Geometry of Lagrangian Cobordisms

SEC Geom/Top Conf
May 2025

The goal of this pair of talks is to give you some insight into Lagrangian cobordisms and their symplectic geometry (NOT topology ☺)

Briefly, we'll be working in symplectic manifolds, like $(\Sigma \times \mathbb{C}, \omega_\Sigma \oplus dx \wedge dy)$. Lagrangian submanifolds, which are ½-dimensional submanifolds w/ zero symplectic area (ie. $\omega|_L = 0$) are of paramount importance in symplectic geometry and topology... and even in smooth topology.

area form on Σ

Part I: Define several types of Lagrangian cobordism, formulate some questions, and sketch some example geometric results

Part II: Describe a method (a novel Floer homology) for obtaining one of these results (joint work w/ Ipsita Datta)

PART I: CONTEXT and QUESTIONS

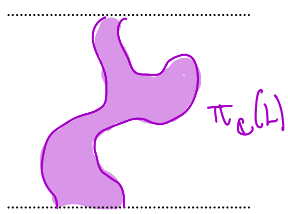
LAGRANGIAN COBORDISMS

We will work in $\Sigma \times (\mathbb{R} \times i[y_-, y_+])$ or $\Sigma \times ([x_-, x_+] \times i\mathbb{R})$, depending on convention.

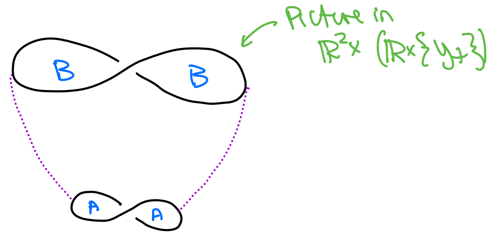
$\bar{\mathbb{C}}$ $\qquad$ $|\mathbb{C}|$

Defn: Let $L^\pm$ be 1-submanifolds of $\Sigma \times \mathbb{R}$. A (general) Lagrangian cobordism L between L^- and L^+ is a cpt, ori, properly embedded Lagr. submanifold of $\Sigma \times \bar{\mathbb{C}}$ s.t. $\partial^\pm L = L^\pm$.

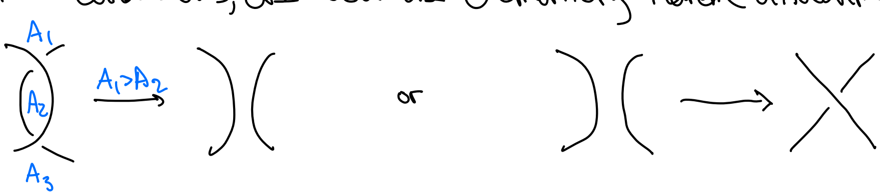
Cartoon picture: Project to $\bar{\mathbb{C}}$



Key example: embedded pieces of Whitney immersion in $\mathbb{R}^2 \times \bar{\mathbb{C}}$



There are diagrammatic constructions due to Lin that look like Reidemeister moves with area conditions, as well as elementary handle attachments, such as

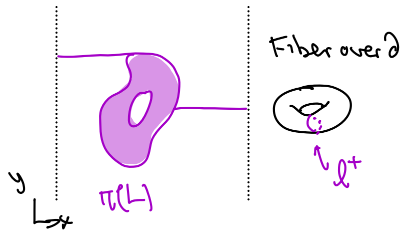


Two important special cases have been studied extensively. Recent extensive study by Biran-Cornea et al.

Defn Let $\ell^{\pm}$ be Lagrangian submanifolds of Σ. An (Arnold) Lagrangian cobordism L is a cpt, ori, properly embedded Lagrangian submanifold of $\Sigma \times |\mathbb{C}|$ that looks like $[x_-; x_- + \varepsilon) \times \ell^-$ near x_-

$(x_+ - \varepsilon, x_+] \times \ell^+$ near x_+

Cartoon picture:



Constructions:
- $\ell \times \gamma$
- Lift of Ham isotopy of ℓ
- Trace of (Polterovich) surgery.

Defn: Let $\Lambda^{\pm}$ be Legendrian submflds of $(\mathbb{R}^3, \alpha_0)$. A (SFT) Lagrangian cobordism L is a cpt, ori, properly embedded Lagrangian submfld of $\mathbb{R}^2 \times |\mathbb{C}|$ (w/ $x_- > 0$) that looks like: $[x_-, x_- + \varepsilon) \times \Lambda^-$ near x_-

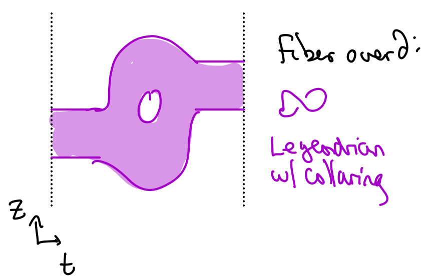
$(x_+ - \varepsilon, x_+] \times \Lambda^+$ near x_+

Not the usual description of the ambient mfld; using $\mathbb{R} \times \mathbb{R}^3 \simeq T^*(\mathbb{R}_+ \times \mathbb{R})$

t

$(a(s), t b(s), c(s))$

Cartoon picture:



Constructions:
- Cylinder over Λ
- lift of Leg isotopy
- Trace of ambient Legendrian surgery.

GEOMETRIC QUESTIONS

Geometric questions about Lagrangian Cobordisms tend to be of the following form:

Especially areas

- How do measurements at the ends restrict the geometry in the middle? or
- How do measurements at the ends compare?

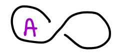
Shadows

The cartoon pictures above suggest the first example of such a question:

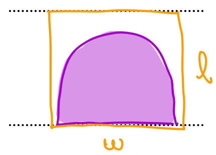
- For given ends, how "large" is the "shadow" $\pi_{\mathbb{C}}(L)$ in $\overline{\mathbb{C}} / |\mathbb{C}|$?

Here are some sample results... (NOT the most general statements! Just want to build intuition + give a flavor!)

For general Lagr cobordisms:

Thm (S-Traynor '10) Suppose $L \subset \mathbb{R}^2 \times \bar{\mathbb{C}}$ is a disk with $L^- =$ 

, $L^+ = \emptyset$.

If: 

Then: $lw \geq A$
("Lagr. nonsqueezing")

Tool: Generating families

For Arnold cobordisms: Let $Out(L)$ be the complement of the union of unbounded components of $\mathbb{C} \setminus \pi(L)$, and the shadow of L be

$$S(L) = Area(Out(L))$$

We may then try to define a distance between Lagrangians in Σ by

$$d(\ell^-, \ell^+) = \inf \{ S(L) : L \text{ cobordism from } \ell^- \to \ell^+ \}$$

see Biran-Cornea-Shelukhin for extensions (Tool: Fukaya category)

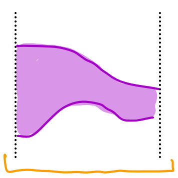
Thm (Cornea-Shelukhin '15) So long as all Lagrangians are "weakly monotone", d is, indeed, a metric (possibly w/ some, but not all, $d = \infty$).

Tool: Lagr Floer homology

The measurements in the results above are, unsurprisingly for the symplectic world, areas. More surprisingly, we can also get measurements for lengths when we restrict to SFT cobordisms.

Thm (S-Traynor '16) Let L be an SFT cobordism from $\Lambda^- =$ 

$(A > 1)$
to $\Lambda^+ =$ 

Then: 

Technique: LCH

Width [Skip if time is short, saying only the first sentence?]

The shadow measurements above are related to a fundamental measurement in symplectic geometry: the (relative) Gromov width. The classical Gromov width of a symplectic manifold is:

$$w(M) = \sup \{ \pi r^2 : B(r) \xhookrightarrow{\omega} (M, \omega) \}$$

The relative Gromov width of a Lagrangian $L \subset M$ insists that the $x_1 \ldots x_n$ plane in $B(r)$ maps to L; you can also insist that the embedding misses another Lagrangian:

$$w(L, L') = \sup \left\{ \frac{\pi r^2}{2} : \exists e : B(r) \xhookrightarrow{\omega} M \text{ s.t. } e^{-1}(L) = \mathbb{R}^n, \ e^{-1}(L') = \emptyset \right\}$$

Thm (Cornea-Shelukhin '15) If L is (relatively) exact, then
$$S(L) \geq w(\ell^-, \ell^+).$$

This is how they get previous result: Use Floer thry to find a J-hol disk $\Rightarrow w(\ell^-, \ell^+) > 0$.

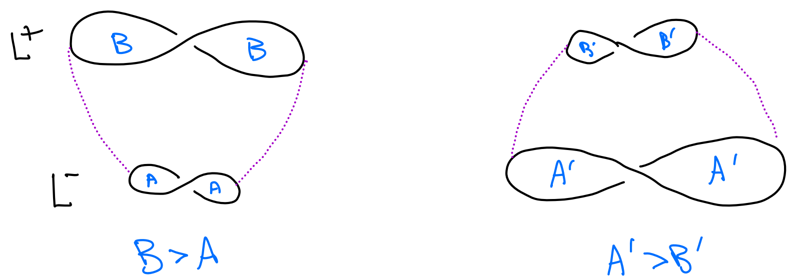
For SFT cobordisms, the relative widths at the ends bound the length of a cobordism

Thm (S-Traynor '20) Under some technical conditions,

$$\mathrm{len}(L) \geq w([0,1]\times\Lambda_+) - w([0,1]\times\Lambda_-)$$

Area and Direction

General Lagrangian cobordisms have an additional piece of geometry: the height of crossings at an end determines the rate of change of areas bounded. We have already seen this in our key example:



This restriction on changes in areas is global, not merely local:

Thm (S-Traynor '10, Datta '23) If a Lagr cobordism L has slices $L^\pm$ as above, then $B > A$.

Tool: Generating families

Tool: Hands-on use of J-hol curves.

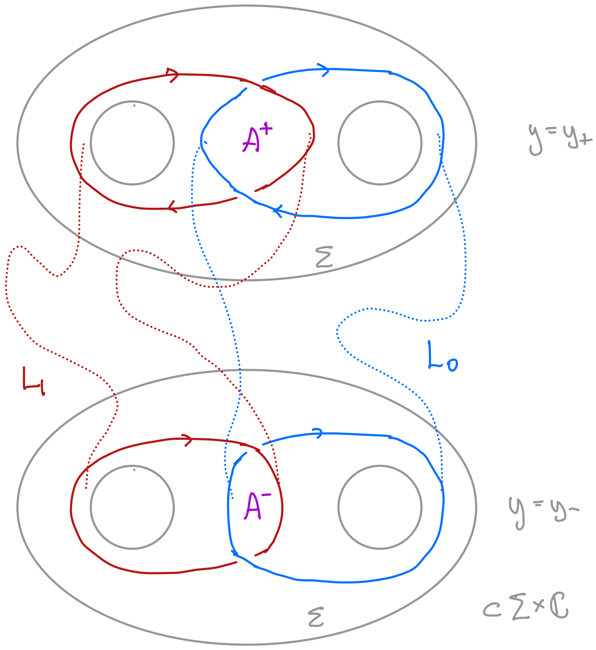
In the second part of this series, we will discuss how to "algebraicize" Datta's J-holomorphic curves into a novel Floer theory in a related setting, and we will use it to solve a related "local to global" problem in comparing geometry at the ends of a pair of general cobordisms.

PART II: FLOER THEORY for LAGRANGIAN COBORDISMS

Work in progress w/ Ipsita Datta

SETTING

The goal for today's talk is to understand this picture as much as we can! Let's start by introducing the objects.



- Symplectic mfld $(\Sigma \times \underbrace{(\mathbb{R} \times i[y_-, y_+])}_{\mathbb{C}}, \omega_\Sigma \oplus dx \wedge dy)$
- Lagrangian submflds L_0, L_1 $(\omega|_{L_i} = 0)$ (oriented, properly embedded, some other technical assumptions to prevent bubbling + define a relative action) "Lagrangian cobordism link"
- Slices $L_i^\pm$, which are links in $\Sigma \times \mathbb{R}$ w/ $\pi_\Sigma(L_i^\pm)$ essential scc's.

Technical assumptions:
- $\omega|_{\pi_2(\Sigma \times \mathbb{C}, L)} = 0$
- $\omega|_{\pi_1(\Omega(L_0, L_1), \ell_0)} = 0$

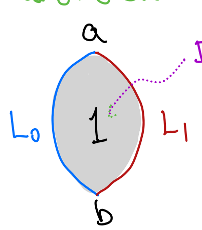
Sample Thm: $A^- < A^+$ (i.e. there is interesting symplectic geometry here)

FLOER THEORY for LAGRANGIAN COBORDISM LINKS

I won't give details, but I'd like to convey what's difficult / interesting and then outline the shape of the theory.

First, we add small collars to shrink the x coordinates at the ends so that $L_i^\pm$ all have x coordinate 0, and, like the Arnold setting, we can think of $L_i^\pm$ as lying in Σ. We call these revised objects Lagrangian tangle links. (and we'll abuse notation...). The ends of each tangle are still embedded, but now $L_0^\pm \cap L_1^\pm \neq \emptyset$ (there are no interior intersections).

The usual Floer story would now do the following:
- Define a chain group $C(L_0, L_1)$, over $\mathbb{Z}/2$, generated by $L_0 \cap L_1$ and filtered by some "action". [Critical points of action functional]
- Define a differential ∂ using rigid J-hol disks in $\Sigma \times \mathbb{C}$ (Index 1) [∇ flowlines of action functional]



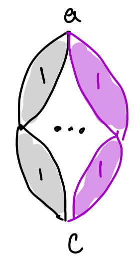
Index (dim = index − 1)

Contributes to $\partial a = b$ and relative action $\mathcal{A}(a, b) = $ Area of disk

Technically:
- J tame, i.e. $\omega(\cdot, J\cdot)$ is nondeg.
- $J = j \oplus i$ near ∂

[Nonempty contractible. Get automatic $\pitchfork$ in this case b/c of dimension]

- Prove $\partial^2=0$ by looking at boundaries of 1D (index 2) moduli spaces of J-hol disks, which tend to be 2-level "broken" disks:



$$\partial^2 a = c + c = 0$$

[Pairing broken ∇ flowlines:

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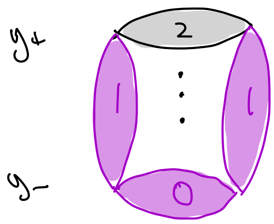
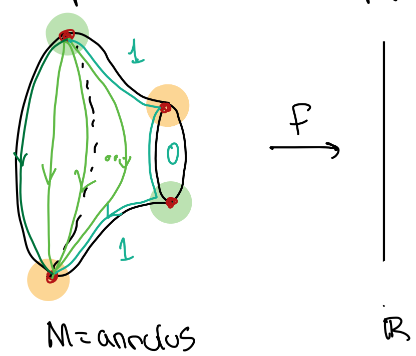
...but this does not work for Lagrangian tangles. We get a phenomenon like this: (This is Daha's thesis!)



This, to say the least, is disconcerting.

We need a new framework. The idea is to mimic Kronheimer-Mrowka's Morse theory for manifolds with boundary, where ∇f // boundary. A similar phenomenon happens there!

Index difference 2

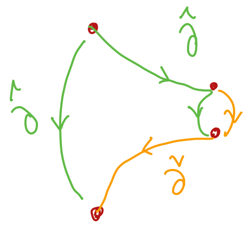


M = annulus

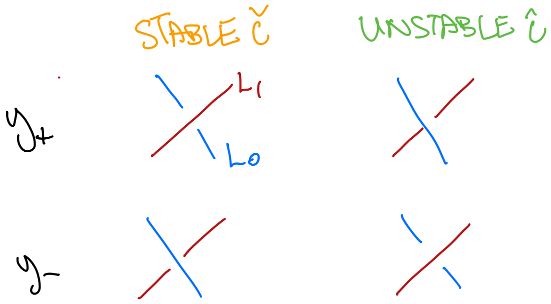
The idea is to form <u>three</u> filtered (by height) complexes:

Stable ($\check{C}$) (flowlines in M go <u>in</u>)
Unstable ($\hat{C}$) (flowlines in M go <u>out</u>)
Boundary $\bar{C} = \hat{C} \oplus \check{C}$.

Define parts of the differential using 2-level flowlines as well as simple flowlines in ∂M (w/ $\bar{\partial}$ only depending on flowlines in ∂M)



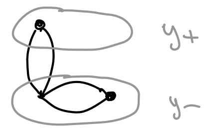
We can use the <u>same</u> framework, with the local geometry of the Lagrangians letting us identify stable and unstable intersections as follows:



$\bar{\partial}$: Riemann mapping thm lets us compute this combinatorially in the boundary
(cf. da Silva - Robbin - Salamon)

$\hat{\partial}$: Some $\bar{\partial}$ disks, plus:

($\check{\partial}$ similar)



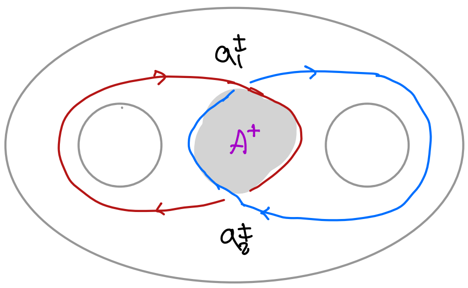
Thm: In the original setup, $\exists$ filtered chain complexes $(\bar{C},\bar{\partial})$, $(\hat{C},\hat{\partial})$, and $(\check{C},\check{\partial})$ related by a LES

$$\to H\bar{C} \to H\check{C} \to H\hat{C} \xrightarrow{\partial_*}$$

Note: Filtration is by ACTION, w/ ∂ lowering action by ω-areas of the T-hol curves.

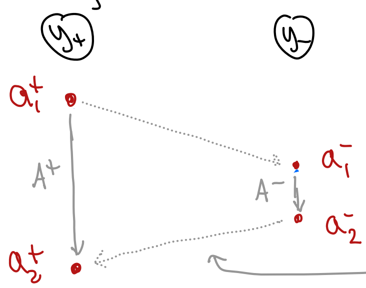
APPLICATION

Going back to the motivating example...



$\bar{\partial} a_1^\pm = a_2^\pm$

So we get $H\bar{C} = 0$, which yields an isomorphism $H\check{C} \simeq H\hat{C}$. A little bit of staring at the algebra tells us that the actions of the intersection points are arranged precisely like those for the Morse function on the annulus above:



Disks forced by algebra.
Make action go down!
[Draw on original]

[In particular, $A^+ > A^-$