Non-flexible loops of loose Legendrians in 3D

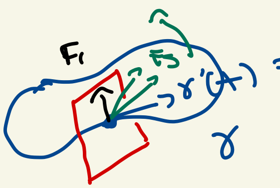
(j.w. in progress with Fabio Gironella)

Main characters: $(M,\xi) \equiv$ Contact 3-manifold

$$\mathcal{L}(M,\xi) \equiv \left\{ \text{Legendrian embeddings in } (M,\xi) \right\}$$

$$\Big\downarrow i$$

$$\mathcal{FL}(M,\xi) \equiv \left\{ \text{Formal Legendrian embeddings in } (M,\xi) \right\}$$



$$\left\{ \begin{array}{ccc} TS^1 & \xhookrightarrow{\ \ } & TM \\ \downarrow & & \downarrow \\ S^1 & \xhookrightarrow{\ \gamma\ } & M \end{array} ,\quad F_0 = d\gamma,\quad F_1(TS^1) \subseteq \xi \right\}$$

Bundle monomorphism; smooth emb.

... same for links

Classical Legendrian knot/link theory concerns the study of the induced map at path-connected components.

$$\pi_0(i) : \pi_0\mathcal{L}(M,\xi) \longrightarrow \pi_0\mathcal{FL}(M,\xi)$$

Two null-homologous Legendrians are formally isotopic $\Leftrightarrow$ Same classical inv.

Three major advances in the study of $\pi_0(i)$:

S_0: * If (M,ξ) is tight then $\pi_0(i)$ is not surjective (Bennequin for (S^3,ξ_{std}), Eliashberg in general) $\rightsquigarrow tb + |rot| \leq -\chi(\Sigma)$!

* If (M,ξ) is OT then $\pi_0(i)$ is surjective.

I_0: The map $\pi_0(i)$ is not injective. (There are Legendrians with same classical invariants but not Legendrian isotopic) (Fraser, Chekanov, Eliashberg, ...)

C_0: If we restrict the smooth knot / link type the map could become injective and the image could be determined $\rightsquigarrow$ Classification of Legendrian knot / links

(Eliashberg–Fraser for unknots in (M, ξ) tight, Etnyre–Honda for torus knots in (S^3, ξ_{std}), ...)

Active area

Today: We study the induced map at higher homotopy groups

Λ base point.

$$\pi_k(i): \pi_k(\mathcal{L}(M,\xi), \Lambda) \longrightarrow \pi_k(\mathcal{FL}(M,\xi), \Lambda)$$

- Motivation:

(1) 1-parameter families of Legendrians define Lagrangians in the symplectization of (M, ξ) (Chantraine, Eliashberg-Gromov)

$\Downarrow$

Legendrian loops are a fundamental tool in the study of Lagrangian fillings in (D^4, ω_{std}) (e.g. Casals–Gao)

(2) Parametric families of Legendrians in $(\mathbb{R}^3, \xi_{std})$ define higher dimensional Legendrians in higher dimensional contact manifolds (e.g. spinning constructions Ekholm-Etnyre-Sullivan, Ekholm-Kalman ...)

(3) The group $\mathrm{Cont}(M, \xi) = \{ \varphi \in \mathrm{Diff}(M) : \varphi_* \xi = \xi \}$ "Contactomorphisms"

is "almost" w.h.e to the space of Legendrian embeddings of certain Legendrian graph in (M, ξ)

skeleton of the page of an adapted OBD

(F, Martínez-Aguinaga, Presas, '21)

(4) Today a new one!

What do we know about $\pi_k(i)$?

S_k: The map $\pi_k(i)$ is not surjective in general ⇝ This is done by direct computation no "deep" reason.

It would be interesting to study this for max-tb representatives

(FMP '21)

C_k: For some Legendrians Λ the map $\pi_k(i)$ becomes injective and the image can be computed.

(FMP '21, F-Min '23)

e.g. $\mathcal{L}($  $, S^3, \xi_{std}) \simeq U(2)$

max-tb torus knots, ...

I_k: It was open...

↑

Today $\underline{\underline{k=1}}$

Thm A: (F-Gironella)

Let (M,ξ) be a closed OT contact 3-manifold.

Then, there exists a (∞-many…) Legendrian (link) $\Lambda \subseteq (M,\xi)$ such that

$$\operatorname{Ker}\left(\pi_1(i): \pi_1\mathcal{L}(M,\xi),\Lambda) \to \pi_1(\mathcal{FL}(M,\xi),\Lambda)\right) \neq \{0\}.$$

Rmks:

- $(M\setminus\Lambda,\xi)$ is OT $\equiv$ Λ is loose.

This should be compared with the folk h-principle

$$\mathcal{L}(M\setminus\Delta_{OT},\xi) \xhookrightarrow{\text{w.h.e.}} \mathcal{FL}(M\setminus\Delta_{OT},\xi)$$

OT disk
FIXED!

which follows from Eliashberg's OT h-principle (see Farias-F '24 for a proof without Eliashberg's Thm)

The invariant: Legendrian loops and contactomorphisms.

Given $\Lambda\subseteq(M,\xi)$ we can perform Legendrian surgery along Λ to obtain a new contact 3-manifold

$$\rightsquigarrow (M(\Lambda),\xi(\Lambda))$$

Similarly, given a loop of Legendrians Λ^s, $s\in S^1$, we can perform a 1-parameter family of Legendrian surgeries to obtain a bundle

$$\begin{array}{c} X^4 \\ \downarrow \pi \\ S^1 \end{array}$$

with fiber $\pi^{-1}(s) = (M(\Lambda^s), \xi(\Lambda^s))$

The monodromy of this bundle is a contactomorphism.

$$\varphi(\Lambda^s) \in \mathrm{Cont}(M(\underset{\Lambda^0}{\overset{\shortparallel}{\Lambda}}), \xi(\underset{\Lambda^0}{\overset{\shortparallel}{\Lambda}}))$$

It follows (details omitted...) that this defines a group homomorphism.

$$\mathcal{L}S_1 : \pi_1(\mathcal{L}(M, \xi), \Lambda) \to \pi_0 \mathrm{Cont}(M(\Lambda), \xi(\Lambda))$$

This is motivation (1) to study Legendrian loops! We can draw contactomorphisms! (Surgery pictures!)

Thm B: (F-Gironella)

Let (N, ξ_N) be a closed OT 3-manifold and $\Lambda \subseteq (N, \xi)$ a Legendrian such that

$$(N(\Lambda), \xi_N(\Lambda)) \cong (S^3, \xi_{std}).$$

Always exists! (Etnyre-Honda)

Let (M, ξ_M) be a closed contact 3-manifold. Then, the surgery homomorphism

$$\mathcal{L}S_1 : \pi_1(\mathcal{L}(M\#N, \xi_M \# \xi_N), \Lambda) \longrightarrow \pi_0 \mathrm{Cont}(M, \xi_M)$$

surjects over every formally trivial contactomorphism.

$$(M\#N(\Lambda), \xi_M \# \xi_N(\Lambda)) \cong (M\#(N(\Lambda)), \xi_M \# (\xi_N(\Lambda))) \cong (M \# S^3, \xi_M \# \xi_{std}) \cong (M, \xi_M)$$

Definition of formal contactomorphism is analogous to formal Legendrian. This are just contactomorphisms that topologically look isotopic to the identity but they may not be trivial.

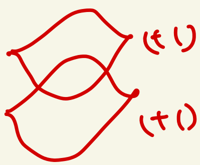
Rmk:

This is the contact analog of a thm of David Gay about diffeos in the 4-sphere and loops of 2-spheres, also a thm of Kupers-Krannich in a more general setup.

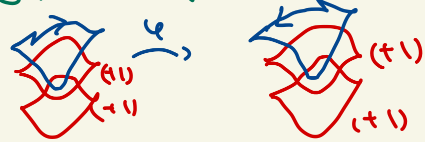
Thm B $\Rightarrow$ Thm A:

- Vogel, Chekanov: there exists an OT 3-sphere (S^3, ξ_{OT}) with a formally trivial but non trivial contactomorphism.

(S^3, ξ_{OT})



The contactomorphism switches two non-isotopic OT disks.



- By Eliashberg's h-principle we can write

$$(M, \xi) \cong (S^3, \xi_{OT}) \# (\hat{M}, \hat{\xi})$$

OT.

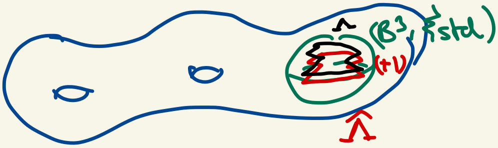
- Apply Thm B for this pair $((\hat{M}, \hat{\xi}) = (N, \xi_N))$ $\boxtimes$

"Proof" of Thm B:

$(M,\xi) = (M,\xi) \# (S^3,\xi_{std})$

$= (M,\xi) \# (N(\Lambda),\xi(\Lambda))$

$\varphi \in Cont(M,\xi)$ formally trivial.



(M,ξ)

Step I: Isotope φ so it fixes a Darboux ball (B^3,ξ_{std})

Step II: Perform a $(+1)$ surgery along $\hat{\Lambda}$ (push-off of Λ_N in) $\subseteq (B^3,\xi_{std})$

Step III: φ naturally extends to $\hat{\varphi}$ in

$$(M \# S^3(\hat{\Lambda}), \xi_M \# \xi_{std}(\hat{\Lambda})) = (M \# N, \xi_M \# \xi_N)$$

is formally trivial and fixes the OT (N,ξ_N)

Step IV: Apply Eliashberg's h-principle to find a contact isotopy

$$\hat{\varphi}_t \in Cont(M \# N, \xi_M \# \xi_N),\ t \in [0,1]$$

between

$$\hat{\varphi}_0 = Id$$
$$\hat{\varphi}_1 = \hat{\varphi}$$

The loop is precisely $\hat{\varphi}_t(\Lambda) = \Lambda^t$.

$(\hat{\varphi}_0(\Lambda) = \Lambda,\ \hat{\varphi}_1(\Lambda) = \Lambda$ b.c. $\hat{\varphi}_1|_N = Id!)$ ☒