, then glue ∂'s together, you get a small SFS

→ picture (Kirby)



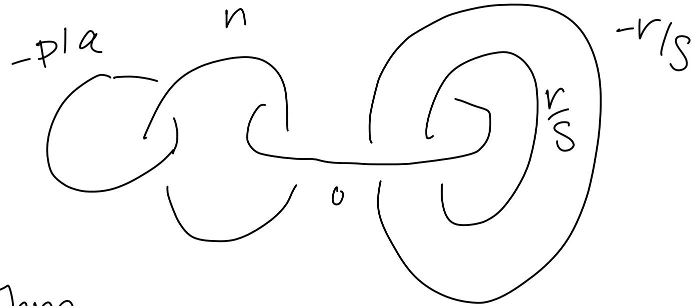
If $L(p,q) = \partial X \to QHB^4$ and $S^1 \times S^2 = \partial(S^1 \times D^3)$



X'

then attach a 1 handle and 2-handle as above to get $\to QHB^4$

So, lots of small SFS bound QHB^4



$p/q \in \mathbb{R}$

Thm

1. No contact str. on $Y(e_0; r_1, r_2, r_3)$ is symp. filled by a $\mathbb{Q}HB^4$ if $e_0 \le -5$

2. If $e_0 = -4, -3$, only 4 families of small SFS that symplectically filled by a $\mathbb{Q}HB^4$ a unique contact structure that does up to orientation

3. We know a lot when $e_0 = -2$.

e.g

(a) If Y is one of 7 families than $\exists!$ contact str. filled by a Symp $\mathbb{Q}HB^4$

(b) If Y is L-space, then at most 4 contact str. could be symp filled by $\mathbb{Q}HB^4$

4. If $e_0 \ge 0$, then at most 4 contact str. could be symp. filled by $\mathbb{Q}HB^4$

5. If $e_0 = -1$, then Y can have arbitrarily many tight structures filled by symp. $\mathbb{Q}HB^4$

Conjecture:

$Y(e_0; r_1, \dots, r_n)$ bounds a symplectic $\mathbb{Q}HB^4$ if and only if $n \leq 4$ and either

(1) $e_0 \leq -2$ and Y = link of a complex surface singularity

(2) $e_0 = -1$ and contact structure comes from the previous construction and there are finitely many such contact structures

(3) $e_0 \geq 0$ " " and unique contact structure

Conjecture:

If Y, a SFS, is a closed, oriented 3 manifold and Y and $-Y$ both bound symplectic $\mathbb{Q}HB^4$, then $Y = S^3$