Contact Morse Theory (SEC Topology 2025, Joe Breen)

Broad goal of talks: Understand Morse theory compatible with contact manifolds

Talk # 1: Background; a.k.a. how I think of contact manifolds
(Giroux, Weinstein, Gompf, Eliashberg, Gabai ...)

0. Preamble
1. Weinstein topology
2. Contact handles
3. Examples

Talk # 2: Recent developments and next steps
(B.-Honda-Huang '23, B.-Christian '24, Avdek-B.-Girella WIP)

3. Examples cnt'd
4. Questions

0 Preamble

Provocative example 1: There are contact structures ξ_1, ξ_2 on S^3 such that:

$$CMor(S^3, \xi_i) = \begin{array}{c}\text{Minimum \# of critical points of a}\\ \text{"contact Morse function" on } (S^3, \xi_i)\end{array} = \begin{cases} 2 & i=1 \\ 4 & i=2 \end{cases} \quad \cdots \; Mor(S^3) = 2$$

In particular, the function witnessing $CMor(S^3, \xi_2)$ has a pair of smoothly canceling critical points of index 1/2 that cannot be "contact canceled."

So Morse theory interacts with contact topology in an interesting way.

Provocative example 2: There are distinct *tight* contact structures $\xi_1, \xi_2, \xi_3, \ldots$ on S^5 such that $CMor(S^5, \xi_i) = 4$.

Question: Currently, there are no classifications of tight contact structures in dim ≥ 5.
In dim 3, convex surface theory (to me: Morse-theoretic) is a key tool.
How can we use contact Morse theory for tight classifications?

The notion of contact-compatible Morse theory is made precise by:

Definition: (Eliashberg–Gromov, Giroux '90s) Let (M^{2n+1}, ξ) be a contact manifold. A contact Morse structure is (f, X) where:

- $f : M \to \mathbb{R}$ is Morse
- X is a contact vector field, i.e. $\mathcal{L}_X \xi = \xi$ (flow preserves ξ)
- X is gradient-like w.r.t. f (for some metric on M)

Contextual results:

Theorem (Giroux–Mohsen '00's) Every (M^{2n+1}, ξ) admits a contact Morse structure.

"Giroux correspondence"

Theorem (B.–Honda–Huang '23) Two (Weinstein) open book decompositions of (M^{2n+1}, ξ) are equivalent up to positive stabilizations.

proved partially using techniques discussed in these talks

① Weinstein topology

Contact Morse theory has a strong interaction with symplectic Morse theory (Weinstein topology).

Definition: (Weinstein, Eliashberg-Gromov '90s) Let (W^{2n}, ω) be a symplectic manifold.
A Weinstein structure is (f, X) where:

- $f: W \to \mathbb{R}$ is Morse
- X is a Liouville vector field, i.e. $\mathcal{L}_X \omega = \omega$ (flow conformally! preserves ω)
- X is gradient-like w.r.t. f (for some metric on W)

Facts:

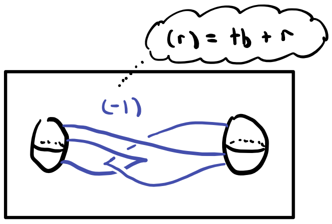
- $\lambda = \iota_X \omega$ satisfies $d\lambda = \omega$ and $\mathcal{L}_X \lambda = \lambda$

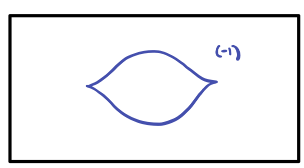
- (W^{2n}, ω, f, X) Weinstein implies critical points of f are at most index n (flow of X expands ω, so not possible to "crunch" more than n directions). Consequently Weinstein mflds either noncompact or with boundary.

- (Donaldson, Giroux) (M^{2n}, ω) closed symplectic. There is a codimension-2 symplectic submfld Γ s.t. $M - \Gamma$ is Weinstein.

- (Eliashberg, Gompf, Weinstein) Weinstein 4-folds are of the form:

  (0-handle) ∪ (1-handles) ∪ (2-handles attached along Legendrian knots with framing tb−1.)

e.g. 

or $T^*S^2 =$ 

- If L^n smooth mfld with handle decomposition, then T^*L^n is Weinstein with a corresponding Weinstein handle decomposition

- (Cieliebak - Eliashberg) Fix W^{2n}. Then

π_0 {space of Weinstein structures on W} $\cong$ π_0 {space of Stein structures on W}

Recall a Stein mfld is a cx submanifold of $\mathbb{C}^N$.

(Conjecturally, the spaces are w.h.e.)

So for purposes of these talks, "Weinstein" ~ "Stein" in case the latter is more familiar.

2 Contact handles.

Consider (M^{2n+1}, ξ) contact.

Definition (contact k-handle, $0 \leq k \leq n$)

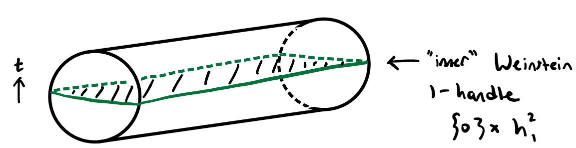
Let $(h_k^{2n}, d\lambda_k, f_k, X_k)$ be a $2n$-dimensional Weinstein handle.

Then $(\tilde{h}_k^{2n+1}, \tilde{\xi}, \tilde{f}_k, \tilde{X}_k) := ([-1,1]_t \times h_k^{2n}, \ker(dt + \lambda_k), t^2 + f_k, t\partial_t + X_k)$.

$$\mathcal{L}_{\tilde{X}_k}(dt + \lambda_k) = \mathcal{L}_{t\partial_t}(dt) + \mathcal{L}_{X_k}\lambda_k = dt + \lambda_k$$

So $\tilde{X}_k$ is a contact vector field

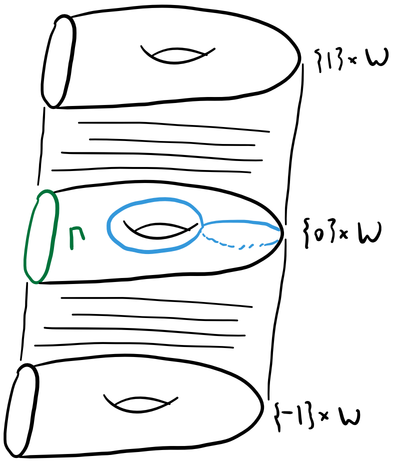
e.g. 3-dimensional contact 1-handle:



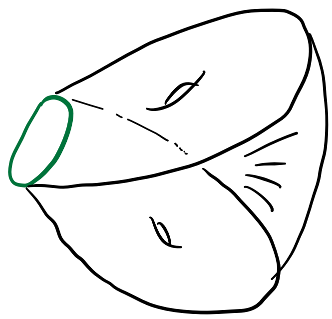
Definition (contact handlebody)

Let $(W^{2n}, d\lambda, X_\lambda)$ be a Weinstein domain (compact w/ ∂ w/ X_λ outwardly $\pitchfork$).

A *contact handlebody over W* is $(H = [-1,1]\times W, \ker(dt+\lambda), t\partial_t + X_\lambda)$.

$H =$ 

or

$H =$ 

Perspectives:

- H^{2n+1} built from contact $0,\dots,n$ handles
- H (trivial) sutured manifold with suture $\Gamma = \{0\}\times\partial W$
- ∂H convex with dividing set $\Gamma = \{0\}\times\partial W$ and $R_\pm \cong W$
- H admits a (trivial) partial open book decomposition with page W and empty monodromy
- $H = J^1(\underbrace{\{0\}\times \mathrm{Skel}(W)}_{\text{Legendrian CW cx}})$

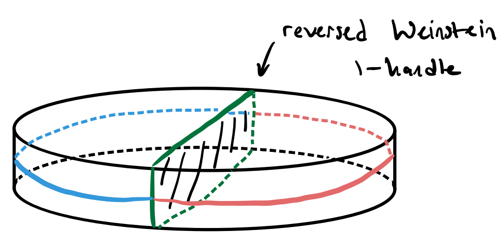
Definition (contact k-handle, $n+1 \le k \le 2n+1$)

Take the model for contact $0 \le 2n+1-k \le n$ handle and reverse the vector field.

Contact $0,\dots,n$ handles : expanding contactization of Weinstein $0,\dots,n$ handles

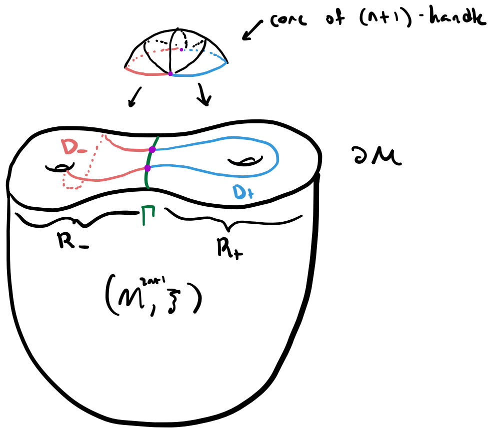
Contact $n,\dots,2n+1$ handles : contracting contactization of reversed Weinstein $0,\dots,n$ handles

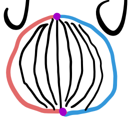
e.g. 3-dimensional contact 2-handle:



- The attaching sphere of a contact $(n+1)$-handle to ∂M is $S^n = D^n_+ \cup D^n_-$.

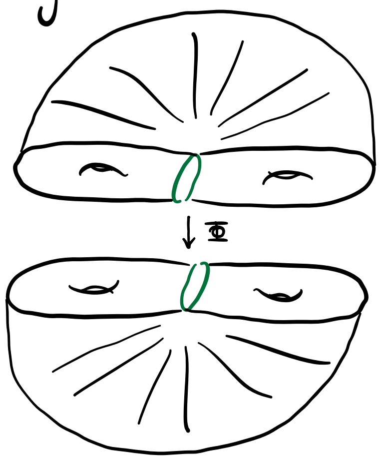
Lagrangian disks in $R_\pm$ of ∂M



- Core disk foliated by Legendrian n-disks from D_+ to D_- : 

$D_- \cap D_+ = S^{n-1}$ Legendrian sphere in Γ^{2n-1}

Theorem (Giroux-Mohsen) Every closed (M^{2n+1}, ξ) admits a decomposition of the form:



(reversed) Contact handlebody $H = [-1,1] \times W^{2n}$ contact $n+1, \dots, 2n+1$ handles

Contact handlebody $H = [-1,1] \times W^{2n}$ contact $0, \dots, n$ handles

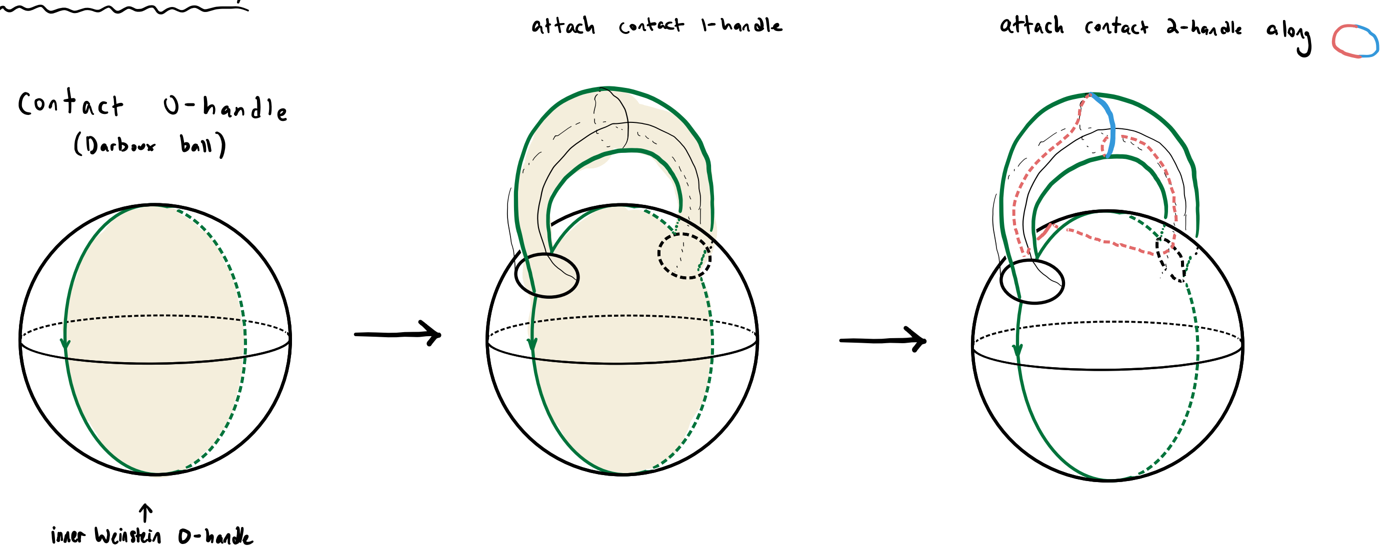
where the gluing map Φ maps Γ to Γ and maps the Weinstein regions of ∂H via exact symplectomorphisms.

Remark: This structure is equivalent to supporting open book decomposition with page W.

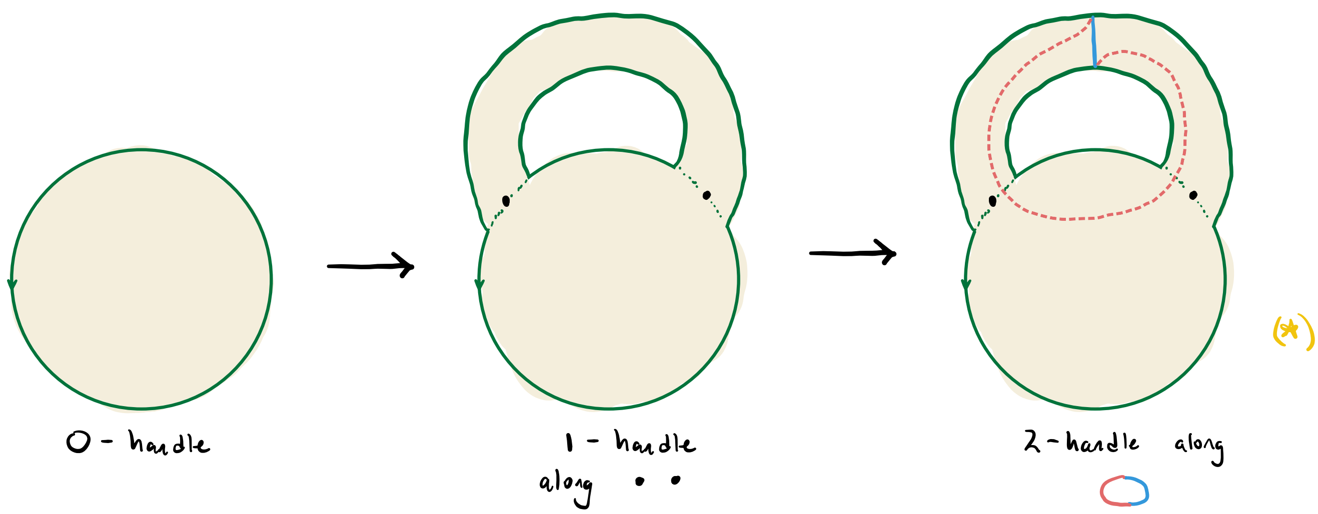
Remark:
- Giroux-Mohsen prove this in all dimensions using Donaldson techniques.
- Honda-Huang '19 reprove using modern version of contact Morse theory.

3 Examples

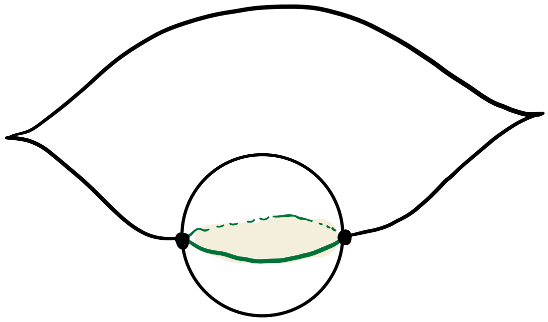
(1) (S^3, ξ_{tight})



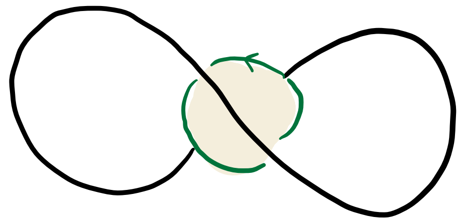
To make this easier to draw / visualize, adopt a "front side" view:



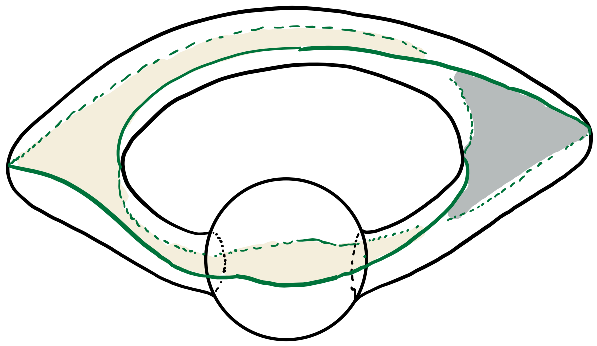
The claim is that the result of these abstract handle attachments is a Darboux ball. One way to see this is to identify the handles explicitly in a Darboux chart:
(the above pictures are abstract pics, not embedded pics)



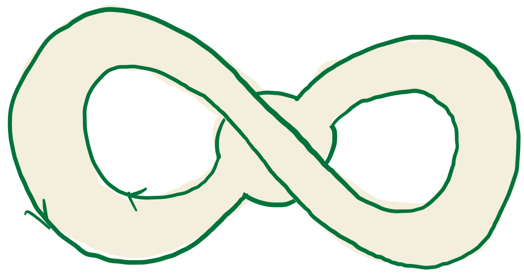
0-handle and core of 1-handle
(front projection)



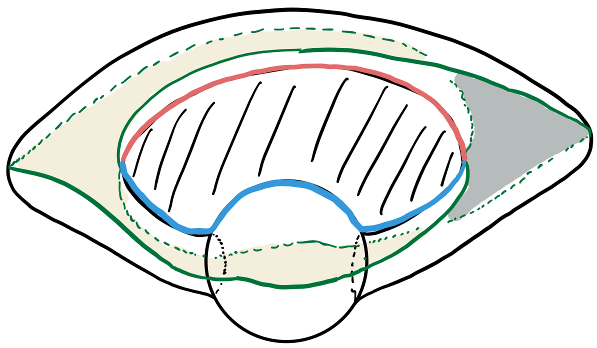
0-handle and core of 1-handle
(Lagrangian projection)



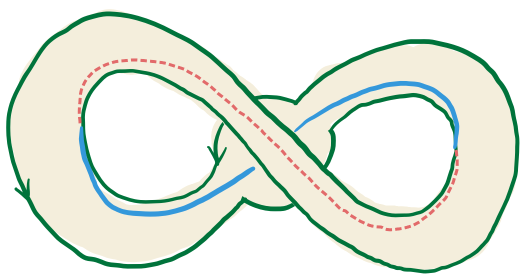
0-handle and core of 1-handle
(front projection)



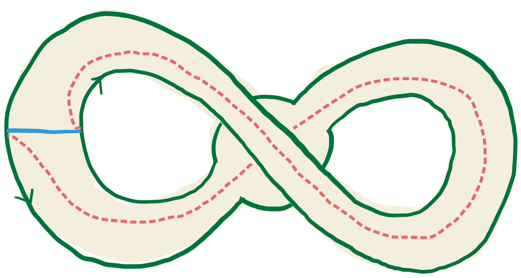
0-handle and core of 1-handle
(Lagrangian projection)



2-handle core disk
(front projection)



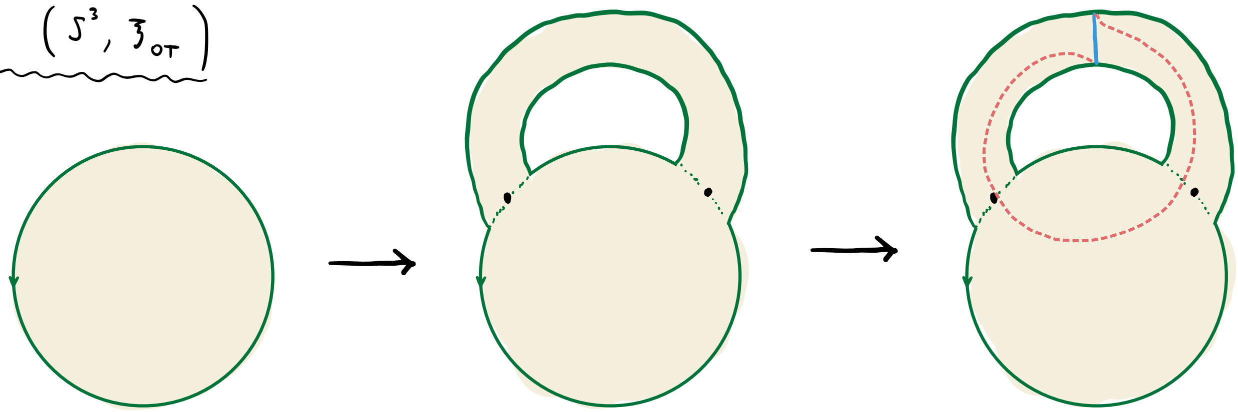
Attaching sphere of 2-handle
(Lagrangian projection)



isotopy of 2-handle attaching sphere; note (abstract) agreement with (*)

So capping off this 0-1-2 handle configuration with a 3-handle (another Darboux ball) gives (S^3, ξ_{tight}).

(2) (S^3, ξ_{OT})

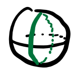


0 - handle

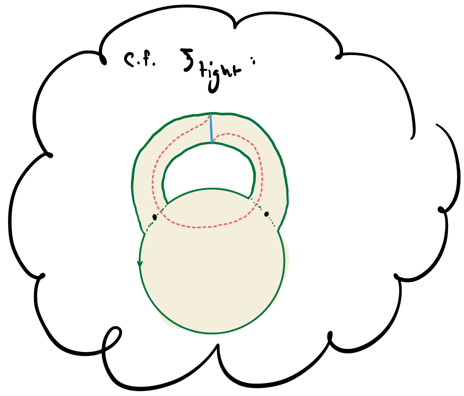
1 - handle along • •

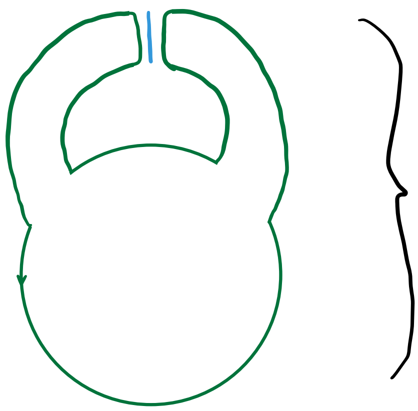
2 - handle along

Claims:

- Diffeo to B^3 (obvious)
- $S^2 = \partial B^3$ has Γ curve:  .

  Indeed, one identifies new Γ after 2-handle attachment by removing a nbd of | (or |):





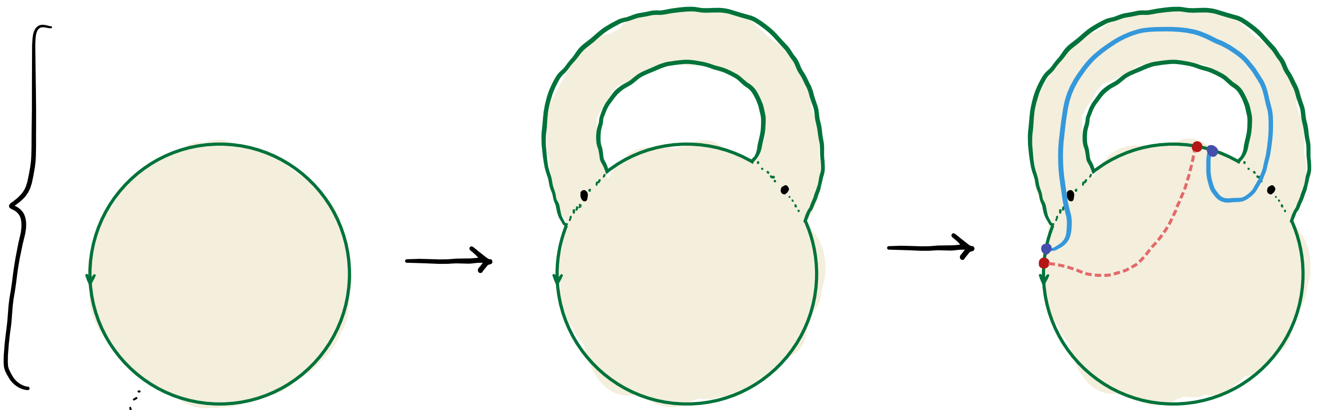
Necessarily an equator on S^2.

- Capping off with a 3-handle gives (S^3, ξ_{OT}).

  Idea: the 2-handle builds in LH Dehn twist monodromy of the corresponding open book (compared to RH Dehn twist in (*))

(3) (S^5, ξ_{st})

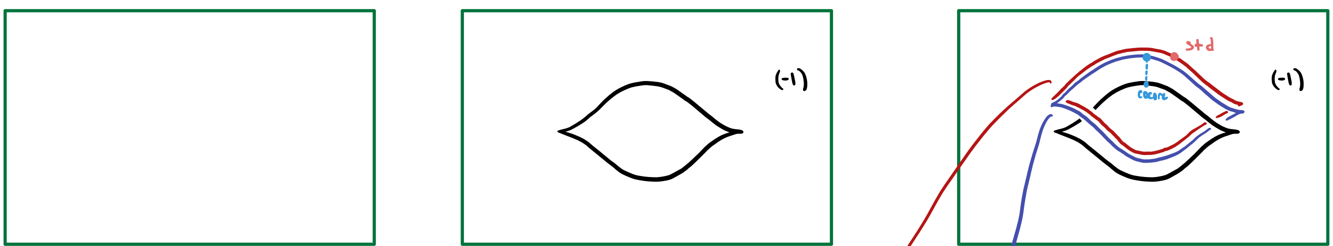
schematic



0 - handle

2 - handle along • •

identify "pre 3-handle attaching sphere data"



front projection of $\Gamma^3 = (S^3, \xi_{st}) = \partial(h_0^4)$

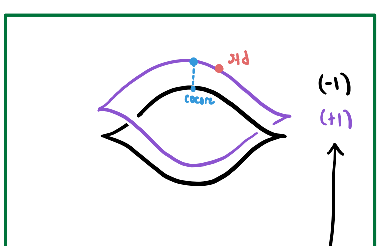
$\mathbb{RP}^3 = \partial(T^*S^2)$

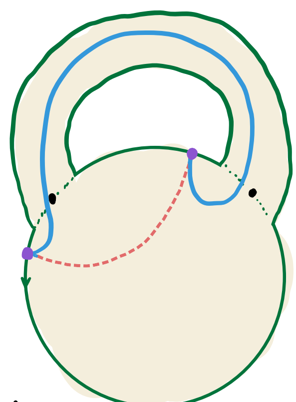
$\mathbb{RP}^3 = \partial(T^*S^2)$

this knot is filled in the 0-handle of R_- by the standard Lagrangian disk

this knot is filled in R_+ of $\partial([-1,1] \times T^*S^2)$ by a positive Reeb pushoff of the co-core of the Weinstein 2-handle

Now join the boundary of the two disks and attach a contact 3-handle along



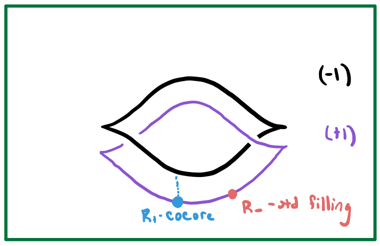


Removing neighborhoods of Lagrangian disks induces contact-(+1) surgery on Γ

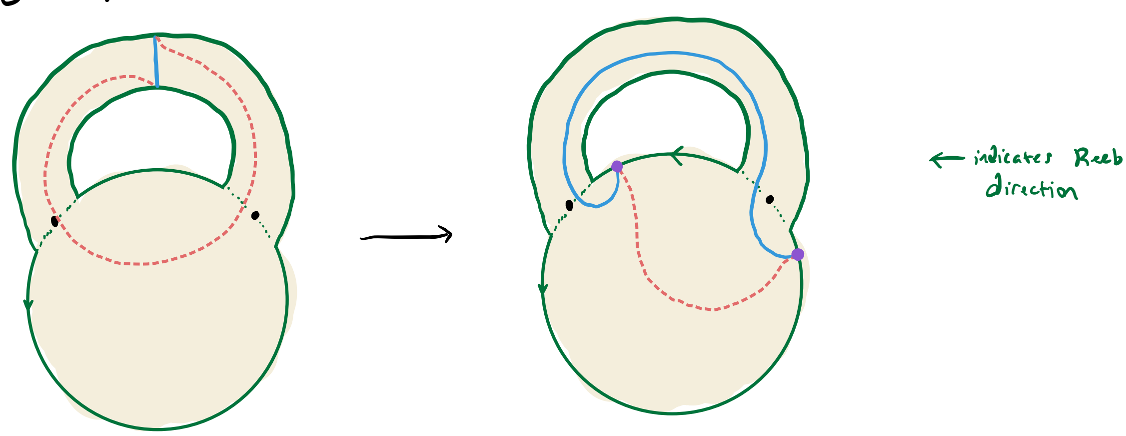
The contact surgeries cancel at the Γ-level, but the stronger claim

(4) (S^5, ξ_{OT})

Contrast (3) with the following 0-2-3-handle configuration:

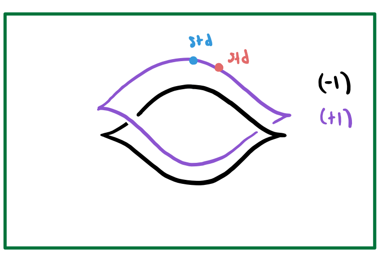


The result is a smooth 5-ball with standard boundary S^4, but not a Darboux ball. The contact structure is OT. Capping off with a 5-handle gives (S^5, ξ_{OT}). Note same schematic in terms of attaching sphere shifts in (2):



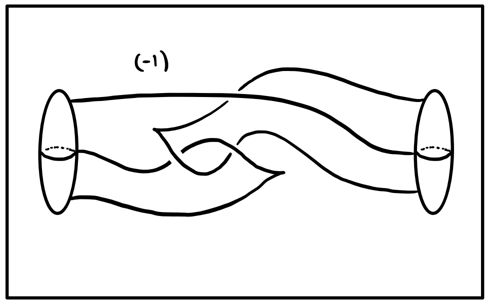
(5) $(S^2 \times S^3, \xi_{st})$

Contrast further with the following 0-2-3-handle configuration:



Capping this off with a 5-handle gives the open book (page $= T^*S^2$, monodromy $=$ id).

(6) (S^5, ξ) mystery (c.f. B.–Christian '24)



Let W^4 be the following Weinstein 4-domain:

Let $H = [-1,1] \times W$ be the contact handlebody over W. As a 5-mfld, the attaching sphere can smoothly be unknotted so that the 1-handle / 2-handle cancel smoothly.

Thus, $H \underset{\text{diffeo}}{\cong} B^5$. (read: W^4 is a contractible 4-fold called a Mazur mfld.)

However, H is not a Darboux ball with std bdry (eg. a contact 0-handle) because e.g. the dividing set Γ of H is not (S^3, ξ_{st}).

Build a contact mfld by taking $H \cup_{id} -H$. As H is diffeo to a ball, the result is diffeo to S^5.

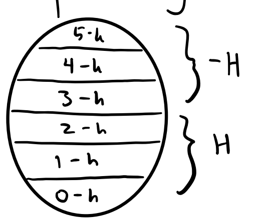
Q: What contact structure is it? ξ_{std} or something else?

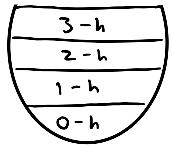
A: (Implicit in work of Ding-Geiges-van Koert, Cieliebak-Eliashberg) This is (S^5, ξ_{std}).

Proof: First, a 6-dimensional proof. By construction, this mfld is supported by the open book (page $= W^4$, monodromy $= id$). This OBD is induced by the (trivial) Lefschetz fibration $W^4 \times D^2$. This is a subcritical (hence flexible) Weinstein mfld diffeo to B^6, hence symplecto to B^6_{std} by h-principle. Thus $\xi = \xi_{std}$.

We give an intrinsic 5-dimensional proof using contact handle calculus.

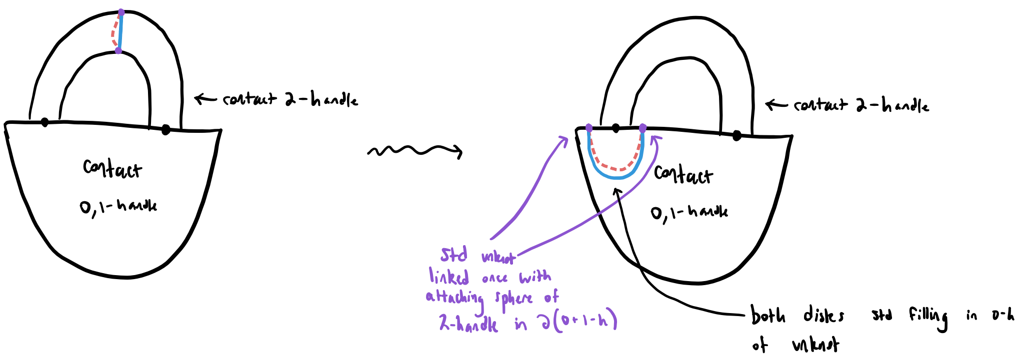
Note that:

$(S^5, \xi) =$ 

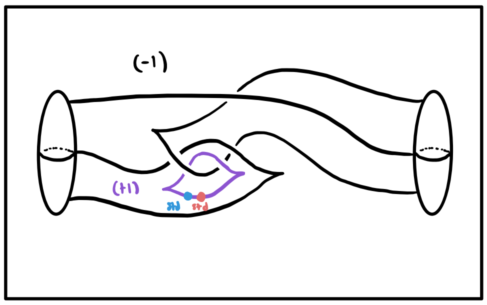
Consider the chunk: 

Since $-H$ is glued onto ∂H via the identity, the 3-handle is attached along (cocore of 2-handle of R_+) $\cup$ (cocore of 2-handle of R_-)

Schematically:

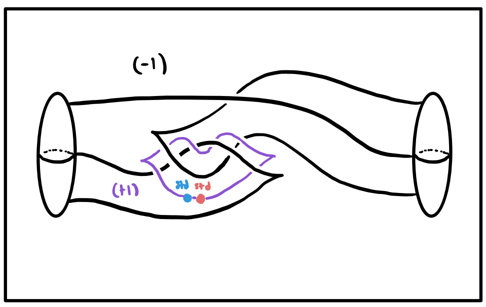


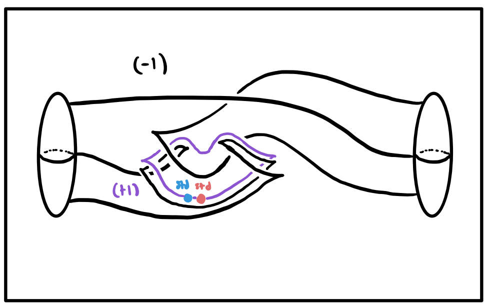
Diagrammatically:

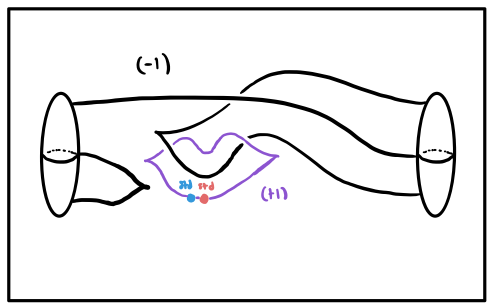


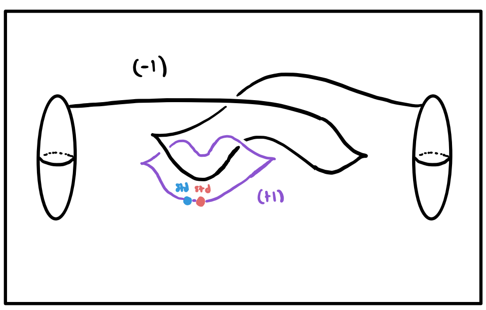
(A)

handleslide contact 2-handle over contact 3-handle

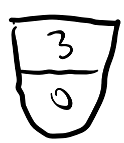


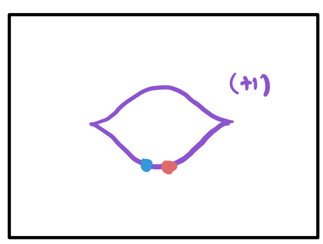






Now the contact 1/2-handles can be canceled, as this corresponds to canceling the underlying Weinstein handles.

This leaves 

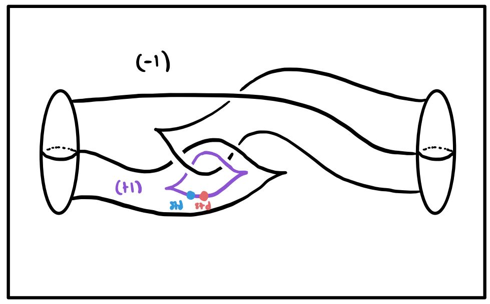
= 

Remembering the 4/5 handles we have $S^5 = \frac{H'}{0}$ } contact handlebody over some W'.

As the dividing set of ∂(0-handle) is (S^3, ξ_{st}), W' must be a Weinstein filling of (S^3, ξ_{st}), hence B^4_{st}.

Therefore, H' is a contact Darboux ball, so $\xi = \xi_{st}$ on S^5. $\square$

Remark: At this stage (before handle slide): the contact $-(\pm 1)$ surgeries cancel at the 3-dimensional level; the corresponding 5-dimensional handles do not.



(✱)

4 Next Steps

In dim 3, convex surface theory (contact Morse theory...) is an important tool in tight classification problems.

Problem 1: (Higher-dimensional Giroux criterion) Characterize when a neighborhood of a convex hypersurface Σ in terms of the dividing set / $R_\pm$.

Problem 2: Let $(S^5, \xi_n) = OB(T^*S^2, \tau^{2n+1})$. These are exotic tight contact structures. Puncture with two Darboux balls to get exotic tight $(S^4 \times [0,1], \xi_n)$. Describe bypass decompositions of these layers to better understand exotica via bypasses.