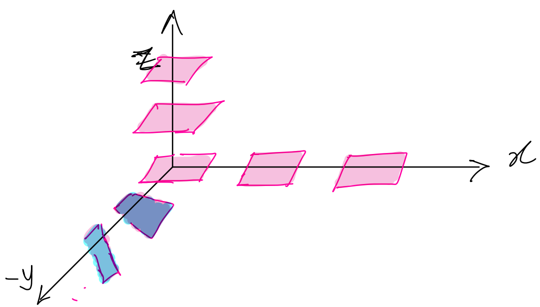
Rationally null homologous knots in Contact 3-Manifolds

– w/ Ipsita Dutta

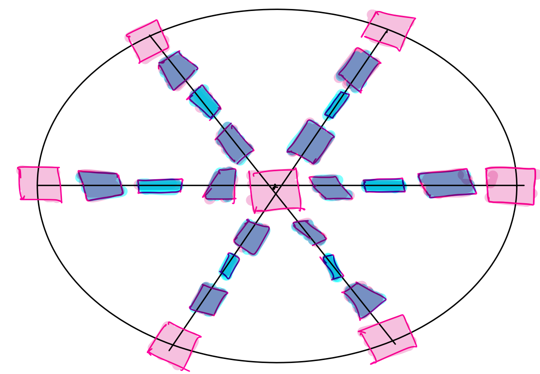
- Hyperplane field ξ on M^3 is a contact structure if $\exists$ 1-form α s.t. $\alpha \wedge d\alpha \neq 0$ & $\xi = \ker \alpha$
- ξ_1 & ξ_2 are contactomorphic if $\exists$ diffeomorphism $f : (M, \xi_1) \longrightarrow (M, \xi_2)$ s.t. $f_*(\xi_1) = \xi_2$

Ex: $\mathbb{R}^3$, $\alpha_1 = dz - y\,dx$
$\ker \alpha_1 = \xi_{std.}$



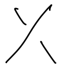
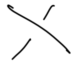
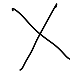
TIGHT

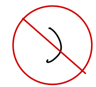
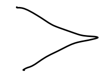
Ex: $\mathbb{R}^3$, $\alpha_2 = \cos r\,dz + r \sin r\,d\theta$



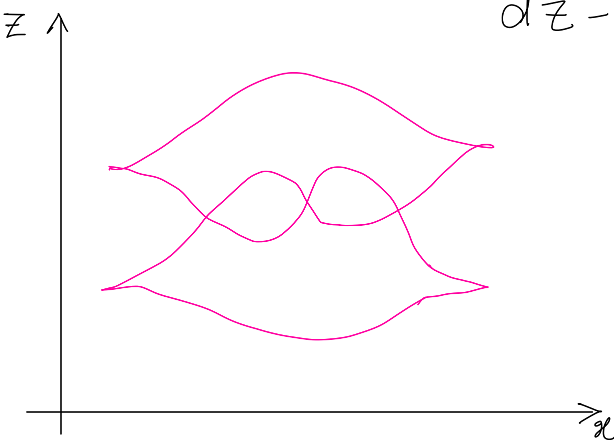
OVERTWISTED

$L^1 \subset M^3$ is a <u>Legendrian submanifold</u> if $T_xL \subset \xi_x \quad \forall x \in L$

-  not allowed -  allowed $\rightsquigarrow$ 

- no vertical tangencies :  $\rightsquigarrow$ 

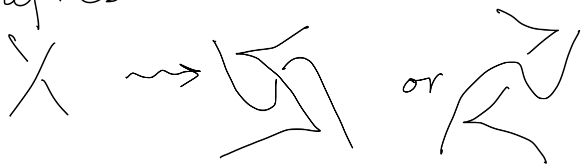
- L_1 & L_2 are equivalent if they are isotopic through a family of Legendrian knots.



$$dz - y\,dx = 0$$

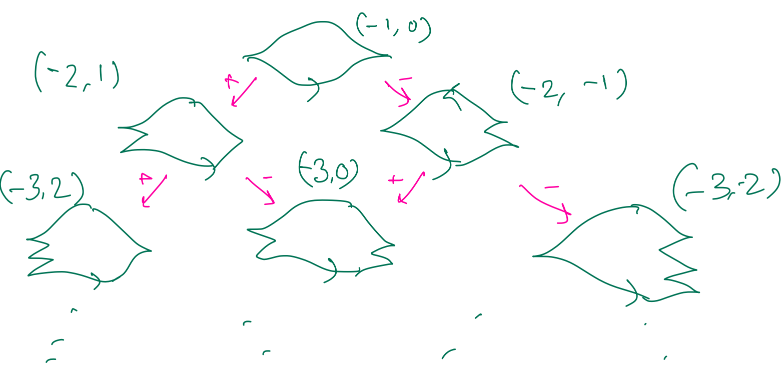
$$y = \frac{dz}{dx}$$

- Any knot type has Leg. representatives :



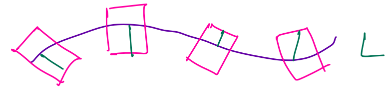
- Multiple representatives

stabilization



Invariants : Thurston – Bennequin

$tb(L)$ = Difference of contact framing & Seifert framing



In front projection :

$= \text{writhe}(L) - (\# \prec)$

Rotation : $rot(L)$ = Euler number of $\xi|_\Sigma$ rel. to v

Σ Seifert surface

($v \in TL$ orients L)

In front proj : $\frac{1}{2}$(# down cusps − # up cusps)

• $\mathcal{L}(K)$ = set of Leg. knots in (S^3, ξ_{std}) top. isotopic to K.

$\Phi : \mathcal{L}(K) \longrightarrow \mathbb{Z} \times \mathbb{Z} \; : \; L \longmapsto (rot(L), tb(L))$

• Classifying Leg. knots is equi to:

Geography: determine image Φ

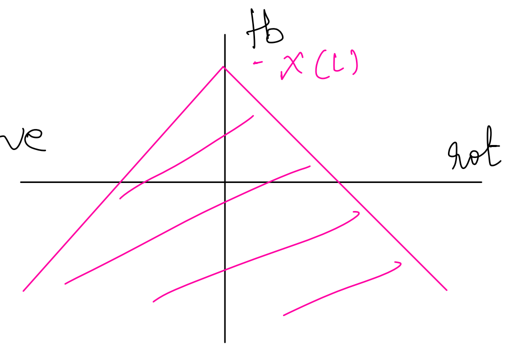
Botany: determine $\Phi^{-1}(rot, tb)$ $\forall (rot, tb) \in im\, \Phi$

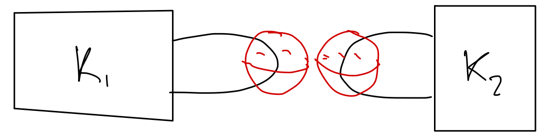
If Φ injective then K is Leg. simple.

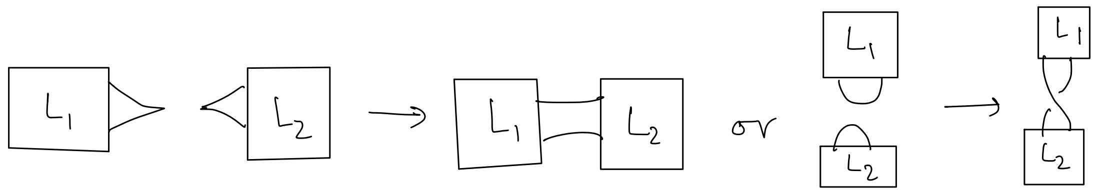
Thm: (Bennequin '82) + (Eliashberg): For a knot $L \subset (M, \text{tight})$

$$tb(L) + |rot(L)| \leq -\chi(\Sigma)$$

Remark: (M, ξ) is tight $\Longleftrightarrow$ $im(\Phi)$ is bdd above for any K



Connect Sum: Topologically: 

Legendrian: 

It is not obvious but this is well-defined

Tight(M): space of tight contact 2-plane fields on M.

Thm (Colin): $\pi_0(\text{Tight}(M_1)) \times \pi_0(\text{Tight}(M_2)) \xrightarrow{\sim} \pi_0(\text{Tight}(M_1 \# M_2))$

For null-homologous leg. knots:

$tb(L_1 \# L_2) = tb(L_1) + tb(L_2) + 1$

$rot(L_1 \# L_2) = rot(L_1) + rot(L_2)$

Thm: (Etnyre-Honda): $(M,\xi) = (M_1,\xi_1) \# \cdots \# (M_n,\xi_n)$ is tight & $K \subset M$ & prime decomposition $K = K_1 \# \cdots \# K_n$ s.t. $K_i \subset (M_i,\xi_i)$. Then

$$C : \left(\frac{\mathcal{L}(K_1) \times \cdots \times \mathcal{L}(K_n)}{\sim} \right) \to \mathcal{L}(K_1 \# \cdots \# K_n).$$

$$(L_1, \ldots, L_n) \mapsto L_1 \# \cdots \# L_n$$ is bijection. $\sim$ is

① $(L_1, \ldots, L_n) \sim \sigma(L_1, \ldots, L_n)$ σ is Permutation of K_i's

② $(L_1, \ldots, S_\pm(L_i), L_{i+1}, \ldots, L_n) \sim (L_1, \ldots, L_i, S_\pm(L_{i+1}), \ldots, L_n)$

cor ①: $\overline{tb}(K_1 \# K_2) = \overline{tb}(K_1) + \overline{tb}(K_2) + 1$

cor ②: For max tb rep. of $\mathcal{L}(K)$, the prime decomposition is unique.

Help in classification of Leg & trans. knots

"restrict to prime knots"
- find all max tb's
- prove everything destab.

Q : Geography & Botany for knots in 3-M?

For $\mathbb{Q}$null homologous knots :

$\mathbb{Q}$ Seifert surface : $\exists \Sigma \subset M$ s.t. $\partial\Sigma = dK$ for some $d \in \mathbb{Z}$

order of Knot $= d > 0$

If L_1 & L_2 $\mathbb{Q}$null-hom, then $L_1 \# L_2$ is $\mathbb{Q}$ null-hom

$\operatorname{ord}(L_i) = d_i \rightsquigarrow \operatorname{ord}(L_1 \# L_2) = \operatorname{lcm}(d_1, d_2) := d$.

Thm (Datta - S) :

$$tb_{\mathbb{Q}}(L_1 \# L_2) = tb_{\mathbb{Q}}(L_1) + tb_{\mathbb{Q}}(L_2) + 1$$

$$rot_{\mathbb{Q}}(L_1 \# L_2) = rot_{\mathbb{Q}}(L_1) + rot_{\mathbb{Q}}(L_2)$$

$$sl_{\mathbb{Q}}(L_1 \# L_2) = sl_{\mathbb{Q}}(L_1) + sl_{\mathbb{Q}}(L_2) + 1$$

Constructing 3-M from S^3 : $F_{p,q} : S^3 \longrightarrow S^3$

$$F_{p,q}(r_1, \theta_1, r_2, \theta_2) = \left(r_1, \theta_1, \frac{2\pi r_2}{p}, \theta_2 + \frac{2q\pi}{p}\right).$$

$$S^3 / \langle F_{p,q} \rangle = L(p,q)$$

$F_{p,q}$ preserves contact str. so we get a ctct str. on $L(p,q)$

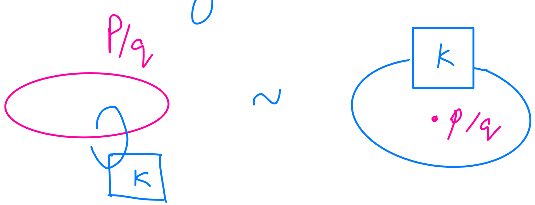
Rmk : $H_1(L p,q) = \mathbb{Z}_p$

Knots are $\mathbb{Q}$ null-homologous.

Another construction :

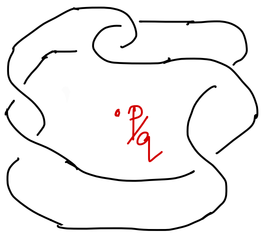
$= L(p,q)$

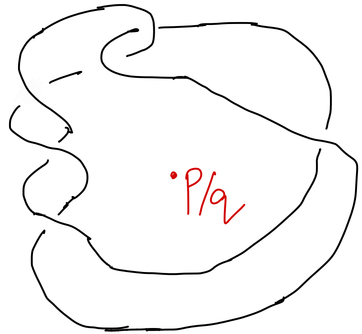
Representing knots.



Non-Simple Prime Null-homologous knots:

Let's see an example

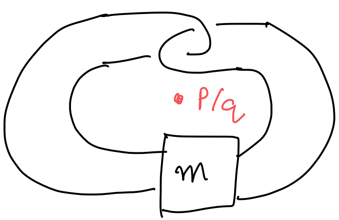




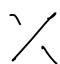
$tb(L) = tb(L')$

$rot(L) = rot(L')$

- Differentiated by LCH (defined by Licata for $L(p,q)$)
  (" by Licata-Sabloff for SFS)

- 

$m \geq 4$



For any n, we have n-many non-simple Leg. representative.

Thank You!

Questions?