Topological Approach to Algebraic Genus Two Singular Fibers

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[joint with Jeremy Van Horn-Morris]

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Genus one fibrations

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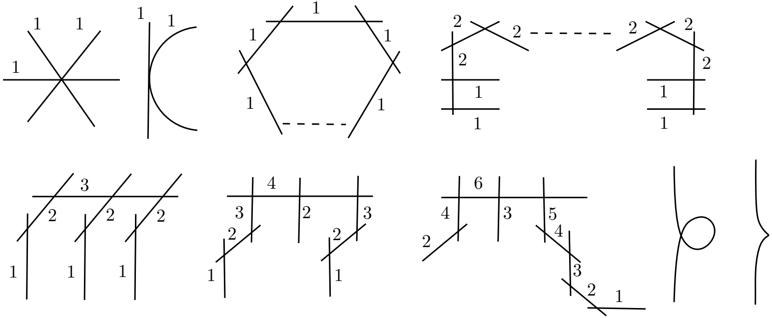

Figure: Kodaira's classification of genus one singularities

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- Our theorem establishes this dictionary in genus 2: We give one-to-one correspondence between the genus 2 singular fibers and their monodromy factorizations.

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- Such families of curves are called *degenerating families of algebraic curves*.

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Our work

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- Symplectically, we find the monodromy factorizations of the Lefschetz fibrations.

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- We give a dictionary between symplectic curve configurations and monodromy factorizations for all singularities of genus two fibrations.
- Our methods and these discovered correspondences between curve configurations and monodromy factorizations can be applied to arbitrary Lefschetz fibrations/pencils.

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- In general $V(f)$ is a singular variety which we resolve. We call the resolution space X_f.
- The fibration lifts to X_f and the singular fiber lifts to its resolution graph.

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Theorem (S., Van Horn-Morris, '23)

The resolution X_f of a singular algebraic variety $V(f)$ where f is

i) $y^2 - x^5 - t^k$, $k = 1, \ldots, 10$,

ii) $y^2 - x^6 - t^k$, $k = 1, \ldots, 6$,

iii) $y^2 - x(x^4 + t^k)$, $k = 1, \ldots, 8$,

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We construct these central fibers and from those, we also obtain N.-U. fibers.

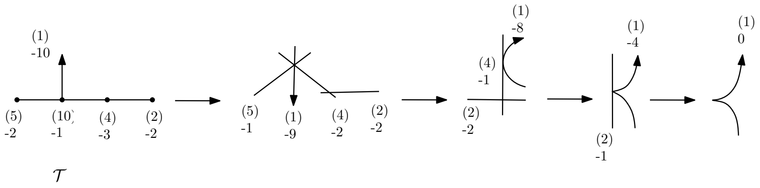


Figure: ϕ_1 - Fiber VIII-1

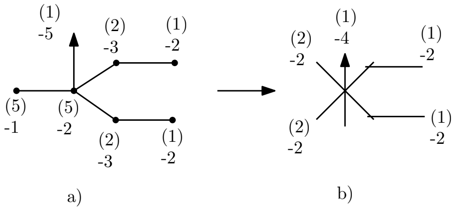
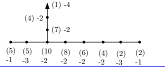
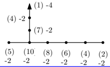
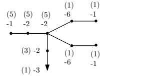
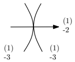
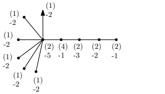
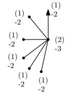
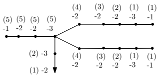
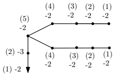
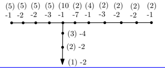
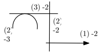
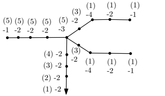
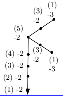
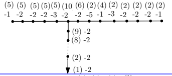
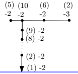
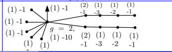


Figure: ϕ_1^2 - Fiber IX-1

ϕ_1^3			: Fiber VIII-2
ϕ_1^4			: Fiber IX-2
ϕ_1^5			: Fiber I*
ϕ_1^6			: Fiber IX-3
ϕ_1^7			: Fiber VIII-3
ϕ_1^8			: Fiber IX-4
ϕ_1^9			: Fiber VIII-4
$\phi_1^{10} = id$			: Generic Fiber

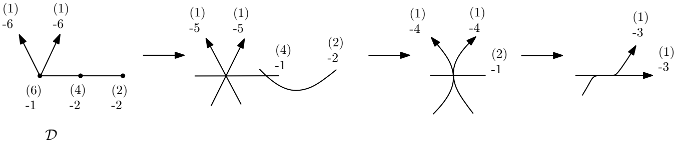


Figure: ϕ_3 - Fiber V

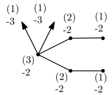
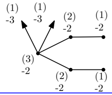
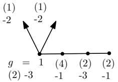
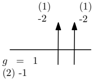
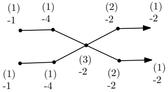
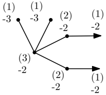
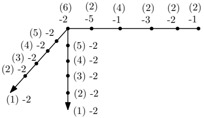
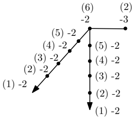
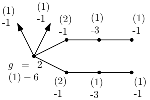
ϕ_3^2			: Fiber III
ϕ_3^3			: Fiber II (Matsumoto)
ϕ_3^4			: Fiber III
ϕ_3^5			: Fiber V*
ϕ_3^6 = id			: Generic Fiber

Figure: ϕ_3^k, $k = 2, \cdots, 6$

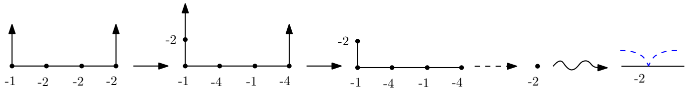


Figure: ϕ_2 and Fiber VII

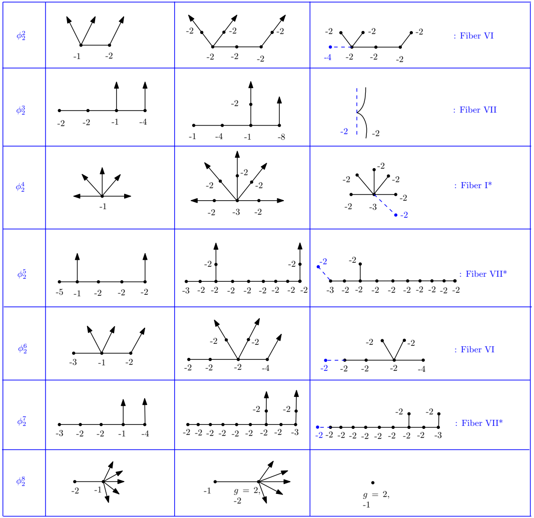


Figure: ϕ_2^k, $k = 2, \cdots, 8$

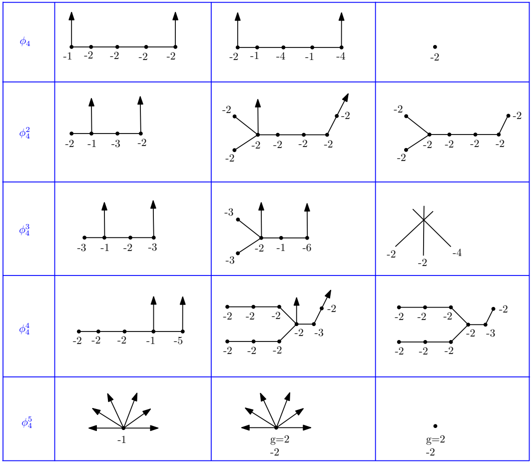


Figure: ϕ_4^k, $k = 1, \cdots, 5$

Our work

Next, we split the fibrations on X_f's into Lefschetz fibrations and give their monodromies:

Our work

Next, we split the fibrations on X_f's into Lefschetz fibrations and give their monodromies:

Theorem (S., Van Horn-Morris, '23)

The genus 2 fibration on the resolution X_f of a singular algebraic variety $V(f)$ where f is

i) $y^2 - x^5 - t^k$, $k = 1, \ldots, 6, 8, 9, 10$,

ii) $y^2 - x^6 - t^k$, $k = 1, 2, 4, 5, 6$,

iii) $y^2 - x(x^4 + t^k)$, $k = 1, \ldots, 8$,

iv) $y^2 - x(x^5 + t^k)$, $k = 1, \ldots, 10$,

splits into a Lefschetz fibration described by one of the following positive words in the mapping class group of the genus 2 surface with either 1 or 2 boundary components:

$$\phi_1, \phi_2, \phi_3, \phi_4, \phi_{\tilde{2}}, \phi_{\tilde{4}}, I, \phi_A, \phi_B, \phi_1^2, \phi_1^3, \phi_2^2, \phi_3^2, \phi_4^2, \phi_{\tilde{2}}^2, \phi_{\tilde{4}}^2,$$
$$\phi_1 I, \phi_2 I, \phi_{\tilde{2}} I, \phi_{\tilde{4}} I, \phi_3\phi_A, \phi_4\phi_B, \tau_\partial$$

Here

- $\phi_1 = \tau_1\tau_2\tau_3\tau_4$,
- $\phi_2 = \tau_1\tau_1\tau_2\tau_3\tau_4$,
- $\phi_{\tilde{2}} = \tau_5\tau_{5'}\tau_4\tau_3\tau_2$,
- $\phi_{\tilde{4}} = \tau_{5'}\tau_5\tau_4\tau_3\tau_2\tau_1$
- I is the hyperelliptic involution on the genus two surface with factorization $I = \tau_1\tau_2\tau_3\tau_4\tau_5\tau_{5'}\tau_4\tau_3\tau_2\tau_1$,

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for the surface with one boundary component, and for the surface with two boundary components we have,

- $\phi_3 = \tau_1\tau_2\tau_3\tau_4\tau_5$,
- $\phi_4 = \tau_1\tau_1\tau_2\tau_3\tau_4\tau_5$,
- $\phi_A = \tau_1\tau_4\tau_3\tau_{a_1}\tau_2\tau_5\tau_1\tau_4\tau_{b_1}\tau_{b_1'}$
- $\phi_B = \tau_4\tau_{a_2}\tau_3\tau_5\tau_2\tau_4\tau_{b_2}\tau_{b_2'}$,

and τ_∂ stands for the boundary (multi-)twist on the surfaces $\Sigma_{2,1}$ and $\Sigma_{2,2}$ (whichever happens to be under consideration).

- Here, we consider factorizations in the mapping class group

$$\mathrm{MCG}(\Sigma_{2,\epsilon}) := \mathrm{Diff}^{+}(\Sigma_{2,\epsilon}, \partial\Sigma_{2,\epsilon})/\text{isotopy rel boundary},$$

where $\epsilon = 1$ or 2 is the number of boundary components of the generic fiber.

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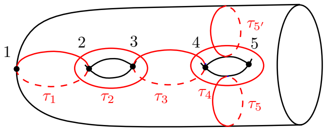
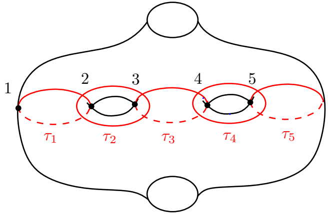
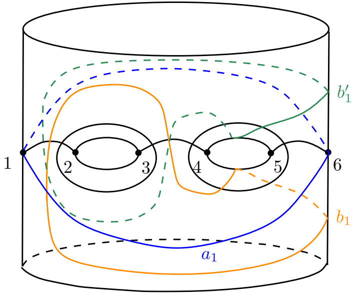
- The Dehn twists $\tau_1, \cdots, \tau_5, \tau_{5'}$ are the standard generators of the hyperelliptic subgroup of the mapping class group of the genus two surface as shown below:

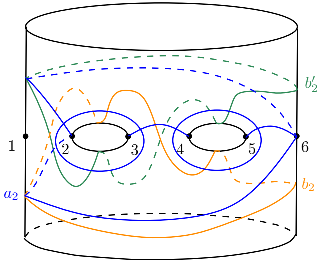




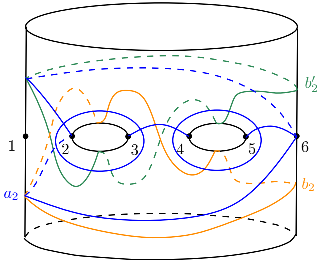
- The curves a_1, b_1 and b_1' from ϕ_A are:



- .. and the curves a_2, b_2 and b_2' from ϕ_B are:



- .. and the curves a_2, b_2 and b_2' from ϕ_B are:



- The monodromies ϕ_i, $i = 1, 2, 3, 4$ are of orders $10, 8, 6, 5$ respectively.

Splitting Algebraic Singular Fibrations via Perturbation of Branch Covers

Splitting Algebraic Singular Fibrations via Perturbation of Branch Covers

- In the above work, we constructed all genus two singular fibers that have monodromies with finite order. We built them in a different way than given by Namikawa-Ueno (N.-U.), but from the same polynomials given by N.-U.

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- In our recent work with Van Horn-Morris we have worked on these singularities. We constructed them from polynomials that are different from the polynomials *suggested* by N.-U. In this way we were able to obtain the exact same fibers as given on their list.

Splitting Algebraic Singular Fibrations via Perturbation of Branch Covers

- That is, for each singularity we obtain the same configuration as given by N.-U., and the multiplicity, self-intersection and genus of each irreducible component of our configurations also agree with those of N.-U. fibers.

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- Let us note that Matsumoto studied one of N.-U. fibers in their list (different from what we studied in this work). He constructed it from a different polynomial. He recovered the same central fiber as given by N.-U., but the total space is not biholomorphic to the total space of the N.-U. fiber.

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Splitting Algebraic Singular Fibrations via Perturbation of Branch Covers

Theorem (S., Van Horn-Morris, '23)

The genus two fibration on the resolution X of a singular algebraic variety $V(f)$ where f is

$$t^k = x(x^3 - y^2),\ k = 1, \ldots, 8,$$

splits into a Lefschetz fibration described by one of the following positive words in the mapping class group of the genus two surface with 2 boundary components:

- $\psi_1 = \tau_4\tau_3\tau_2\tau_1\tau_{1'}$,
- $\psi_1^2 = (\tau_4\tau_3\tau_2\tau_1\tau_{1'})^2$,
- $\psi_{\tilde{1}} = \tau_2\tau_3\tau_4\tau_5\tau_{5'}\tau_{\partial_1}$,
- $\psi_1^4 = \psi_{\tilde{1}}\psi_1$,
- $\psi_1^5 = \psi_{\tilde{1}}\psi_1^2$,
- $\psi_1^6 = \psi_{\tilde{1}}^2$,
- $\psi_1^7 = \psi_{\tilde{1}}^2\psi_1$,
- $\psi_1^8 = \tau_{\partial_1}^3\tau_{\partial_2}$

where the labeling agrees with the labeling of the Dehn twist curves on the surface $\Sigma_{2,2}$ shown in the figure below and τ_{∂_i} stand for the boundary multitwists on the surface $\Sigma_{2,2}$.

Splitting Algebraic Singular Fibrations via Perturbation of Branch Covers

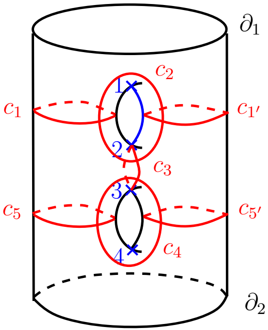

Figure: The Dehn twists τ_i are about the curves c_i (or ∂_j) for $i = 1, 1', 2, 3, 4, 5, 5', \partial_1, \partial_2$.

Splitting Algebraic Singular Fibrations via Perturbation of Branch Covers

We also give a new lift of the hyperelliptic involution to the mapping class group of $\Sigma_{2,2}$:

Splitting Algebraic Singular Fibrations via Perturbation of Branch Covers

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Proposition

The following relation holds in the mapping class group of the surface $\Sigma_{2,2}$

$$(\tau_2\tau_3\tau_4\tau_5\tau_{5'}\tau_4\tau_3\tau_2\tau_1\tau_{1'})^2 = \tau_{\partial_1}\tau_{\partial_2}$$

and moreover, the mapping class element represented by $\widetilde{I} = \tau_2\tau_3\tau_4\tau_5\tau_{5'}\tau_4\tau_3\tau_2\tau_1\tau_{1'}$ *is isotopic to the involution on* $\Sigma_{2,2}$ *with four fixed points whose quotient is the annulus.*

THANKS!