

# Stochastic Functional Kolmogorov Equations (I): Persistence

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## Abstract

This work (Part (I)) together with its companion (Part (II)) develops a new framework for stochastic functional Kolmogorov equations, which are nonlinear stochastic differential equations depending on the current as well as the past states. Because of the complexity of the results, it seems to be instructive to divide our contributions to two parts. In contrast to the existing literature, our effort is to advance the knowledge by allowing delay and past dependence, yielding essential utility to a wide range of applications. A long-standing question of fundamental importance pertaining to biology and ecology is: What are the minimal necessary and sufficient conditions for long-term persistence and extinction (or for long-term coexistence of interacting species) of a population? Regardless of the particular applications encountered, persistence and extinction are properties shared by Kolmogorov systems. While there are many excellent treaties of stochastic-differential-equation-based Kolmogorov equations, the work on stochastic Kolmogorov equations with past dependence is still scarce. Our aim here is to answer the aforementioned basic question. This work, Part (I), is devoted to characterization of persistence, whereas its companion, Part (II), is devoted to extinction. The main techniques used in this paper include the newly developed functional Itô formula and asymptotic coupling and Harris-like theory for infinite dimensional systems specialized to functional equations. General theorems for stochastic functional Kolmogorov equations are developed first. Then a number of applications are examined to obtain new results substantially covering, improving, and extending the existing literature. Furthermore, these conditions reduce to that of Kolmogorov systems when there is no past dependence.

**Keywords.** Stochastic functional equation, Kolmogorov system, ecological and biological application, invariance measure, extinction, persistence.

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## 1 Introduction

This work develops a novel framework of systems of stochastic functional Kolmogorov equations. Our main motivation stems from a wide variety of applications in ecology and biology. A long-standing question of fundamental importance pertaining to biology and ecology is: What are the minimal (necessary and sufficient) conditions for long-term persistence and extinction (or for long-term coexistence of interacting species) of a population? It turns out that persistence and extinction are phenomena go far beyond biological and ecological systems. In fact, such long-term properties are shared by all processes of Kolmogorov type. We focus on the issues for such systems that involve stochastic disturbances and past dependence in the dynamics. The problems are substantially more difficult compared to systems without delay or past dependence because one has to treat infinite dimensional processes.

An  $n$ -dimensional deterministic Kolmogorov system is an autonomous system of equations to depict the dynamics of  $n$  interacting populations, which takes the form

$$\dot{x}_i(t) = x_i(t)f_i(x_1(t), \dots, x_n(t)), \quad i = 1, \dots, n, \quad (1.1)$$

where  $f_i(\cdot)$  are functions satisfying suitable conditions. Realizing that fluctuations of the environment make the dynamics of populations inherently stochastic, much effort has been placed on the study of stochastic Kolmogorov equations. As an example, consider a simple 2-dimensional Kolmogorov equations with stochastic effects:

$$\begin{cases} dx(t) = x(t)f_1(x(t), y(t))dt + x(t)g_1(x(t), y(t))dB_1(t) \\ dy(t) = y(t)f_2(x(t), y(t))dt + y(t)g_2(x(t), y(t))dB_2(t), \end{cases} \quad (1.2)$$

where  $B_1(t)$  and  $B_2(t)$  are two Brownian motions (independent or not). The formulation readily generalizes to  $n$ -dimensional stochastic Kolmogorov equations, which are used extensively in the modeling and analysis of ecological and biological systems such as Lotka-Volterra predator-prey models, Lotka-Volterra competitive model, replicator dynamic systems, stochastic epidemic models, and stochastic chemostat models, among others. The study of such systems has encompassed the

central issues of persistence and extinction as well as the existence of invariant measures. Apart from ecological and biological systems, numerous problems arising in mathematical physics, statistical mechanics, and many related fields, use Kolmogorov nonlinear stochastic differential equations. We mention a simple one-dimensional generalized Ginzburg-Landau equation

$$dx(t) = x(t)[a(t) - b(t)x^k(t)] + x(t)\sigma(x(t))dB(t), \quad x(0) = x_0 > 0, \quad (1.3)$$

$k \geq 2$  is a positive integer,  $B(t)$  is a real-valued Brownian motion. Such equations have been used in the theory of bistable systems, chemical turbulence, phase transitions in non-equilibrium systems, nonlinear, optics with dissipation, thermodynamics, and hydrodynamic systems, etc.

Because of its prevalence in applications, Kolmogorov systems have attracted much attention in the past decades; substantial progress has been made. To proceed, let us briefly recall some of the developments to date. Some of the early mathematical formulations were introduced by Verhulst [60] for logistic models, by Lotka and Volterra [38, 61] for Lotka-Volterra systems, and by Kermack and McKendrick [28, 29] for infectious diseases modeling using ordinary differential equations in the last century. The study on mathematical models has stimulated subsequent work with attention devoted to analyzing and predicting the behavior of the populations in a longtime horizon. Subsequently, not only deterministic systems, but also stochastic systems have been studied. Resurgent effort has been devoted to finding the corresponding classification by means of threshold levels. Fast forward, Imhof studied long-run behavior of the stochastic replicator dynamics in [27], whereas Hofbauer and Imhof concentrated on time averages, recurrence, and transience for stochastic replicator dynamics in [26]. By now, Kolmogorov stochastic population systems (using stochastic differential equations or difference equations) together with their longtime behavior have been relatively well understood; see [5, 51, 53] for Kolmogorov stochastic systems in compact domains and [4, 24] for certain general Kolmogorov systems in non-compact domains. Variants of Kolmogorov systems such as epidemic models [12, 15, 58], tumor-immune system [59] and chemostat models [43] etc. have also been studied. In contrast to numerous papers that used Lyapunov function methods to analyze the underlying systems with limited success, Benaïm [4], Benaïm and Lobry [5], Benaïm and Strickler [6], Chesson and Ellner [9], Evans, Henning, and Schreiber [19], and Schreiber and Benaïm [53] initiated the study by examining the corresponding boundary behavior and considered the stochastic rate of growth; see also Du, Nguyen, and Yin [16]. For the most recent development and substantial progress, we refer to Benaïm [4], Henning and Nguyen [24], Schreiber and Benaïm [53], and references therein.

Our study in this work is to consider a class of  $n$ -dimensional stochastic functional Kolmogorov systems; our effort is to substantially advance the existing literature by allowing delay and past dependence, which in turn, provides essential utility to a wide range of applications. Why is it important to consider systems with delays as well as stochastic functional Kolmogorov systems? Mainly, the delays or past dependence are unavoidable in natural phenomena and dynamical systems; the framework of stochastic functional differential equations is more realistic, more effective, and more general for the population dynamics in real life than a stochastic differential equation counterpart. In population dynamics, some delay mechanisms studied in the literature include age structure, feeding times, replenishment or regeneration time for resources [13]. Although there are many excellent treatises of Kolmogorov stochastic differential equations, the work on Kolmogorov stochastic differential equations with delay is relatively scarce. A few exceptions are the study on stochastic delay Lotka-Volterra competitive models [2, 34], the work on stochastic delay Lotka-Volterra predator-prey models [22, 31, 32, 33, 64], the treatment of stochastic delay epidemic SIR models [8, 35, 36, 37, 39], and the study on stochastic delay chemostat models [54, 55, 65]. Nevertheless, other than the specific models and applications treated, there has not been a unified framework and a systematic treatment for Kolmogorov stochastic functional differential systems

yet. Moreover, most of the existing results involving delay are not as sharp as desired. Our effort in this paper takes up the aforementioned issues.

It should be noted that from Kolmogorov stochastic differential equation-type models to that of stochastic functional differential equation models requires a big leap. There are certain essential difficulties. While the solutions of stochastic differential equations are Markovian processes, the solutions of stochastic differential equations with delay is non-Markov. Then one uses the so-called segment processes for the delay equations. However, such segment processes live in an infinite dimensional space. Many of the known results in the usual stochastic differential equation setup are no longer applicable. Because Kolmogorov systems are highly nonlinear, analyzing such systems with delay becomes even more difficult. New methods and techniques need to be developed to carry out the analysis. This brings us to the current work.

In this paper, we first set up the problem in a unified form, introduce the methodology to characterize the longtime behavior of the underlying system, to establish general results for the persistence, and to demonstrate the utility in a number of applications arising in ecology and biology. Our goal is to obtain sharp results under mild and verifiable conditions, which is useful for a wide variety of stochastic functional Kolmogorov systems. In view of the progress and challenges, this work combines the techniques of functional analysis (in particular, the functional Itô formula) in [10, 11], stochastic differential equations (SDEs) in infinite dimension, as well as the methods of asymptotic couplings [23], to develop a new framework for treating functional Kolmogorov systems. It will substantially generalize the methods in [4, 24]. Our results will cover, improve, and outperform the aforementioned existing results for Kolmogorov systems with and without delays. It should be mentioned that in the case of replicator dynamics, it seems to be no investigation of delayed stochastic systems to date to the best of our knowledge.

Although the models with functional stochastic differential equations are more realistic and more general, the analysis of such systems become far more difficult. Perhaps, part of the difficulties in studying stochastic delay systems is that there had been virtually no bona fide operators and functional Itô formulas except some general setup in a Banach space such as [42] before 2009. In [17], Dupire generalized the Itô formula to a functional setting by using pathwise functional derivatives. The Itô formula developed has substantially eased the difficulties and encouraged subsequent development with a wide range of applications. His work was developed further by Cont and Fournié [10, 11]. Using the newly developed functional Itô formula enables us to analyze effectively the segment processes in the stochastic functional Kolmogorov equations.

Because of the non-Markovian property of the solution processes due to delay and the use of memory segment functions, one needs to analyze the corresponding stochastic equations in an infinite dimensional space. Handling occupation measures in an infinite dimensional space to obtain the tightness and characterize its limit is more challenging, so is to prove the uniqueness of the invariant measure. The associated Markov semigroups are often not strong Feller, even in some simple cases. Because of the absence of the strong Feller property, Doob's method to prove uniqueness of the invariant measure is no longer applicable; see [50]. There are some recent works on asymptotic analysis for functional stochastic differential equations; for example, see [3] and references therein. Most notably, in [23], Hairer, Mattingly, and Scheutzow developed a necessary and sufficient condition for uniqueness of the invariant measure using asymptotic couplings, provided sufficient conditions for weak convergence to the invariant measure, and obtained a Harris-like theory for general infinite-dimensional spaces. By using ideas from this abstract theory and our subtle estimates for certain coupled systems, we are able to prove the uniqueness of invariant measure for the Kolmogorov systems. To characterize the longtime behavior of the underlying system under natural conditions and to develop a systematic method for this kind problem, we use the intuition from dynamical system theory, in which we need to examine the corresponding problem on the

boundary and reveal the behavior of the process when it is close to the boundary. Nevertheless, the behavior of solutions near the boundary for functional Kolmogorov systems requires more delicate analysis than that for systems without delay. Even if the current state is close to the boundary, its history may not be.

The rest of the paper is organized as follows. Section 2 presents the formulation of our problem as well as mathematical definitions and terminologies; and proceeds to state our main results of the persistence of the stochastic functional Kolmogorov systems. Section 3 examines basic properties of Kolmogorov equations with delays, including well-posedness of the system, positivity of solutions. Also obtained are the tightness of families of occupation measures and the convergence to the corresponding invariant measures. To obtain the desired theory, a number of key auxiliary results are provided. Then the conditions for persistence of Kolmogorov systems are given in Section 4. Finally, Section 5 provides several applications involving Kolmogorov dynamical systems and detailed account on how to use our results on stochastic functional Kolmogorov equations to treat each of the application examples.

## 2 Main Results

To help the reading, we first provide a glossary of symbols and notation to be used in this paper.

$r$	a fixed positive number
$ \cdot $	Euclidean norm
$\mathcal{C}([a; b]; \mathbb{R}^n)$	set of $\mathbb{R}^n$ -valued continuous functions defined on $[a; b]$
$\mathcal{C}$	$:= \mathcal{C}([-r; 0]; \mathbb{R}^n)$
$\varphi = (\varphi_1, \dots, \varphi_n)$	$\in \mathcal{C}$
$\mathbf{x} = (x_1, \dots, x_n)$	$:= \varphi(0) \in \mathbb{R}^n$
$\ \varphi\ $	$:= \sup\{ \varphi(t)  : t \in [-r, 0]\}$
$\mathbf{X}_t$	$:= \mathbf{X}_t(s) := \{\mathbf{X}(t+s) : -r \leq s \leq 0\}$ (segment function)
$X_{i,t}$	$:= X_{i,t}(s) := \{X_i(t+s) : -r \leq s \leq 0\}$
$\mathcal{C}_+$	$:= \{\varphi = (\varphi_1, \dots, \varphi_n) \in \mathcal{C} : \varphi_i(s) \geq 0 \forall s \in [-r, 0], i = 1, \dots, n\}$
$\partial\mathcal{C}_+$	$:= \{\varphi = (\varphi_1, \dots, \varphi_n) \in \mathcal{C} : \ \varphi_i\  = 0 \text{ for some } i = 1, \dots, n\}$
$\mathcal{C}_+^\circ$	$:= \{\varphi \in \mathcal{C}_+ : \varphi_i(s) > 0, \forall s \in [-r, 0], i = 1, \dots, n\} \neq \mathcal{C}_+ \setminus \partial\mathcal{C}_+$
$\ \varphi\ _\alpha$	$:= \ \varphi\  + \sup_{-r \leq s \neq t \leq 0} \frac{ \varphi(t) - \varphi(s) }{(t-s)^\alpha}, \text{ for some } 0 < \alpha < 1$
$\mathcal{C}^\alpha$	space of Hölder continuous functions endowed with the norm $\ \cdot\ _\alpha$
$\Gamma$	$n \times n$ matrix
$\Gamma^\top$	transpose of $\Gamma$
$\mathbf{B}(t)$	$= (B_1(t), \dots, B_n(t))^\top$ , a vector of independent standard Brownian motions
$\mathbf{E}(t)$	$= (E_1(t), \dots, E_n(t))^\top := \Gamma^\top \mathbf{B}(t)$
$\Sigma$	$= (\sigma_{ij})_{n \times n} := \Gamma^\top \Gamma$
$\mathcal{M}$	set of ergodic invariant probability measures of $\mathbf{X}_t$ supported on $\partial\mathcal{C}_+$
$\text{Conv}(\mathcal{M})$	convex hull of $\mathcal{M}$
$\mathbf{0}$	the zero constant function in $\mathcal{C}$
$\delta^*$	the Dirac measure concentrated at $\mathbf{0}$
$\mathbf{1}_A$	the indicator function of set $A$
$D_{\varepsilon, R}$	$:= \{\varphi \in \mathcal{C}_+ : \ \varphi\  \leq R, x_i \geq \varepsilon \forall i; \mathbf{x} := \varphi(0)\}, \varepsilon, R > 0$
$\mathbb{D}$	space of Cadlag functions mapping $[-r, 0]$ to $\mathbb{R}^n$
$A_0, A_1, A_2, \gamma_0, \gamma_b, M$	constants satisfying Assumption 2.1
$\mathbf{c}, h(\cdot), \mu$	vector, function and probability measure satisfying Assumption 2.1

$\tilde{K}, b_1, b_2$	constants satisfying Assumption 2.2
$h_1(\cdot), \mu_1$	function and probability measure satisfying Assumption 2.2
$D_0, d_0$	constants satisfying Assumption 2.4
$I$	a subset of $\{1, \dots, n\}$
$I^c$	$:= \{1, \dots, n\} \setminus I$
$\mathcal{C}_+^I$	$:= \{\varphi \in \mathcal{C}_+ : \ \varphi_i\  = 0 \text{ if } i \in I^c\}$
$\mathcal{C}_+^{I,\circ}$	$:= \{\varphi \in \mathcal{C}_+ : \ \varphi_i\  = 0 \text{ if } i \in I^c \text{ and } \varphi_i(s) > 0 \text{ for all } s \in [-r, 0] \text{ if } i \in I\}$
$\partial \mathcal{C}_+^I$	$:= \{\varphi = (\varphi_1, \dots, \varphi_n) \in \mathcal{C} : \ \varphi_i\  = 0 \text{ if } i \in I^c \text{ and } \ \varphi_i\  = 0 \text{ for some } i \in I\}$
$\mathcal{M}^I, \mathcal{M}^{I,\circ}, \partial \mathcal{M}^I$	sets of ergodic measures on $\mathcal{C}_+^I, \mathcal{C}_+^{I,\circ}, \partial \mathcal{C}_+^I$ respectively
$I_\pi$	the subset of $\{1, \dots, n\}$ such that $\pi(\mathcal{C}_+^{I_\pi, \circ}) = 1, \pi \in \mathcal{M}$
$\gamma, p_0, A$	constants satisfying the condition in Lemma 3.1
$\rho = (\rho_1, \dots, \rho_n)$	vector satisfying the condition in Lemma 3.1
$V_\rho(\varphi)$	$:= \left(1 + \mathbf{c}^\top \mathbf{x}\right) \prod_{i=1}^n x_i^{\rho_i} \exp \left\{ A_2 \int_{-r}^0 \mu(ds) \int_s^0 e^{\gamma(u-s)} h(\varphi(u)) du \right\}$
$V_0(\varphi)$	$:= \left(1 + \mathbf{c}^\top \mathbf{x}\right) \exp \left\{ A_2 \int_{-r}^0 \mu(ds) \int_s^0 e^{\gamma(u-s)} h(\varphi(u)) du \right\}$
$\mathcal{C}_{V,M}$	$:= \{\varphi \in \mathcal{C}_+ : A_2 \gamma \int_{-r}^0 \mu(ds) \int_s^0 e^{\gamma(u-s)} h(\varphi(u)) du \leq A_0 \text{ and }  \mathbf{x}  \leq M\}$
$H_1$	constant satisfying (3.29) and (3.36)
$\rho^*, \kappa^*$	vector and constant satisfying (4.1)
$n^*$	constant satisfying $\gamma_0(n^* - 1) - A_0 > 0$
$p_1$	constant satisfying condition (3.4) and $p_1 > p_0$
$H_1^*, R_{V,M}, \varepsilon^*$	constants determined in (4.12), (4.15) and (4.16), respectively
$T^*, \hat{\delta}$	constants determined in Lemma 4.3
$\mathcal{C}_V(\hat{\delta})$	$:= \{\varphi \in \mathcal{C}_+^\circ \cap \mathcal{C}_{V,M} \text{ and }  \varphi_i(0)  \leq \hat{\delta} \text{ for some } i\}$

Consider a stochastic delay Kolmogorov system

$$\begin{cases} dX_i(t) = X_i(t)f_i(\mathbf{X}_t)dt + X_i(t)g_i(\mathbf{X}_t)dE_i(t), & i = 1, \dots, n, \\ \mathbf{X}_0 = \phi \in \mathcal{C}_+, \end{cases} \quad (2.1)$$

and denoted by  $\mathbf{X}^\phi(t)$  its solution. For convenience, we usually suppress the superscript “ $\phi$ ” and use  $\mathbb{P}_\phi$  and  $\mathbb{E}_\phi$  to denote the probability and expectation given the initial value  $\phi$ , respectively. We also assume that the initial value is non-random. Denoted by  $\{\mathcal{F}_t\}_{t \geq 0}$  the filtration satisfying the usual conditions and assume that the  $n$ -dimensional Brownian motion  $\mathbf{B}(t)$  is adapted to  $\{\mathcal{F}_t\}_{t \geq 0}$ . Note that a segment process is also referred to as a memory segment function. Throughout the rest of the paper, we assume the following assumptions hold.

**Assumption 2.1.** The coefficients of (2.1) satisfy

- (1)  $\text{diag}(g_1(\varphi), \dots, g_n(\varphi)) \Gamma^\top \Gamma \text{diag}(g_1(\varphi), \dots, g_n(\varphi)) = (g_i(\varphi)g_j(\varphi)\sigma_{ij})_{n \times n}$  is a positive definite matrix for any  $\varphi \in \mathcal{C}_+$ .
- (2)  $f_i(\cdot), g_i(\cdot) : \mathcal{C}_+ \rightarrow \mathbb{R}$  are Lipschitz continuous in each bounded set of  $\mathcal{C}_+$  for any  $i = 1, \dots, n$ .
- (3) There exist  $\mathbf{c} = (c_1, \dots, c_n) \in \mathbb{R}^n$  with  $c_i > 0, \forall i$ , and  $\gamma_b, \gamma_0 > 0, A_0 > 0, A_1 > A_2 > 0, M > 0$ , a continuous function  $h : \mathbb{R}^n \rightarrow \mathbb{R}_+$  and a probability measure  $\mu$  concentrated on

$[-r, 0]$  such that for any  $\varphi \in \mathcal{C}_+$

$$\begin{aligned} & \frac{\sum_{i=1}^n c_i x_i f_i(\varphi)}{1 + \mathbf{c}^\top \mathbf{x}} - \frac{1}{2} \frac{\sum_{i,j=1}^n \sigma_{ij} c_i c_j x_i x_j g_i(\varphi) g_j(\varphi)}{(1 + \mathbf{c}^\top \mathbf{x})^2} + \gamma_b \sum_{i=1}^n \left( |f_i(\varphi)| + g_i^2(\varphi) \right) \\ & \leq A_0 \mathbf{1}_{\{|\mathbf{x}| < M\}} - \gamma_0 - A_1 h(\mathbf{x}) + A_2 \int_{-r}^0 h(\varphi(s)) \mu(ds), \end{aligned} \quad (2.2)$$

where  $\mathbf{x} := \varphi(0)$ . We assume without loss of generality that  $h : \mathbb{R}^n \rightarrow [1, \infty)$ , otherwise, we can always change  $\gamma_0$  and  $A_1, A_2$  to fulfill this requirement.

**Assumption 2.2.** One of following assumptions holds:

- (a) There is a constant  $\tilde{K}$  such that for any  $\varphi \in \mathcal{C}_+$ ,  $\mathbf{x} = \varphi(0)$

$$\sum_{i=1}^n |f_i(\varphi)| + \sum_{i=1}^n g_i^2(\varphi) \leq \tilde{K} \left[ h(\mathbf{x}) + \int_{-r}^0 h(\varphi(s)) \mu(ds) \right]. \quad (2.3)$$

- (b) There exist  $b_1, b_2 > 0$ , a function  $h_1 : \mathbb{R}^n \rightarrow [1, \infty]$ , and a probability measure  $\mu_1$  on  $[-r, 0]$  such that for any  $\varphi \in \mathcal{C}_+$ ,  $\mathbf{x} = \varphi(0)$

$$b_1 h_1(\mathbf{x}) \leq \sum_{i=1}^n |f_i(\varphi)| + \sum_{i=1}^n g_i^2(\varphi) \leq b_2 \left[ h_1(\mathbf{x}) + \int_{-r}^0 h_1(\varphi(s)) \mu_1(ds) \right]. \quad (2.4)$$

*Remark 1.* Let us comment on the above assumptions.

- The above assumptions (and additional assumptions provided later) are not restrictive, and are easily verifiable. Such conditions are widely used in popular models in the literature; see Section 5.
- Parts (2) and (3) of Assumption 2.1 guarantee the existence and uniqueness of a strong solution to (2.1). We need part (1) of Assumption 2.1 to ensure that the solution to (2.1) is a non-degenerate diffusion. Moreover, as will be seen later that (3) implies the tightness of the family of transition probabilities associated with the solution to (2.1). One difficulty stems from the positive term  $A_2 \int_{-r}^0 h(\varphi(s)) \mu(ds)$  on the right-hand side of (2.2), which cannot be relaxed in practice.
- Assumption 2.2 plays an important role in guaranteeing the  $\pi$ -uniform integrability of the function  $\sum_i (|f_i(\cdot)| + g_i^2(\cdot))$ , for any invariant measure  $\pi$ . It will become clear in Lemma 3.4 and Lemma 3.5 as well as the remaining parts of the paper.

As was alluded to, persistence and extinction are concepts of vital importance in biology and ecology. It turns out that such concepts are features shared by all stochastic functional Kolmogorov systems. While the termination of a species in biology is referred to as extinction, the moment of extinction is generally considered to be the death of the last individual of the species. In contrast to extinction, we have the persistence of a species. To proceed, similar to [24, 52, 53], we define persistence and extinction as follows.

**Definition 2.1.** Let  $\mathbf{X}(t) = (X_1(t), \dots, X_n(t))^\top$  be the solution of (2.1). The process  $\mathbf{X}$  is strongly stochastically persistent if for any  $\varepsilon > 0$ , there exists an  $R > 0$  such that for any  $\phi \in \mathcal{C}_+^\circ$

$$\liminf_{t \rightarrow \infty} \mathbb{P}_\phi \{ R^{-1} \leq |X_i(t)| \leq R \} \geq 1 - \varepsilon \text{ for all } i = 1, \dots, n. \quad (2.5)$$

**Definition 2.2.** With  $\mathbf{X}(t)$  given in Definition 2.1, for  $\phi \in \mathcal{C}_+^\circ$  and some  $i \in \{1, \dots, n\}$ , we say  $X_i$  goes extinct with probability  $p_\phi > 0$  if

$$\mathbb{P}_\phi \left\{ \lim_{t \rightarrow \infty} X_i(t) = 0 \right\} = p_\phi.$$

Let  $\mathcal{M}$  be the set of ergodic invariant probability measures of  $\mathbf{X}_t$  supported on the boundary  $\partial\mathcal{C}_+$ . Note that if we let  $\delta^*$  be the Dirac measure concentrated at  $\mathbf{0}$ , then  $\delta^* \in \mathcal{M}$  so that  $\mathcal{M} \neq \emptyset$ . For a subset  $\widetilde{\mathcal{M}} \subset \mathcal{M}$ , denote by  $\text{Conv}(\widetilde{\mathcal{M}})$  the convex hull of  $\widetilde{\mathcal{M}}$ , that is, the set of probability measures  $\pi$  of the form  $\pi(\cdot) = \sum_{\nu \in \widetilde{\mathcal{M}}} p_\nu \nu(\cdot)$  with  $p_\nu \geq 0$  and  $\sum_{\nu \in \widetilde{\mathcal{M}}} p_\nu = 1$ .

**Assumption 2.3.** For any  $\pi \in \text{Conv}(\mathcal{M})$ , we have

$$\max_{i=1, \dots, n} \{\lambda_i(\pi)\} > 0,$$

where

$$\lambda_i(\pi) := \int_{\partial\mathcal{C}_+} \left( f_i(\varphi) - \frac{\sigma_{ii} g_i^2(\varphi)}{2} \right) \pi(d\varphi). \quad (2.6)$$

**Theorem 2.1.** Assume that Assumptions 2.1, 2.2, and 2.3 hold. For any  $\varepsilon > 0$ , there exists a positive number  $R_* = R_*(\varepsilon)$  such that for all  $\phi \in \mathcal{C}_+^\circ$ ,

$$\liminf_{t \rightarrow \infty} \mathbb{P}_\phi \{ R_*^{-1} \leq |X_i(t)| \leq R_* \} \geq 1 - \varepsilon \text{ for all } i = 1, \dots, n.$$

It is well-recognized that the nondegeneracy of the diffusion is not sufficient to imply the strong Feller property as well as the uniqueness of an invariant probability measure of stochastic delay systems. The following assumption is needed to obtain the uniqueness of an invariant probability measure.

**Assumption 2.4.** The following conditions hold:

- (i) There are some constants  $D_0, d_0 > 0$  such that for any  $\varphi^{(1)}, \varphi^{(2)} \in \mathcal{C}_+^\circ$ ,  $i \in \{1, \dots, n\}$ ,

$$\begin{aligned} \left| f_i(\varphi^{(1)}) - f_i(\varphi^{(2)}) \right| &\leq D_0 \left| \mathbf{x}^{(1)} - \mathbf{x}^{(2)} \right| \left| 1 + \mathbf{x}^{(1)} + \mathbf{x}^{(2)} \right|^{d_0} \\ &\quad + D_0 \int_{-r}^0 \left| \varphi^{(1)}(s) - \varphi^{(2)}(s) \right| \left| 1 + \varphi^{(1)}(s) + \varphi^{(2)}(s) \right|^{d_0} \mu(ds), \end{aligned} \quad (2.7)$$

where  $\mathbf{x}^{(1)} := \varphi^{(1)}(0)$ ,  $\mathbf{x}^{(2)} := \varphi^{(2)}(0)$ .

- (ii) The conditions in (i) above holds with  $f_i(\cdot)$  replaced by  $g_i(\cdot)$  and  $g_i^2(\cdot)$ .

- (iii) The inverse of matrix  $(g_i(\varphi)g_j(\varphi)\sigma_{ij})_{n \times n}$  is uniformly bounded in  $\mathcal{C}_+^\circ$ .

**Proposition 2.1.** Under Assumptions 2.1 and 2.4, the solution of equation (2.1) has at most one invariant probability measure on  $\mathcal{C}_+^\circ$ .

**Theorem 2.2.** Under Assumptions 2.1, 2.2, 2.3, and 2.4, system (2.1) has a unique invariant measure concentrated on  $\mathcal{C}_+^\circ$ .

*Remark 2.* Assumption 2.3 means that all invariant measures on the boundary are repellers (because the maximum Lyapunov exponent of an invariant measure is positive), which guarantees that the solution in the interior cannot stay long near the boundary. As a result, the species coexist.

### 3 Preliminaries and Key Technical Results

#### 3.1 Existence, uniqueness, positivity, and key estimates of the solutions

To begin, we state the functional Itô formula for our processes; see [11] for more details. Let  $\mathbb{D}$  be the space of càdlàg functions  $\varphi : [-r, 0] \mapsto \mathbb{R}^n$ . For  $\varphi \in \mathbb{D}$ , with  $s \geq 0$  and  $y \in \mathbb{R}^n$ , we define horizontal and vertical perturbations as

$$\varphi_h(t) = \begin{cases} \varphi(t+s) & \text{if } t \in [-r, -s], \\ \varphi(0) & \text{if } s \in [-s, -0], \end{cases}$$

and

$$\varphi^y(t) = \begin{cases} \varphi(t) & \text{if } t \in [-r, 0), \\ \varphi(0) + y & \text{if } t = 0, \end{cases}$$

respectively. The horizontal and vertical partial derivatives of  $V : \mathbb{D} \rightarrow \mathbb{R}$  at  $\varphi$  are defined as

$$\partial_t V(\varphi) = \lim_{s \rightarrow 0} \frac{V(\varphi_s) - V(\varphi)}{s},$$

and

$$\partial_k V(\varphi) = \lim_{s \rightarrow 0} \frac{V(\varphi^{se_k}) - V(\varphi)}{s}, \quad (3.1)$$

respectively, if the limits exist. In (3.1),  $e_k$  is the standard unit vector in  $\mathbb{R}^n$  whose  $k$ -th component is 1 and all other components are 0. Let  $\mathbb{F}$  be the family of functions  $V(\cdot) : \mathbb{D} \mapsto \mathbb{R}$  satisfying that

- $V$  is continuous, that is, for any  $\varepsilon > 0$ ,  $\varphi \in \mathbb{D}$  there is a  $\delta > 0$  such that  $|V(\varphi) - V(\varphi')| < \varepsilon$  as long as  $\|\varphi - \varphi'\| < \delta$ .
- The derivatives  $V_t$ ,  $V_x = (\partial_k V)$ , and  $V_{xx} = (\partial_{kl} V)$  exist and are continuous.
- $V$ ,  $V_t$ ,  $V_x = (\partial_k V)$  and  $V_{xx} = (\partial_{kl} V)$  are bounded in each set  $\{\varphi \in \mathbb{D} : \|\varphi\| \leq R\}$ ,  $R > 0$ .

Let  $V(\cdot) \in \mathbb{F}$ , we define the operator

$$\mathcal{L}V(\varphi) = \partial_t V(\varphi) + \sum_{i=1}^n \varphi_i(0) f_i(\varphi) \partial_i V(\varphi) + \frac{1}{2} \sum_{i,j=1}^n \varphi_i(0) \varphi_j(0) \sigma_{ij} g_i(\varphi) g_j(\varphi) \partial_{kl} V(\varphi). \quad (3.2)$$

We have the functional Itô formula (see [10, 11]) as follows

$$dV(\mathbf{X}_t) = \left( \mathcal{L}V(\mathbf{X}_t) \right) dt + \sum_{i=1}^n X_i(t) g_i(\mathbf{X}_t) \partial_i V(\mathbf{X}_t) dE_i(t). \quad (3.3)$$

**Lemma 3.1.** *For any  $\gamma < \gamma_b$  and  $p_0 > 0$ ,  $\boldsymbol{\rho} = (\rho_1, \dots, \rho_n) \in \mathbb{R}^n$  satisfying*

$$|\rho| < \min \left\{ \frac{\gamma_b}{2}, \frac{1}{n}, \frac{\gamma_b}{4\sigma^*} \right\} \text{ and } p_0 < \min \left\{ 1, \frac{\gamma_b}{8n\sigma^*} \right\}, \quad (3.4)$$

where  $\sigma^* := \max\{\sigma_{ij} : 1 \leq i, j \leq n\}$ , let

$$V_{\boldsymbol{\rho}}(\varphi) := \left( 1 + \mathbf{c}^\top \mathbf{x} \right) \prod_{i=1}^n x_i^{\rho_i} \exp \left\{ A_2 \int_{-r}^0 \mu(ds) \int_s^0 e^{\gamma(u-s)} h(\varphi(u)) du \right\}.$$

Then, we have

$$\begin{aligned} \mathcal{L}V_{\rho}^{p_0}(\varphi) \leq & p_0 V_{\rho}^{p_0}(\varphi) \left[ A_0 \mathbf{1}_{\{|\mathbf{x}| < M\}} - \gamma_0 - Ah(\mathbf{x}) \right. \\ & \left. - A_2 \gamma \int_{-r}^0 \mu(ds) \int_s^0 e^{\gamma(u-s)} h(\varphi(u)) du - \frac{\gamma_b}{2} \sum_{i=1}^n \left( |f_i(\varphi)| + g_i^2(\varphi) \right) \right], \end{aligned} \quad (3.5)$$

where  $\mathbf{x} := \varphi(0)$  and  $A$  is a positive number satisfying  $A < A_1 - A_2 \int_{-r}^0 e^{-\gamma s} \mu(ds)$ . Recall that  $\mathbf{c}$ ,  $M$ ,  $A_0$ ,  $A_1$ ,  $A_2$ ,  $\gamma_0$ ,  $\gamma_b$ ,  $h(\cdot)$ , and  $\mu(\cdot)$  are defined in Assumption 2.1(3).

*Proof.* Let

$$\begin{aligned} U_{\rho}(\varphi) &= \ln V_{\rho}(\varphi) \\ &= \ln \left( 1 + \mathbf{c}^{\top} \mathbf{x} \right) + \sum_{i=1}^n \rho_i \ln x_i + A_2 \int_{-r}^0 \mu(ds) \int_s^0 e^{\gamma(u-s)} h(\varphi(u)) du. \end{aligned}$$

By [44, Remark 2.2] and direct calculation, we have

$$\begin{aligned} \partial_t U_{\rho}(\varphi) &= A_2 h(\mathbf{x}) \int_{-r}^0 e^{-\gamma s} \mu(ds) - A_2 \int_{-r}^0 h(\varphi(s)) \mu(ds) - A_2 \gamma \int_{-r}^0 \mu(ds) \int_s^0 e^{\gamma(u-s)} h(\varphi(u)) du, \\ \partial_i U_{\rho}(\varphi) &= \frac{c_i}{1 + \mathbf{c}^{\top} \mathbf{x}} + \frac{\rho_i}{x_i}; \quad \partial_{ij} U_{\rho}(\varphi) = \frac{-c_i c_j}{\left( 1 + \mathbf{c}^{\top} \mathbf{x} \right)^2} + \frac{-\delta_{ij} \rho_i}{x_i^2}, \end{aligned}$$

where

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

As a consequence, we obtain from the functional Itô formula that

$$\begin{aligned} \mathcal{L}U_{\rho}(\varphi) &= A_2 h(\mathbf{x}) \int_{-r}^0 e^{-\gamma s} \mu(ds) - A_2 \int_{-r}^0 h(\varphi(s)) \mu(ds) - A_2 \gamma \int_{-r}^0 \mu(ds) \int_s^0 e^{\gamma(u-s)} h(\varphi(u)) du \\ &\quad + \frac{\sum_{i=1}^n c_i x_i f_i(\varphi)}{1 + \mathbf{c}^{\top} \mathbf{x}} - \frac{1}{2} \sum_{i,j=1}^n \frac{c_i c_j \sigma_{ij} x_i x_j g_i(\varphi) g_j(\varphi)}{\left( 1 + \mathbf{c}^{\top} \mathbf{x} \right)^2} + \sum_{i=1}^n \rho_i \left( f_i(\varphi) - \sigma_{ii} g_i^2(\varphi) \right). \end{aligned} \quad (3.6)$$

Therefore, by the fact  $V_{\rho}^{p_0}(\varphi) = e^{p_0 U_{\rho}(\varphi)}$  and an application of the functional Itô formula, we get

$$\begin{aligned} \mathcal{L}V_{\rho}^{p_0}(\varphi) &= p_0 V_{\rho}^{p_0}(\varphi) \left( A_2 h(\mathbf{x}) \int_{-r}^0 e^{-\gamma s} \mu(ds) - A_2 \int_{-r}^0 h(\varphi(s)) \mu(ds) \right. \\ &\quad \left. - A_2 \gamma \int_{-r}^0 \mu(ds) \int_s^0 e^{\gamma(u-s)} h(\varphi(u)) du \right. \\ &\quad \left. + \frac{\sum_{i=1}^n c_i x_i f_i(\varphi)}{1 + \mathbf{c}^{\top} \mathbf{x}} - \frac{1}{2} \sum_{i,j=1}^n \frac{c_i c_j \sigma_{ij} x_i x_j g_i(\varphi) g_j(\varphi)}{\left( 1 + \mathbf{c}^{\top} \mathbf{x} \right)^2} \right. \\ &\quad \left. + \sum_{i=1}^n \rho_i \left( f_i(\varphi) - \sigma_{ii} g_i^2(\varphi) \right) \right. \\ &\quad \left. + \frac{1}{2} p_0 \sum_{i,j=1}^n \left( \frac{c_i x_i}{1 + \mathbf{c}^{\top} \mathbf{x}} + \rho_i \right) \left( \frac{c_j x_j}{1 + \mathbf{c}^{\top} \mathbf{x}} + \rho_j \right) \sigma_{ij} g_i(\varphi) g_j(\varphi) \right). \end{aligned}$$

Since

$$\begin{aligned}
& \frac{1}{2}p_0 \sum_{i,j=1}^n \left( \frac{c_i x_i}{1 + \mathbf{c}^\top \mathbf{x}} + \rho_i \right) \left( \frac{c_j x_j}{1 + \mathbf{c}^\top \mathbf{x}} + \rho_j \right) \sigma_{ij} g_i(\boldsymbol{\varphi}) g_j(\boldsymbol{\varphi}) \\
& \leq \frac{1}{4}p_0 \sum_{i,j=1}^n (1 + \rho_i)(1 + \rho_j) \sigma_{ij} (g_i^2(\boldsymbol{\varphi}) + g_j^2(\boldsymbol{\varphi})) \\
& \leq 2p_0 n \sigma^* \sum_{i=1}^n g_i^2(\boldsymbol{\varphi}),
\end{aligned}$$

and  $|\rho_i| < \frac{\gamma_b}{2}$ ;  $|\rho_i| \sigma^* + 2p_0 n \sigma^* < \frac{\gamma_b}{2} \forall i = 1, \dots, n$ , using Assumption 2.1, we have

$$\begin{aligned}
\mathcal{L}V_{\boldsymbol{\rho}}^{p_0}(\boldsymbol{\varphi}) & \leq p_0 V_{\boldsymbol{\rho}}^{p_0}(\boldsymbol{\varphi}) \left( A_0 \mathbf{1}_{\{|\mathbf{x}| < M\}} - \gamma_0 - h(\mathbf{x}) \left( A_1 - A_2 \int_{-r}^0 e^{-\gamma s} \mu(ds) \right) \right. \\
& \quad \left. - A_2 \gamma \int_{-r}^0 \mu(ds) \int_s^0 e^{\gamma(u-s)} h(\boldsymbol{\varphi}(u)) du - \frac{\gamma_b}{2} \sum_{i=1}^n (|f_i(\boldsymbol{\varphi})| + g_i^2(\boldsymbol{\varphi})) \right) \\
& \leq p_0 V_{\boldsymbol{\rho}}^{p_0}(\boldsymbol{\varphi}) \left( A_0 \mathbf{1}_{\{|\mathbf{x}| < M\}} - \gamma_0 - A h(\mathbf{x}) \right. \\
& \quad \left. - A_2 \gamma \int_{-r}^0 \mu(ds) \int_s^0 e^{\gamma(u-s)} h(\boldsymbol{\varphi}(u)) du - \frac{\gamma_b}{2} \sum_{i=1}^n (|f_i(\boldsymbol{\varphi})| + g_i^2(\boldsymbol{\varphi})) \right).
\end{aligned}$$

The proof is complete.  $\square$

**Theorem 3.1.** *For any initial condition  $\boldsymbol{\phi} \in \mathcal{C}_+$ , there exists a unique global solution of (2.1). It remains in  $\mathcal{C}_+$  (resp.  $\mathcal{C}_+^\circ$ ), provided  $\boldsymbol{\phi} \in \mathcal{C}_+$  (resp.,  $\boldsymbol{\phi} \in \mathcal{C}_+^\circ$ ). Moreover, for any  $p_0, \boldsymbol{\rho}$  satisfying condition (3.4), we have*

$$\mathbb{E}_{\boldsymbol{\phi}} V_{\boldsymbol{\rho}}^{p_0}(\mathbf{X}_t) \leq V_{\boldsymbol{\rho}}^{p_0}(\boldsymbol{\phi}) e^{A_0 p_0 t}. \quad (3.7)$$

In addition, if  $\rho_i \geq 0, \forall i$ , then

$$\mathbb{E}_{\boldsymbol{\phi}} V_{\boldsymbol{\rho}}^{p_0}(\mathbf{X}_t) \leq V_{\boldsymbol{\rho}}^{p_0}(\boldsymbol{\phi}) e^{-\gamma_0 p_0 t} + \overline{M}_{p_0, \boldsymbol{\rho}}, \quad (3.8)$$

where

$$\overline{M}_{p_0, \boldsymbol{\rho}} := \frac{A_0}{\gamma_0} \sup_{\boldsymbol{\varphi} \in \mathcal{C}_{V,M}} V_{\boldsymbol{\rho}}^{p_0}(\boldsymbol{\varphi}) < \infty \text{ provided } \rho_i \geq 0 \forall i,$$

and  $\mathcal{C}_{V,M} = \{\boldsymbol{\varphi} \in \mathcal{C}_+ : A_2 \gamma \int_{-r}^0 \mu(ds) \int_s^0 e^{\gamma(u-s)} h(\boldsymbol{\varphi}(u)) du \leq A_0 \text{ and } |\mathbf{x}| \leq M\}$ .

*Proof.* We carry out the proof for the existence and uniqueness of the solution with initial value  $\boldsymbol{\phi} \in \mathcal{C}_+^\circ$ . The other cases can be handled similarly. Let  $\boldsymbol{\rho}^{(1)} = (\rho_1^{(1)}, \dots, \rho_n^{(1)}) \in \mathbb{R}^n$  with  $\rho_i^{(1)} < 0 \forall i = 1, \dots, n$  satisfy the conditions (3.4). We define the following stopping times  $\tau_k^{(1)} = \inf\{t \geq 0 : V_{\boldsymbol{\rho}^{(1)}}^{p_0}(\mathbf{X}_t) \geq k\}$  and  $\tau_\infty^{(1)} = \lim_{k \rightarrow \infty} \tau_k^{(1)}$ . It is easily seen that

$$\lim_{m \rightarrow \infty} \inf \left\{ V_{\boldsymbol{\rho}^{(1)}}^{p_0}(\boldsymbol{\varphi}) : x_i \vee x_i^{-1} > m \text{ for some } i \in \{1, \dots, n\}, \mathbf{x} := \boldsymbol{\varphi}(0), \boldsymbol{\varphi} \in \mathcal{C}_+^\circ \right\} = \infty. \quad (3.9)$$

The existence and uniqueness of local solutions can be seen in [41] due to the local Lipschitz continuity of the coefficients. To prove the solution is global and remains in  $\mathcal{C}_+^\circ$ , since (3.9), it is sufficient to prove that  $\tau_\infty^{(1)} = \infty$  a.s. We obtain from (3.5) that

$$\mathcal{L}V_{\boldsymbol{\rho}^{(1)}}^{p_0}(\boldsymbol{\varphi}) \leq A_0 p_0 V_{\boldsymbol{\rho}^{(1)}}^{p_0}(\boldsymbol{\varphi}), \quad \forall \boldsymbol{\varphi} \in \mathcal{C}_+^\circ.$$

Hence, by the functional Itô formula, we get

$$\begin{aligned}\mathbb{E}_\phi V_{\boldsymbol{\rho}^{(1)}}^{p_0}(\mathbf{X}_{t \wedge \tau_k^{(1)}}) &= V_{\boldsymbol{\rho}^{(1)}}^{p_0}(\phi) + \mathbb{E}_\phi \int_0^{t \wedge \tau_k^{(1)}} \mathcal{L} V_{\boldsymbol{\rho}^{(1)}}^{p_0}(X_s) ds \\ &\leq V_{\boldsymbol{\rho}^{(1)}}^{p_0}(\phi) + p_0 A_0 \int_0^t \mathbb{E}_\phi V_{\boldsymbol{\rho}^{(1)}}^{p_0}(\mathbf{X}_{s \wedge \tau_k^{(1)}}) ds.\end{aligned}$$

Combined with Gronwall's inequality, this implies that

$$\mathbb{E}_\phi V_{\boldsymbol{\rho}^{(1)}}^{p_0}(\mathbf{X}_{t \wedge \tau_k^{(1)}}) \leq V_{\boldsymbol{\rho}^{(1)}}^{p_0}(\phi) e^{p_0 A_0 t}, \quad \forall t \geq 0. \quad (3.10)$$

As a consequence,

$$\mathbb{P}_\phi \left\{ V_{\boldsymbol{\rho}^{(1)}}^{p_0}(\mathbf{X}_{t \wedge \tau_k^{(1)}}) \geq k \right\} \leq \frac{V_{\boldsymbol{\rho}^{(1)}}^{p_0}(\phi) e^{p_0 A_0 t}}{k} \rightarrow 0 \text{ as } k \rightarrow \infty,$$

which forces  $\tau_\infty^{(1)} > t$  a.s. for any  $t > 0$  and hence,  $\tau_\infty^{(1)} = \infty$  a.s.

Now, we consider the second part. For any  $p_0, \boldsymbol{\rho}$  satisfying (3.4), by applying (3.5), one has

$$\mathcal{L} V_{\boldsymbol{\rho}}^{p_0}(\varphi) \leq A_0 p_0 V_{\boldsymbol{\rho}}^{p_0}(\varphi) \text{ for all } \varphi \in \mathcal{C}_+^\circ.$$

Thus, from (3.10), we get

$$\mathbb{E}_\phi V_{\boldsymbol{\rho}}^{p_0}(\mathbf{X}_t) \leq V_{\boldsymbol{\rho}}^{p_0}(\phi) e^{A_0 p_0 t}.$$

If  $\rho_i \geq 0 \forall i$ , a consequence of (3.5) is

$$\mathcal{L} V_{\boldsymbol{\rho}}^{p_0}(\varphi) \leq \gamma_0 p_0 \overline{M}_{p_0, \boldsymbol{\rho}} - \gamma_0 p_0 V_{\boldsymbol{\rho}}^{p_0}(\varphi). \quad (3.11)$$

In (3.11), we have used the fact

$$A_0 \mathbf{1}_{\{|\mathbf{x}| < M\}} - A_2 \gamma \int_{-r}^0 \mu(ds) \int_s^0 e^{\gamma(u-s)} h(\varphi(u)) du \leq 0, \text{ if } \varphi \notin \mathcal{C}_{V,M}.$$

By a standard argument (see e.g., [40, Theorem 5.2, p.157]), we can obtain (3.8) from (3.11). The proof is complete.  $\square$

**Lemma 3.2.** *For any  $R_1 > 0$ ,  $T > r$ , and  $\varepsilon > 0$ , there exists an  $R_2 > 0$  such that*

$$\mathbb{P}_\phi \left\{ \|\mathbf{X}_t\| \leq R_2, \forall t \in [r, T] \right\} > 1 - \varepsilon,$$

*for any initial point  $\phi$  satisfying  $V_0(\phi) < R_1$ , where  $V_0$  is defined as in Lemma 3.1 corresponding to  $\boldsymbol{\rho} = \mathbf{0} = (0, \dots, 0)$ .*

*Proof.* As the proof of Theorem 3.1, we define the following stopping times  $\tau_k^{(2)} = \inf\{t \geq 0 : V_0^{p_0}(\mathbf{X}_t) \geq k\}$  and  $\tau_\infty^{(2)} = \lim_{k \rightarrow \infty} \tau_k^{(2)}$ . Analogous (3.10), we obtain

$$\mathbb{E}_\phi V_{\mathbf{0}}^{p_0}(\mathbf{X}_{t \wedge \tau_k^{(2)}}) \leq V_{\mathbf{0}}^{p_0}(\phi) e^{p_0 A_0 t}, \quad \forall t \geq 0.$$

Therefore, for any  $R_1, T, \varepsilon > 0$ , and initial condition  $\phi$  satisfying  $V_0(\phi) < R_1$ , there exists a finite constant  $k_0$  such that

$$\frac{V_{\mathbf{0}}^{p_0}(\phi) e^{p_0 A_0 T}}{k_0} < \varepsilon,$$

and

$$\mathbb{P}_\phi \left\{ V_0^{p_0}(\mathbf{X}_{T \wedge \tau_{k_0}^{(2)}}) \geq k_0 \right\} \leq \frac{V_0^{p_0}(\phi) e^{p_0 A_0 T}}{k_0} < \varepsilon.$$

That means  $\mathbb{P}_\phi \{ \tau_{k_0}^{(2)} \geq T \} > 1 - \varepsilon$  or

$$\mathbb{P}_\phi \{ V_0^{p_0}(\mathbf{X}_t) \leq k_0 \ \forall t \in [0, T] \} > 1 - \varepsilon.$$

Note that  $V_0^{p_0}(\mathbf{X}_t) \geq 1 + \sum_{i=1}^n c_i X_i(t)$  and  $c_i > 0 \ \forall i = 1, \dots, n$ . Therefore, it is easily seen that there exists a finite constant  $R_2$  satisfying

$$\mathbb{P}_\phi \left\{ \|\mathbf{X}_t\| \leq R_2, \ \forall t \in [r, T] \right\} > 1 - \varepsilon.$$

□

**Lemma 3.3.** *There is a sufficiently small  $\alpha > 0$  such that for any  $R > 0$  and  $\varepsilon > 0$ , there exists  $R_3 = R_3(R, \varepsilon) > 0$  satisfying*

$$\text{if } \|\phi\| \leq R \text{ then } \mathbb{P}_\phi \{ \|\mathbf{X}_t\|_{2\alpha} \leq R_3 \ \forall t \in [r, 3r] \} \geq 1 - \frac{\varepsilon}{2}. \quad (3.12)$$

As a consequence, for any  $R > 0$  and  $\varepsilon > 0$ , there exists an  $R_4 = R_4(\varepsilon, R) > 0$  satisfying that

$$\text{if } V_0(\phi) \leq R \text{ then } \mathbb{P}_\phi \{ \|\mathbf{X}_t\|_{2\alpha} \leq R_4 \ \forall t \in [2r, 3r] \} \geq 1 - \varepsilon. \quad (3.13)$$

*Proof.* For any  $R$  and  $\varepsilon > 0$ , by slightly modifying the proof of Lemma 3.2, there exists an  $\tilde{R} > 0$  depending only on  $R$  such that

$$\mathbb{P}_\phi \{ \|\mathbf{X}_t\| \leq \tilde{R}, \text{ for all } t \in [0, 3r] \} \geq 1 - \frac{\varepsilon}{4} \text{ if } \|\phi\| \leq R. \quad (3.14)$$

Denote by  $f_i^{\tilde{R}}(\cdot)$  and  $g_i^{\tilde{R}}(\cdot)$  the truncated functions, where

$$f_i^{\tilde{R}}(\varphi) = \begin{cases} f_i(\varphi) & \text{if } \|\varphi\| < \tilde{R}, \\ f_i\left(\frac{R_1 \varphi}{\|\varphi\|}\right) & \text{otherwise,} \end{cases}$$

and  $g_i^{\tilde{R}}(\cdot)$  is defined similarly. Then  $f_i^{\tilde{R}}(\cdot)$  and  $g_i^{\tilde{R}}(\cdot)$  are globally Lipschitz and bounded. Let  $\tilde{\mathbf{X}}(t) = (\tilde{X}_1(t), \dots, \tilde{X}_n(t))$  be the solution of (2.1) when we replace  $f_i(\cdot)$  and  $g_i(\cdot)$  by  $f_i^{\tilde{R}}(\cdot)$  and  $g_i^{\tilde{R}}(\cdot)$ , respectively. By a standard argument, it is easy to obtain that

$$\mathbb{E}_\phi \left| \tilde{X}_i(t) \right|^4 \leq \tilde{K} \ \forall 0 \leq t \leq 3r, \|\phi\| \leq R$$

where  $\tilde{K}$  is a constant depending only on  $R$  and  $\tilde{R}$ . On the other hand, by Burkholder's inequality we have

$$\mathbb{E}_\phi \left| \tilde{X}_i(t) - \tilde{X}_i(s) \right|^4 \leq C_1 \mathbb{E}_\phi \left| \int_s^t \tilde{X}_i(y) dy \right|^4 + C_1 \mathbb{E}_\phi \left( \int_s^t |\tilde{X}_i(y)|^2 dy \right)^2, \ \forall 0 \leq s \leq t \leq 3r, \|\phi\| \leq R,$$

where  $C_1$  depends only on  $T$ ,  $R$ , and  $\tilde{R}$ . Hence, by Hölder's inequality, we obtain for  $0 \leq s \leq t \leq 3r, \|\phi\| \leq R$  that

$$\mathbb{E}_\phi \left| \tilde{X}_i(t) - \tilde{X}_i(s) \right|^4 \leq 2C_1(t-s)^2 \mathbb{E}_\phi \int_0^s \left| \tilde{X}_i(y) \right|^4 dy \leq C_2(t-s)^2,$$

where  $C_2$  is a constant depending only on  $R$  and  $\tilde{R}$ . As a consequence of the Kolmogorov-Chentsov theorem,  $\{\tilde{\mathbf{X}}(t) : 0 \leq t \leq 3r\}$  has Hölder-continuous sample paths with an exponent  $2\alpha \in (0, \frac{1}{2})$ . Moreover, there is a  $R_3 = R_3(R, \varepsilon)$  satisfying

$$\mathbb{P}_\phi \left\{ \sup_{0 \leq t \leq 3r} |\tilde{\mathbf{X}}(t)| + \sup_{0 \leq s \leq t \leq 3r} \frac{|\tilde{\mathbf{X}}(t) - \tilde{\mathbf{X}}(s)|}{(t-s)^{2\alpha}} \leq R_3 \right\} \geq 1 - \frac{\varepsilon}{4}, \|\phi\| \leq R,$$

which implies

$$\mathbb{P}_\phi \left\{ \|\tilde{\mathbf{X}}_t\|_{2\alpha} \leq R_3 \ \forall t \in [r, 3r] \right\} \geq 1 - \frac{\varepsilon}{4}, \|\phi\| \leq R. \quad (3.15)$$

Combining (3.14) and (3.15) implies that

$$\mathbb{P}_\phi \{ \|\mathbf{X}_t\|_{2\alpha} \leq R_3 \ \forall t \in [r, 3r] \} \geq 1 - \frac{\varepsilon}{2}, \text{ provided } \|\phi\| < R,$$

and the first part of the proposition is proved.

Now, we consider the second part. By Lemma 3.2, there is an  $R_5 = R_5(\varepsilon, R)$  such that

$$\mathbb{P}_\phi \{ \|\mathbf{X}_t\| < R_5 \ \forall t \in [r, 3r] \} \geq 1 - \frac{\varepsilon}{2} \text{ if } V_0(\phi) < R. \quad (3.16)$$

Hence, the second conclusion follows from the first part, (3.16), and the Markov property of  $(\mathbf{X}_t)$ .  $\square$

**Proposition 3.1.** *The following results hold.*

- (i) Let  $\rho_1^{(3)}$  be a fixed constant satisfying  $0 < \rho_1^{(3)} < \min \{ \frac{\gamma_b}{2}, \frac{1}{n}, \frac{\gamma_b}{4\sigma^*} \}$ . For any  $T > r$  and  $m > 0$  there exists a finite constant  $K_{m,T}$  such that

$$\mathbb{E}_\phi \|X_{i,t}\|^{p_0 \rho_1^{(3)}} \leq K_{m,T} \phi_i^{p_0 \rho_1^{(3)}}(0), \ \forall t \in [r, T], i = 1, \dots, n,$$

given that

$$|\phi(0)| + \int_{-r}^0 \mu(ds) \int_s^0 e^{\gamma(u-s)} h(\phi(u)) du < m,$$

where  $\mathbf{X}_t =: (X_{1,t}, \dots, X_{n,t})$  and  $\phi =: (\phi_1, \dots, \phi_n)$  is the initial value.

- (ii) For any  $T > r$ ,  $\varepsilon > 0$ ,  $R > 0$ , there exists an  $\varepsilon_1 > 0$  such that

$$\mathbb{P} \left\{ \left\| \mathbf{X}_T^{\phi_1} - \mathbf{X}_T^{\phi_2} \right\| \leq \varepsilon \right\} \geq 1 - \varepsilon \text{ whenever } V_0(\phi_i) < R, \|\phi_1 - \phi_2\| \leq \varepsilon_1. \quad (3.17)$$

Moreover, the solution  $(\mathbf{X}_t)$  has the Feller property in  $\mathcal{C}_+$ .

*Proof.* Let  $\rho^{(3)} = (\rho_1^{(3)}, 0, \dots, 0)$ . Then  $\rho^{(3)}$  satisfies condition (3.4). By the functional Itô formula, we obtain

$$\begin{aligned} V_{\rho^{(3)}}^{\frac{p_0}{2}}(\mathbf{X}_t) &= V_{\rho^{(3)}}^{\frac{p_0}{2}}(\phi) + \int_0^t \mathcal{L} V_{\rho^{(3)}}^{\frac{p_0}{2}}(\mathbf{X}_s) ds \\ &\quad + \int_0^t \frac{p_0}{2} V_{\rho^{(3)}}^{\frac{p_0}{2}}(\mathbf{X}_s) \sum_{i=1}^n \left( \frac{c_i X_i(s)}{1 + \sum_{i'=1}^n c_{i'} X_{i'}(s)} + \delta_{1i} \rho_1^{(3)} \right) g_i(\mathbf{X}_s) dE_i(s), \end{aligned} \quad (3.18)$$

where  $\delta_{1i} = 1$  if  $i = 1$  and otherwise,  $\delta_{1i} = 0$ . Therefore, combining with (3.5) leads to that

$$\begin{aligned} V_{\rho^{(3)}}^{\frac{p_0}{2}}(\mathbf{X}_t) &\leq V_{\rho^{(3)}}^{\frac{p_0}{2}}(\phi) + A_0 p_0 \int_0^t V_{\rho^{(3)}}^{\frac{p_0}{2}}(\mathbf{X}_s) ds \\ &\quad + \int_0^t \frac{p_0}{2} V_{\rho^{(3)}}^{\frac{p_0}{2}}(\mathbf{X}_s) \sum_{i=1}^n \left( \frac{c_i X_i(s)}{1 + \sum_{i'=1}^n c_{i'} X_{i'}(s)} + \delta_{1i} \rho_1^{(3)} \right) g_i(\mathbf{X}_s) dE_i(s). \end{aligned} \quad (3.19)$$

In the estimates to follow, in fact we need the terms in (3.20) to be finite, which can be done by first using estimates for the solution at stopping time  $t \wedge \tau_k$  with  $\tau_k$  being the first time such that  $|g(\mathbf{X}_s)| \vee V_{\rho^{(3)}}(\mathbf{X}_s) > k$ , and letting  $k \rightarrow \infty$ . Since it is a standard argument, we omit it for brevity. We obtain from (3.19) that

$$\begin{aligned} \mathbb{E}_\phi \sup_{t \in [0, T]} \left[ V_{\rho^{(3)}}^{\frac{p_0}{2}}(\mathbf{X}_t) \right]^2 &\leq C^{(1)} V_{\rho^{(3)}}^{p_0}(\phi) + C^{(1)} \mathbb{E}_\phi \int_0^T \sup_{s' \in [0, s]} \left[ V_{\rho^{(3)}}^{\frac{p_0}{2}}(\mathbf{X}_{s'}) \right]^2 ds \\ &\quad + C^{(1)} \mathbb{E}_\phi \sup_{t \in [0, T]} \left| \int_0^t V_{\rho^{(3)}}^{\frac{p_0}{2}}(\mathbf{X}_s) \sum_{i=1}^n \left( \frac{c_i X_i(s)}{1 + \sum_{i'=1}^n c_{i'} X_{i'}(s)} + \delta_{1i} \rho_1^{(3)} \right) g_i(\mathbf{X}_s) dE_i(s) \right|^2, \end{aligned} \quad (3.20)$$

where  $C^{(1)}$  is a constant, independent of  $\phi$ . The Burkholder-Davis-Gundy inequality and the Hölder inequality imply that

$$\begin{aligned} \mathbb{E}_\phi \sup_{t \in [0, T]} \left| \int_0^t V_{\rho^{(3)}}^{\frac{p_0}{2}}(\mathbf{X}_s) \sum_{i=1}^n \left( \frac{c_i X_i(s)}{1 + \sum_{i'=1}^n c_{i'} X_{i'}(s)} + \delta_{1i} \rho_1^{(3)} \right) g_i(\mathbf{X}_s) dE_i(s) \right|^2 \\ \leq 16n\sigma^* \mathbb{E}_\phi \int_0^T V_{\rho^{(3)}}^{p_0}(\mathbf{X}_s) \sum_{i=1}^n g_i^2(\mathbf{X}_s) ds, \end{aligned} \quad (3.21)$$

for a constant  $C_p^{(2)}$ , independent of  $\phi$ . In the display above, we have used

$$\sum_{i,j=1}^n \left( \frac{c_i X_i(s)}{1 + \sum_{i'=1}^n c_{i'} X_{i'}(s)} + 1 \right) \left( \frac{c_j X_j(s)}{1 + \sum_{i'=1}^n c_{i'} X_{i'}(s)} + 1 \right) \sigma_{ij} g_i(\mathbf{X}_s) g_j(\mathbf{X}_s) \leq 4n\sigma^* \sum_{i=1}^n g_i^2(\mathbf{X}_s).$$

It follows from (3.20) and (3.21) that

$$\begin{aligned} \mathbb{E}_\phi \sup_{t \in [0, T]} V_{\rho^{(3)}}^{p_0}(\mathbf{X}_t) &\leq C^{(1)} V_{\rho^{(3)}}^{p_0}(\phi) + C^{(1)} \mathbb{E}_\phi \int_0^T \sup_{s' \in [0, s]} V_{\rho^{(3)}}^{p_0}(\mathbf{X}_{s'}) ds \\ &\quad + 16n\sigma^* C^{(1)} \mathbb{E}_\phi \int_0^T V_{\rho^{(3)}}^{p_0}(\mathbf{X}_s) \sum_{i=1}^n g_i^2(\mathbf{X}_s) ds. \end{aligned} \quad (3.22)$$

On the other hand, by (3.7), we get

$$\mathbb{E}_\phi V_{\rho^{(3)}}^{p_0}(\mathbf{X}_t) \leq V_{\rho^{(3)}}^{p_0}(\phi) e^{p_0 A_0 t}, \quad \forall t \geq 0.$$

Therefore, we obtain from the functional Itô formula and (3.5) that

$$\begin{aligned} 0 &\leq \mathbb{E}_\phi V_{\rho^{(3)}}^{p_0}(\mathbf{X}_T) = V_{\rho^{(3)}}^{p_0}(\phi) + \mathbb{E}_\phi \int_0^T \mathcal{L} V_{\rho^{(3)}}^{p_0}(\mathbf{X}_s) ds \\ &\leq V_{\rho^{(3)}}^{p_0}(\phi) + \mathbb{E}_\phi \int_0^T \left( p_0 A_0 V_{\rho^{(3)}}^{p_0}(\mathbf{X}_s) - \frac{\gamma_b}{2} V_{\rho^{(3)}}^{p_0}(\mathbf{X}_s) \sum_{i=1}^n g_i^2(\mathbf{X}_s) \right) ds \\ &\leq K_T^{(1)} V_{\rho^{(3)}}^{p_0}(\phi) - \frac{\gamma_b}{2} \mathbb{E}_\phi \int_0^T V_{\rho^{(3)}}^{p_0}(\mathbf{X}_s) \sum_{i=1}^n g_i^2(\mathbf{X}_s) ds, \end{aligned}$$

where  $K_T^{(1)}$  is a finite constant depending only on  $T$ . It follows that

$$\mathbb{E}_\phi \int_0^T V_{\rho^{(3)}}^{p_0}(\mathbf{X}_s) \sum_{i=1}^n g_i^2(\mathbf{X}_s) ds \leq K_T^{(2)} V_{\rho^{(3)}}^{p_0}(\phi), \text{ for some constant } K_T^{(2)}. \quad (3.23)$$

Combining (3.22) and (3.23) yields that

$$\mathbb{E}_\phi \sup_{t \in [0, T]} V_{\rho^{(3)}}^{p_0}(\mathbf{X}_t) \leq K_T^{(3)} V_{\rho^{(3)}}^{p_0}(\phi) + K_T^{(3)} \mathbb{E}_\phi \int_0^T \sup_{s' \in [0, s]} V_{\rho^{(3)}}^{p_0}(\mathbf{X}_{s'}) ds, \quad (3.24)$$

for some constant  $K_T^{(3)}$  independent of  $\phi$ . It is clear that there exists  $K_{m,T}^{(4)}$  such that

$$V_{\rho^{(3)}}^{p_0}(\phi) \leq K_{m,T}^{(4)} \phi_1^{p_0 \rho_1^{(3)}}(0), \quad (3.25)$$

given that

$$|\phi(0)| + \int_{-r}^0 \mu(ds) \int_s^0 e^{\gamma(u-s)} h(\phi(u)) du < m.$$

Combining (3.24), (3.25), and Gronwall's inequality, we have that

$$\mathbb{E}_\phi \sup_{t \in [0, T]} V_{\rho^{(3)}}^{p_0}(\mathbf{X}_t) \leq K_{m,T}^{(5)} \phi_1^{p_0 \rho_1^{(3)}}(0), \quad (3.26)$$

where  $K_{m,T}^{(5)}$  is a finite constant independent of  $\phi$ . Note that

$$V_{\rho^{(3)}}^{p_0}(\mathbf{X}_t) \geq X_1^{p_0 \rho_1^{(3)}}(t). \quad (3.27)$$

It follows from (3.26) and (3.27) that

$$\mathbb{E}_\phi \|X_{1t}\|^{p_0 \rho_1^{(3)}} \leq K_{m,T}^{(5)} \phi_1^{p_0 \rho_1^{(3)}}(0), \quad \forall t \in [r, T].$$

Hence, by a similar argument, we obtain

$$\mathbb{E}_\phi \|X_{it}\|^{p_0 \rho_i^{(3)}} \leq K_{m,T} \phi_i^{p_0 \rho_i^{(3)}}(0), \quad \forall t \in [r, T], i = 1, \dots, n,$$

for some constant  $K_{m,T}$  depending only on  $m, T$ . As a result, the first part of the Theorem is proved.

Finally, since our coefficients are Lipschitz continuous in each bounded set of  $\mathcal{C}_+$ , by using (3.7) and the truncation argument, the second conclusion is easily obtained (it is similar to the proof of Lemma 3.3). In addition, the Feller property can be obtained by slightly modifying the proof in [40, Lemma 2.9.4 and Theorem 2.9.3].  $\square$

### 3.2 Tightness, weak convergence of occupation measures, and uniform integrability

Let  $\rho = \mathbf{0}$ . We obtain from (3.5) that for all  $\varphi \in \mathcal{C}_+$ ,  $\mathbf{x} := \varphi(0)$ ,

$$\mathcal{L}V_{\mathbf{0}}^{p_0}(\varphi) \leq \gamma_0 p_0 \overline{M}_{p_0, \mathbf{0}} - A p_0 V_{\mathbf{0}}^{p_0}(\varphi) h(\mathbf{x}),$$

where  $\overline{M}_{p_0, \mathbf{0}}$  is defined as in Theorem 3.1. Hence, by the functional Itô formula, we have

$$\mathbb{E}_\phi V_{\mathbf{0}}^{p_0}(\mathbf{X}_t) \leq V_{\mathbf{0}}^{p_0}(\phi) + \gamma_0 p_0 \overline{M}_{p_0, \mathbf{0}} t - \mathbb{E}_\phi \int_0^t A p_0 V_{\mathbf{0}}^{p_0}(\mathbf{X}_s) h(\mathbf{X}(s)) ds.$$

Since  $V_{\mathbf{0}}(\varphi) \geq 1 + \mathbf{c}^\top \mathbf{x}$ , we get

$$\int_0^T \mathbb{E}_\phi \left( 1 + \sum_{i=1}^n c_i X_i(t) \right)^{p_0} h(\mathbf{X}(t)) dt \leq \frac{1}{A p_0} (V_{\mathbf{0}}^{p_0}(\phi) + T \gamma_0 p_0 \overline{M}_{p_0, \mathbf{0}}) \text{ for all } T \geq 0. \quad (3.28)$$

A consequence of (3.28) is that there is a constant  $H_1$  such that

$$\begin{aligned} \int_r^T \mathbb{E}_\phi \left( \left( 1 + \sum_{i=1}^n c_i X_i(t) \right)^{p_0} h(\mathbf{X}(t)) + \int_{-r}^0 \left( 1 + \sum_{i=1}^n c_i X_i(t+s) \right)^{p_0} h(\mathbf{X}(t+s)) \mu(ds) \right) dt \\ \leq H_1 (T + V_{\mathbf{0}}^{p_0}(\phi)), \quad \forall T \geq r. \end{aligned} \quad (3.29)$$

On the other hand, using (3.5) again, we have

$$\mathcal{L} V_{\mathbf{0}}^{p_0}(\varphi) \leq \gamma_0 p_0 \overline{M}_{\mathbf{0}, p_0} - \gamma_0 p_0 V_{\mathbf{0}}^{p_0}(\varphi) \text{ for all } \varphi \in \mathcal{C}_+. \quad (3.30)$$

Therefore, similarly to the process of getting (3.28), we obtain

$$\int_0^T \mathbb{E}_\phi V_{\mathbf{0}}^{p_0}(\mathbf{X}_t) dt \leq \frac{1}{p_0 \gamma_0} (T \gamma_0 p_0 \overline{M}_{p_0, \mathbf{0}} + V_{\mathbf{0}}^{p_0}(\phi)), \quad \forall T \geq 0. \quad (3.31)$$

Combining (3.31) and the Markov inequality leads to that for any  $\varepsilon, R > 0$  there exists a finite constant  $R_1 = R_1(\varepsilon, R)$  such that

$$\frac{1}{t} \int_0^t \mathbb{E}_\phi \mathbf{1}_{\{V_{\mathbf{0}}^{p_0}(\mathbf{X}_s) < R_1\}} ds \geq 1 - \frac{\varepsilon}{2}, \text{ provided } V_{\mathbf{0}}(\phi) < R. \quad (3.32)$$

Because of (3.32), Lemma 3.3, and the Markov property of  $\mathbf{X}_t$ , for any  $\varepsilon, R > 0$ , there exists a compact subset  $\mathcal{K} = \mathcal{K}(\varepsilon, R) := \{\varphi : \|\varphi\|_{2\alpha} \leq R_4\}$  of  $\mathcal{C}_+$  satisfying

$$\frac{1}{t} \int_{2r}^{t+2r} \mathbb{E}_\phi \mathbf{1}_{\{\mathbf{X}_s \in \mathcal{K}\}} ds \geq 1 - \varepsilon, \text{ provided } V_{\mathbf{0}}(\phi) < R. \quad (3.33)$$

In the above,  $R_4 = R_4(\varepsilon, R)$  is determined as in Lemma 3.3; the compactness of  $\mathcal{K}$  in  $\mathcal{C}$  follows the Sobolev embedding theorem.

For each  $t > r$ , define the following occupation measures

$$\Pi_t^\phi(\cdot) := \frac{1}{t} \mathbb{E}_\phi \int_r^t \mathbf{1}_{\{\mathbf{X}_s \in \cdot\}} ds. \quad (3.34)$$

Then it follows from (3.33) that for  $V_{\mathbf{0}}(\phi) < R$ ,

$$\left\{ \Pi_t^\phi(\cdot) : t \geq 2r \right\} \text{ is tight.} \quad (3.35)$$

Note that  $\Pi_t^\phi(\cdot)$  defined in (3.34) is a subprobability measure for each  $t > 2r$ . However, its weak\*-limit is still a probability measure.

**Lemma 3.4.** *Under Assumption 2.2(b), there is a constant, still denoted by  $H_1$  (for simplicity of notation) such that*

$$\begin{aligned} & \int_r^T \mathbb{E}_\phi \left( \left( 1 + \sum_{i=1}^n c_i X_i(t) \right)^{p_0} h_1(\mathbf{X}(t)) + \int_{-r}^0 \left( 1 + \sum_{i=1}^n c_i X_i(t+s) \right)^{p_0} h_1(\mathbf{X}(t+s)) \mu_1(ds) \right) dt \\ & \leq H_1 (T + V_0^{p_0}(\phi)), \quad \forall T \geq r. \end{aligned} \quad (3.36)$$

*Proof.* By (3.5), we have

$$\mathcal{L}V_0^{p_0}(\varphi) \leq \gamma_0 p_0 \overline{M}_{p_0, \mathbf{0}} - \frac{p_0 \gamma_b}{2} V_0^{p_0}(\varphi) \sum_{i=1}^n \left( |f_i(\varphi)| + g_i^2(\varphi) \right).$$

In view of the functional Itô formula,

$$\mathbb{E}_\phi V_0^{p_0}(\mathbf{X}_t) \leq V_0^{p_0}(\phi) + \gamma_0 p_0 \overline{M}_{p_0, \mathbf{0}} t - \frac{p_0 \gamma_b}{2} \mathbb{E}_\phi \int_0^t \left( 1 + \sum_{i=1}^n c_i X_i(s) \right)^{p_0} \sum_{i=1}^n \left( |f_i(\mathbf{X}_s)| + g_i^2(\mathbf{X}_s) \right) ds. \quad (3.37)$$

Therefore, we get

$$\int_0^T \mathbb{E}_\phi \left( 1 + \sum_{i=1}^n c_i X_i(t) \right)^{p_0} \sum_{i=1}^n \left( |f_i(\mathbf{X}_s)| + g_i^2(\mathbf{X}_s) \right) ds \leq \frac{2}{p_0 \gamma_b} (V_0^{p_0}(\phi) + T \gamma_0 p_0 \overline{M}_{p_0, \mathbf{0}}) \quad \text{for all } T \geq 0. \quad (3.38)$$

In view of (2.4) and (3.38),

$$\int_0^T \mathbb{E}_\phi \left( 1 + \sum_{i=1}^n c_i X_i(t) \right)^{p_0} h_1(\mathbf{X}_s) ds \leq \frac{2}{b_1 p_0 \gamma_b} (V_0^{p_0}(\phi) + T \gamma_0 p_0 \overline{M}_{p_0, \mathbf{0}}) \quad \text{for all } T \geq 0. \quad (3.39)$$

Hence, we obtain (3.36).  $\square$

*Remark 3.* It is easily seen that  $\sum_i |f_i(\varphi)| + g_i^2(\varphi)$  is uniformly integrable owing to either (3.29) and Assumption 2.2(a) or (3.36) and Assumption 2.2(b). Lemma 3.4 reveals that Assumption 2.2(b) can play the same role as Assumption 2.2(a) in guaranteeing the uniform integrability of  $\sum_i |f_i(\varphi)| + g_i^2(\varphi)$ . Hence, from now on, when we assume Assumption 2.2 holds, without loss of generality, we can assume that Assumption 2.2(a) holds.

**Lemma 3.5.** *Assume that  $(\phi_k)_{k \in \mathbb{N}} \subset \mathcal{C}_+$ ,  $(T_k)_{k \in \mathbb{N}} \subset \mathbb{R}_+$  are such that  $V_0(\phi_k) \leq R$ ,  $T_k > r$ ,  $\lim_{k \rightarrow \infty} T_k = \infty$ , and the sequence  $(\Pi_{T_k}^{\phi_k})_{k \in \mathbb{N}}$  converges weakly to a probability measure  $\pi$ . Then  $\pi$  is an invariant measure and moreover*

$$\lim_{k \rightarrow \infty} \int_{\mathcal{C}} G(\varphi) \Pi_{T_k}^{\phi_k}(d\varphi) = \int_{\mathcal{C}} G(\varphi) \pi(d\varphi), \quad (3.40)$$

for any function  $G : \mathcal{C}_+ \rightarrow \mathbb{R}$  satisfying

$$|G(\varphi)| \leq K_G \left( (1 + \mathbf{c}^\top \mathbf{x})^p h(\mathbf{x}) + \int_{-r}^0 \left( 1 + \sum_{i=1}^n c_i \varphi_i(s) \right)^p h(\varphi(s)) \mu(ds) \right) \quad \text{where } \mathbf{x} := \varphi(0), \quad (3.41)$$

for some  $p < p_0$ . Likewise, if Assumption 2.2 (b) holds, we also have (3.40) for  $G$  satisfying

$$|G(\varphi)| \leq K_G \left( (1 + \mathbf{c}^\top \mathbf{x})^p h_1(\mathbf{x}) + \int_{-r}^0 \left( 1 + \sum_{i=1}^n c_i \varphi_i(s) \right)^p h_1(\varphi(s)) \mu_1(ds) \right) \quad \text{where } \mathbf{x} := \varphi(0).$$

*Proof.* For the proof of  $\pi$  being an invariant measure, we refer to [20, Theorem 9.9], or [19, Proposition 6.4] with a slight modification. We proceed to prove the second conclusion. For any  $\varepsilon > 0$ , let  $l_\varepsilon$  be sufficiently large such that for any  $\varphi$  satisfying  $|\mathbf{x}| + \int_{-r}^0 |\varphi(s)|\mu(ds) \geq 2l_\varepsilon$ ,

$$\frac{(1 + \mathbf{c}^\top \mathbf{x})^p h(\mathbf{x}) + \int_{-r}^0 \left(1 + \sum_{i=1}^n c_i \varphi_i(s)\right)^p h(\varphi(s))\mu(ds)}{(1 + \mathbf{c}^\top \mathbf{x})^{p_0} h(\mathbf{x}) + \int_{-r}^0 \left(1 + \sum_{i=1}^n c_i \varphi_i(s)\right)^{p_0} h(\varphi(s))\mu(ds)} \leq \frac{\varepsilon}{K_G H_1 (1 + R^{p_0})}. \quad (3.42)$$

The above inequality follows from the fact that

$$\lim_{|\mathbf{x}| \rightarrow \infty} \frac{(1 + \mathbf{c}^\top \mathbf{x})^p h(\mathbf{x})}{(1 + \mathbf{c}^\top \mathbf{x})^{p_0} h(\mathbf{x})} = 0$$

and

$$\lim_{\int_{-r}^0 |\varphi(s)|\mu(ds) \rightarrow \infty} \frac{\int_{-r}^0 (1 + \sum_{i=1}^n c_i \varphi_i(s))^p h(\varphi(s))\mu(ds)}{\int_{-r}^0 (1 + \sum_{i=1}^n c_i \varphi_i(s))^{p_0} h(\varphi(s))\mu(ds)} = 0 \quad (\text{because } h(\cdot) \geq 1).$$

Denote by  $u_{l_\varepsilon} : \mathcal{C} \rightarrow [0, 1]$  a continuous function satisfying

$$u_{l_\varepsilon}(\varphi) = \begin{cases} 1 & \text{if } |\mathbf{x}| + \int_{-r}^0 |\varphi(s)|\mu(ds) \leq 2l_\varepsilon, \\ 0 & \text{if } |\mathbf{x}| + \int_{-r}^0 |\varphi(s)|\mu(ds) \geq 4l_\varepsilon. \end{cases}$$

By Tonelli's theorem, we get that

$$\begin{aligned} & \int_{\mathcal{C}} \left( (1 + \mathbf{c}^\top \mathbf{x})^{p_0} h(\mathbf{x}) + \int_{-r}^0 \left(1 + \sum_{i=1}^n c_i \varphi_i(s)\right)^{p_0} h(\varphi(s))\mu(ds) \right) \Pi_{T_k}^{\phi_k}(d\varphi) \\ &= \frac{1}{T_k} \int_r^{T_k} \mathbb{E}_{\phi_k} \left( \left(1 + \sum_{i=1}^n c_i X_i(t)\right)^{p_0} h(\mathbf{X}(t)) + \int_{-r}^0 \left(1 + \sum_{i=1}^n c_i X_i(t+s)\right) h(\mathbf{X}(t+s))\mu(ds) \right) dt. \end{aligned} \quad (3.43)$$

Because of (3.41), (3.42), (3.43), and (3.29), one gets

$$\begin{aligned} & \int_{\mathcal{C}} (1 - u_{l_\varepsilon}(\varphi)) |G(\varphi)| \Pi_{T_k}^{\phi_k}(d\varphi) \\ & \leq K_G \int_{\mathcal{C}} (1 - u_{l_\varepsilon}(\varphi)) \left( (1 + \mathbf{c}^\top \mathbf{x})^p h(\mathbf{x}) + \int_{-r}^0 \left(1 + \sum_{i=1}^n c_i \varphi_i(s)\right)^p h(\varphi(s))\mu(ds) \right) \Pi_{T_k}^{\phi_k}(d\varphi) \\ & \leq \frac{\varepsilon}{H_1 (1 + R^{p_0})} \int_{\mathcal{C}} (1 - u_{l_\varepsilon}(\varphi)) \left( (1 + \mathbf{c}^\top \mathbf{x})^{p_0} h(\mathbf{x}) + \int_{-r}^0 \left(1 + \sum_{i=1}^n c_i \varphi_i(s)\right)^{p_0} h(\varphi(s))\mu(ds) \right) \Pi_{T_k}^{\phi_k}(d\varphi) \\ & \leq \varepsilon. \end{aligned} \quad (3.44)$$

Similarly, because of (3.29) and  $\pi$  being invariant, we have

$$\int_{\mathcal{C}} (1 - u_{l_\varepsilon}(\varphi)) |G(\varphi)| \pi(d\varphi) \leq \varepsilon. \quad (3.45)$$

The weak convergence of  $\Pi_{T_k}^{\phi_k}$  to  $\pi$  implies

$$\lim_{k \rightarrow \infty} \int_{\mathcal{C}} u_{l_\varepsilon}(\varphi) |G(\varphi)| \Pi_{T_k}^{\phi_k}(d\varphi) = \int_{\mathcal{C}} u_{l_\varepsilon}(\varphi) |G(\varphi)| \pi(d\varphi). \quad (3.46)$$

Combining (3.44), (3.45), and (3.46) yields that

$$\limsup_{k \rightarrow \infty} \left| \int_{\mathcal{C}} |G(\varphi)| \Pi_{T_k}^{\phi_k}(d\varphi) - \int_{\mathcal{C}} |G(\varphi)| \pi(d\varphi) \right| \leq 2\varepsilon.$$

Hence, the proof of the lemma is concluded by letting  $\varepsilon \rightarrow 0$ .  $\square$

**Lemma 3.6.** *Let  $Y$  be a random variable,  $\theta_0 > 0$  be a constant, and suppose*

$$\mathbb{E} \exp(\theta_0 Y) + \mathbb{E} \exp(-\theta_0 Y) \leq K_1$$

*for some finite constant  $K_1$ . Then the log-Laplace transform  $\eta(\theta) = \ln \mathbb{E} \exp(\theta Y)$  is twice differentiable on  $\left[0, \frac{\theta_0}{2}\right)$  and*

$$\begin{aligned} \frac{d\eta}{d\theta}(0) &= \mathbb{E}Y, \\ 0 \leq \frac{d^2\eta}{d\theta^2}(\theta) &\leq K_2, \theta \in \left[0, \frac{\theta_0}{2}\right), \end{aligned}$$

*for some  $K_2 > 0$  depending only on  $K_1$ .*

## 4 Persistence

This section is devoted to proving Theorem 2.1 and Theorem 2.2. It is shown in [53, Lemma 4], by the min-max principle that Assumption 2.3 is equivalent to the existence of  $\boldsymbol{\rho}^* = (\rho_1^*, \dots, \rho_n^*)$  with  $\rho_i^* > 0$  such that

$$\inf_{\pi \in \mathcal{M}} \left\{ \sum_{i=1}^n \rho_i^* \lambda_i(\pi) \right\} := 2\kappa^* > 0. \quad (4.1)$$

By rescaling if necessary, we can assume that  $|\boldsymbol{\rho}^*|$  is sufficiently small to satisfy condition (3.4).

**Lemma 4.1.** *Assume Assumptions 2.1 and 2.2 hold. For any invariant measure  $\pi$ , one has*

$$\int_{\mathcal{C}_+} Q_0(\varphi) \pi(d\varphi) = 0,$$

where

$$\begin{aligned} Q_0(\varphi) &= A_2 h(\mathbf{x}) \int_{-r}^0 e^{-\gamma s} \mu(ds) - A_2 \int_{-r}^0 h(\varphi(s)) \mu(ds) \\ &\quad - A_2 \gamma \int_{-r}^0 \mu(ds) \int_s^0 e^{\gamma(u-s)} h(\varphi(u)) du \\ &\quad + \frac{\sum_{i=1}^n c_i x_i f_i(\varphi)}{1 + \mathbf{c}^\top \mathbf{x}} - \frac{1}{2} \sum_{i,j=1}^n \frac{c_i c_j \sigma_{ij} x_i x_j g_i(\varphi) g_j(\varphi)}{(1 + \mathbf{c}^\top \mathbf{x})^2}. \end{aligned}$$

*Proof.* Because of (3.29), (3.36), Lemma 3.5, and Assumption 2.2,  $Q_0$  is  $\pi$ -integrable. By the strong law of large numbers (see, e.g., [30, Theorem 4.2]) we have

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t Q_0(\mathbf{X}_s) ds = \int_{\mathcal{C}_+} Q_0(\varphi) \pi(d\varphi), \quad \mathbb{P}_{\pi^-} \text{ a.s. }, \quad (4.2)$$

and

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \frac{\sum_{i,j} \sigma_{ij} c_i c_j X_i(s) X_j(s) g_i(\mathbf{X}_s) g_j(\mathbf{X}_s)}{(1 + \sum_i c_i X_i(s))^2} ds \\ = \int_{\mathcal{C}_+} \frac{\sum_{i,j} \sigma_{ij} c_i c_j x_i x_j g_i(\boldsymbol{\varphi}) g_j(\boldsymbol{\varphi})}{(1 + \mathbf{c}^\top \mathbf{x})^2} \pi(d\boldsymbol{\varphi}) < \infty \quad \mathbb{P}_\pi\text{-a.s.}, \text{ where } \mathbf{x} := \boldsymbol{\varphi}(0). \end{aligned}$$

The above limit tells us that if we let  $\langle L, L \rangle_t$  be the quadratic variation of the local martingale

$$L_t := \int_0^t \frac{\sum_i c_i X_i(s) g_i(\mathbf{X}_s) dE_i(s)}{1 + \sum_i c_i X_i(s)},$$

then

$$\limsup_{t \rightarrow \infty} \frac{\langle L, L \rangle_t}{t} = \int_{\mathcal{C}_+} \frac{\sum_{i,j} \sigma_{ij} c_i c_j x_i x_j g_i(\boldsymbol{\varphi}) g_j(\boldsymbol{\varphi})}{(1 + \mathbf{c}^\top \mathbf{x})^2} \pi(d\boldsymbol{\varphi}) < \infty \quad \mathbb{P}_\pi\text{-a.s.}$$

Applying the strong law of large numbers for local martingales (see [40, Theorem 1.3.4]),

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \frac{\sum_i c_i X_i(s) g_i(\mathbf{X}_s) dE_i(s)}{1 + \sum_i c_i X_i(s)} = 0 \quad \mathbb{P}_\pi\text{-a.s.} \quad (4.3)$$

As in (3.6), we have  $\mathcal{L}U_0(\boldsymbol{\varphi}) = Q_0(\boldsymbol{\varphi})$ , where

$$U_0(\boldsymbol{\varphi}) = \ln(1 + \mathbf{c}^\top \mathbf{x}) + A_2 \int_{-r}^0 \mu(ds) \int_s^0 e^{\gamma(u-s)} h(\boldsymbol{\varphi}(u)) du, \quad \mathbf{x} := \boldsymbol{\varphi}(0). \quad (4.4)$$

Combining (4.2), (4.3), (4.4), and the functional Itô formula yields that

$$0 \leq \lim_{t \rightarrow \infty} \frac{U_0(\mathbf{X}_t)}{t} = \int_{\mathcal{C}_+} Q_0(\boldsymbol{\varphi}) \pi(d\boldsymbol{\varphi}) \quad \mathbb{P}_\pi\text{-a.s.} \quad (4.5)$$

A simple contradiction argument coupled with (4.5) and (3.8) leads to that

$$\int_{\mathcal{C}_+} Q_0(\boldsymbol{\varphi}) \pi(d\boldsymbol{\varphi}) = 0.$$

□

**Lemma 4.2.** *Assume Assumptions 2.1, 2.2, and 2.3 hold. Let  $\boldsymbol{\rho}^*$  be as in (4.1). For any compact set  $\mathcal{K}$  of  $\mathcal{C}_+$ , there exists a  $T_{\mathcal{K}} > r$  such that for any  $T \geq T_{\mathcal{K}}$  and  $\boldsymbol{\phi} \in \partial\mathcal{C}_+ \cap \mathcal{K}$ , we have*

$$\int_r^T \mathbb{E}_{\boldsymbol{\phi}} Q_{\boldsymbol{\rho}^*}(\mathbf{X}_t) dt \leq -\kappa^* T, \quad (4.6)$$

where

$$Q_{\boldsymbol{\rho}^*}(\boldsymbol{\varphi}) := Q_0(\boldsymbol{\varphi}) - \sum_{i=1}^n \rho_i^* \left( f_i(\boldsymbol{\varphi}) - \frac{\sigma_{ii} g_i^2(\boldsymbol{\varphi})}{2} \right).$$

*Proof.* We prove the lemma by using a contradiction argument. Suppose that we can find  $\boldsymbol{\phi}_k \in \partial\mathcal{C}_+ \cap \mathcal{K}$  and  $T_k > r$ ,  $T_k \uparrow \infty$  such that

$$\int_r^{T_k} \mathbb{E}_{\boldsymbol{\phi}_k} Q_{\boldsymbol{\rho}^*}(\mathbf{X}_t) dt \geq -\kappa^* T_k. \quad (4.7)$$

Since  $\mathbb{E}_\phi |Q_{\rho^*}(\mathbf{X}_t)| \leq H_1 \tilde{K}(t + V_0^{p_0}(\phi))$  due to (3.29), (3.36), and Assumption 2.2, we can apply Tonelli's theorem to obtain

$$\int_{\mathcal{C}_+} Q_{\rho^*}(\varphi) \Pi_{T_k}^{\phi_k}(d\varphi) = \frac{1}{T_k} \int_r^{T_k} \mathbb{E}_{\phi_k} Q_{\rho^*}(\mathbf{X}_t) dt.$$

Under Assumption 2.2, as a consequence of Lemma 3.5,

$$\lim_{k \rightarrow \infty} \frac{1}{T_k} \int_r^{T_k} \mathbb{E}_{\phi_k} Q_{\rho^*}(\mathbf{X}_t) dt = \int_{\mathcal{C}_+} Q_{\rho^*}(\varphi) \pi(d\varphi), \quad (4.8)$$

where the invariant measure  $\pi$  is the weak limit of  $\{\Pi_{T_k}^{\phi_k}\}$ . Since the initial values lie on the boundary,  $\pi$  is supported in  $\partial\mathcal{C}_+$ . This combined with Lemma 4.1 implies that

$$\int_{\mathcal{C}_+} Q_{\rho^*}(\varphi) \pi(d\varphi) = - \sum_{i=1}^n \rho_i^* \lambda_i(\pi). \quad (4.9)$$

Thus, we obtain from (4.1), (4.8), and (4.9) that

$$\lim_{k \rightarrow \infty} \frac{1}{T_k} \int_r^{T_k} \mathbb{E}_{\phi_k} Q_{\rho^*}(\mathbf{X}_t) dt \leq -2\kappa^*. \quad (4.10)$$

Combining (4.7) and (4.10) leads to a contradiction. As a result, the Lemma is proved.  $\square$

Now, let  $n^*$  be sufficiently large to satisfy

$$\gamma_0(n^* - 1) - A_0 > 0, \quad (4.11)$$

and  $p_1 > p_0$  but (3.4) still holds. Under Assumption 2.2, a consequence of (3.29) is that there is an  $H_1^*$  satisfying

$$\int_r^T \mathbb{E}_\phi Q_{\rho^*}(\mathbf{X}_t) dt \leq H_1^*(T + V_0^{p_0}(\phi)) \text{ for all } T \geq r. \quad (4.12)$$

Because of (3.8), we have

$$\mathbb{E}_\phi V_0^{p_1}(\mathbf{X}_t) \leq V_0^{p_1}(\phi) e^{-\gamma_0 p_1 t} + \overline{M}_{p_1, \mathbf{0}}. \quad (4.13)$$

Note that

$$\text{if } \phi \in \mathcal{C}_{V,M} \text{ then } V_0(\phi) \leq (1 + M|\mathbf{c}|)e^{A_0}, \quad (4.14)$$

where  $\mathcal{C}_{V,M}$  is defined as in Theorem 3.1. Equations (4.13) and (4.14) imply that

$$\mathbb{E}_\phi V_0^{p_1}(\mathbf{X}_t) \leq R_{V,M} \text{ if } \phi \in \mathcal{C}_{V,M}, t \geq 0 \quad (4.15)$$

for some  $R_{V,M} > 0$ . Let  $\varepsilon^* \in (0, \frac{1}{9})$  be such that

$$R_{V,M}^{\frac{p_0}{p_1}}(\varepsilon^*)^{\frac{p_1 - p_0}{p_1}} + \varepsilon^* H_1^* T \leq \frac{\kappa^*}{10} (T - 2r) \text{ for any } T \geq 3r. \quad (4.16)$$

To take care of the case when  $X_i(t)$  is small but the norm of the segment function  $\|X_t\|$  is not, we derive the following Lemma.

**Lemma 4.3.** Assume Assumptions 2.1, 2.2, and 2.3 hold. There exist a  $\widehat{\delta} > 0$  and  $T^* > 0$  such that for any  $T \in [T^*, n^*T^*]$ ,

$$\mathbb{E}_\phi \int_0^T Q_{\rho^*}(\mathbf{X}_t) dt \leq -\frac{1}{2}\kappa^*T, \text{ if } \phi \in \mathcal{C}_V(\widehat{\delta}), \quad (4.17)$$

where

$$\mathcal{C}_V(\widehat{\delta}) := \left\{ \phi \in \mathcal{C}_+^\circ \cap \mathcal{C}_{V,M} \text{ such that } |\phi_i(0)| \leq \widehat{\delta} \text{ for some } i \right\}.$$

*Proof.* For any event  $\mathcal{A}$  with  $\mathbb{P}_\phi(\mathcal{A}) \geq 1 - \varepsilon$ , we obtain from (4.12), the Hölder inequality, and (4.15) that

$$\begin{aligned} \mathbb{E}_\phi \mathbf{1}_{\mathcal{A}^c} \int_{3r}^T Q_{\rho^*}(\mathbf{X}_t) dt &\leq \mathbb{E} \mathbf{1}_{\mathcal{A}^c} H_1^*(T + V_0^{p_0}(\mathbf{X}_{2r})) \\ &\leq (\varepsilon^*)^{\frac{p_1-p_0}{p_1}} (\mathbb{E}_\phi V_0^{p_1}(\mathbf{X}_{2r}))^{\frac{p_0}{p_1}} + \varepsilon^* H_1^*T \\ &\leq R_{V,M}^{\frac{p_0}{p_1}} (\varepsilon^*)^{\frac{p_1-p_0}{p_1}} + \varepsilon^* H_1^*T \text{ if } \phi \in \mathcal{C}_{V,M}, \end{aligned} \quad (4.18)$$

where  $\mathcal{A}^c = \Omega \setminus \mathcal{A}$ . Applying Lemma 3.3 implies that there is a compact set  $\mathcal{K}^* = \mathcal{K}^*(\varepsilon^*) := \{\varphi \in \mathcal{C}_+ : \|\varphi\| \leq R, \|\varphi\|_{2\alpha} - \|\varphi\| \leq R_4\}$  ( $R_4 = R_4(\varepsilon^*)$  is as in Lemma 3.3) such that

$$\mathbb{P}_\phi \{\mathbf{X}_t \in \mathcal{K}^* \text{ for all } t \in [2r, 3r]\} \geq 1 - \frac{\varepsilon^*}{2} \text{ if } \phi \in \mathcal{C}_{V,M}. \quad (4.19)$$

In view of Lemma 4.2, there exists  $T^* = T^*(\mathcal{K}^*) > 0$  such that for all  $T \geq T^* - 3r$ ,

$$\int_r^T \mathbb{E}_\phi Q_{\rho^*}(\mathbf{X}_t) dt \leq -\kappa^*T \text{ if } \phi \in \partial\mathcal{C}_+ \cap \mathcal{K}^*. \quad (4.20)$$

Without loss of generality, we can choose  $T^* > 3r$  sufficiently large such that for all  $T \geq T^*$ ,

$$-\frac{7}{10}\kappa^*(T - 2r) + 3A_0r \leq -\frac{1}{2}\kappa^*T. \quad (4.21)$$

In view of the Feller property of  $\mathbf{X}_t$  and (4.20), we obtain that there is a  $\delta_1 > 0$  such that for all  $T \in [T^* - 3r, n^*T^*]$

$$\int_r^T \mathbb{E}_\phi Q_{\rho^*}(\mathbf{X}_t) dt \leq -\frac{9}{10}\kappa^*T \text{ if } \phi \in \mathcal{K}^*, \text{dist}(\phi, \partial\mathcal{C}_+) < \delta_1. \quad (4.22)$$

By virtue of (4.19), part (i) of Proposition 3.1, and the structure of  $\mathcal{K}^*$ , there exists a  $\widehat{\delta} > 0$  such that

$$\mathbb{P}_\phi(\mathcal{A}) \geq 1 - \varepsilon^* \text{ if } \phi \in \mathcal{C}_V(\widehat{\delta}), \quad (4.23)$$

where

$$\mathcal{A} = \{\text{dist}(\mathbf{X}_{2r}, \partial\mathcal{C}_+) < \delta_1, \mathbf{X}_{2r} \in \mathcal{K}^*\}.$$

Combining (4.22) and (4.23) leads to that for all  $T \in [T^*, n^*T^*]$

$$\mathbb{E}_\phi \mathbf{1}_{\mathcal{A}} \int_{3r}^T Q_{\rho^*}(\mathbf{X}_t) dt \leq -\frac{9}{10}\kappa^*(T - 2r)(1 - \varepsilon^*) \leq -\frac{8}{10}\kappa^*(T - 2r) \text{ if } \phi \in \mathcal{C}_V(\widehat{\delta}). \quad (4.24)$$

We obtain from (4.18), (4.16), and (4.24) that for all  $T \in [T^*, n^*T^*]$

$$\mathbb{E}_\phi \int_{3r}^T Q_{\rho^*}(\mathbf{X}_t) dt \leq -\frac{7}{10}\kappa^*(T - 2r), \phi \in \mathcal{C}_V(\widehat{\delta}). \quad (4.25)$$

Using the functional Itô formula, Jensen's inequality, and (3.7), we have

$$\begin{aligned} \int_0^{3r} \mathbb{E}_\phi Q_{\rho^*}(\mathbf{X}_s) ds &= \frac{1}{p_0} \mathbb{E}_\phi \left( \ln V_{-\rho^*}^{p_0}(\mathbf{X}_{3r}) - \ln V_{-\rho^*}^{p_0}(\phi) \right) \\ &\leq \frac{1}{p_0} \ln \frac{\mathbb{E}_\phi V_{-\rho^*}^{p_0}(\mathbf{X}_{3r})}{V_{-\rho^*}^{p_0}(\phi)} \leq \frac{\ln e^{3A_0 p_0 r}}{p_0} = 3A_0 r. \end{aligned} \quad (4.26)$$

Therefore, we obtain from (4.25), (4.26), and (4.21) that

$$\mathbb{E}_\phi \int_0^T Q_{\rho^*}(\mathbf{X}_t) dt \leq -\frac{7}{10} \kappa^*(T - 2r) + 3A_0 r \leq -\frac{1}{2} \kappa^* T, \text{ for all } T \in [T^*, n^* T^*] \text{ if } \phi \in \mathcal{C}_V(\widehat{\delta}).$$

The lemma is proved.  $\square$

**Proposition 4.1.** *Assume that Assumptions 2.1, 2.2, and 2.3 hold. Then there are  $\theta \in (0, \frac{p_0}{2})$  and  $\tilde{K}_\theta > 0$  such that for any  $T \in [T^*, n^* T^*]$  and  $\phi \in \mathcal{C}_+^\circ \cap \mathcal{C}_{V,M}$ ,*

$$\mathbb{E}_\phi V_{-\rho^*}^\theta(\mathbf{X}_T) \leq V_{-\rho^*}^\theta(\phi) \exp\left(-\frac{1}{4} \theta \kappa^* T\right) + \tilde{K}_\theta, \quad (4.27)$$

where  $-\rho^* = (-\rho_1^*, \dots, -\rho_n^*)$ .

*Proof.* By the functional Itô formula, we obtain that

$$\begin{aligned} \ln V_{-\rho^*}(\mathbf{X}_T) &= \ln V_{-\rho^*}(\phi) + \int_0^T Q_{\rho^*}(\mathbf{X}_t) dt \\ &\quad + \int_0^T \left( \frac{\sum_i c_i X_i(t) g(\mathbf{X}_t) dE_i(t)}{1 + \sum_i c_i X_i(t)} - \sum_i \rho_i^* g_i(\mathbf{X}_t) dE_i(t) \right) \\ &=: \ln V_{-\rho^*}(\phi) + z(T). \end{aligned} \quad (4.28)$$

Because of (4.28) and (3.7), we have

$$\mathbb{E}_\phi \exp(p_0 z(T)) = \frac{\mathbb{E}_\phi V_{-\rho^*}^{p_0}(\mathbf{X}_T)}{V_{-\rho^*}^{p_0}(\phi)} \leq e^{A_0 p_0 T}. \quad (4.29)$$

Another consequence of (3.7) is that

$$\frac{\mathbb{E}_\phi V_{\rho^*}^{p_0}(\mathbf{X}_T)}{V_{\rho^*}^{p_0}(\phi)} \leq e^{A_0 p_0 T}. \quad (4.30)$$

We obtain from the definition of  $V_{\rho^*}^{p_0}(\varphi)$  that

$$\begin{aligned} V_{-\rho^*}^{-p_0}(\varphi) &= \left(1 + \mathbf{c}^\top \mathbf{x}\right)^{-2p_0} \exp\left\{-2p_0 A_2 \int_{-r}^0 \mu(ds) \int_s^0 e^{\gamma(u-s)} h(\varphi(u)) du\right\} V_{\rho^*}^{p_0}(\varphi) \\ &\leq V_{\rho^*}^{p_0}(\varphi), \text{ where } \mathbf{x} := \varphi(0). \end{aligned} \quad (4.31)$$

Applying (4.31) and (4.30) to (4.28) yields

$$\begin{aligned} \mathbb{E}_\phi \exp(-p_0 z(T)) &= \frac{\mathbb{E}_\phi V_{-\rho^*}^{-p_0}(\mathbf{X}_T)}{V_{-\rho^*}^{-p_0}(\phi)} \\ &\leq \frac{\mathbb{E}_\phi V_{\rho^*}^{p_0}(\mathbf{X}_T)}{V_{\rho^*}^{p_0}(\phi)} \left(1 + \mathbf{c}^\top \phi(0)\right)^{-2p_0} \exp\left\{-2p_0 A_2 \int_{-r}^0 \mu(ds) \int_s^0 e^{\gamma(u-s)} h(\phi(u)) du\right\} \\ &\leq \left(1 + \mathbf{c}^\top \phi(0)\right)^{-2p_0} \exp\left\{-2p_0 A_2 \int_{-r}^0 \mu(ds) \int_s^0 e^{\gamma(u-s)} h(\phi(u)) du\right\} \exp(A_0 p_0 T). \end{aligned} \quad (4.32)$$

In view of (4.29) and (4.32), an application of Lemma 3.6 for  $z(T)$  implies that there is  $\tilde{K}_2 \geq 0$  such that

$$0 \leq \frac{d^2 \tilde{\eta}_{\phi,T}}{d\theta^2}(\theta) \leq \tilde{K}_2 \text{ for all } \theta \in \left[0, \frac{p_0}{2}\right), \phi \in \mathcal{C}_V(\hat{\delta}), T \in [T^*, n^*T^*],$$

where

$$\tilde{\eta}_{\phi,T}(\theta) = \ln \mathbb{E}_\phi \exp(\theta z(T)).$$

Hence, using Lemma 3.6 and (4.17) yields

$$\frac{d\tilde{\eta}_{\phi,T}}{d\theta}(0) = \mathbb{E}_\phi z(T) \leq -\frac{1}{2}\kappa^*T \text{ for } \phi \in \mathcal{C}_V(\hat{\delta}), T \in [T^*, n^*T^*].$$

By a Taylor expansion around  $\theta = 0$ , for  $\phi \in \mathcal{C}_V(\hat{\delta}), T \in [T^*, n^*T^*]$ , and  $\theta \in [0, \frac{p_0}{2})$ , we have

$$\tilde{\eta}_{\phi,T}(\theta) \leq -\frac{1}{2}\kappa^*T\theta + \theta^2 \tilde{K}_2.$$

Now, if we choose  $\theta \in (0, \frac{p_0}{2})$  satisfying  $\theta < \frac{\kappa^*T^*}{4\tilde{K}_2}$ , we get

$$\tilde{\eta}_{\phi,T}(\theta) \leq -\frac{1}{4}\kappa^*T\theta \text{ for all } \phi \in \mathcal{C}_V(\hat{\delta}), T \in [T^*, n^*T^*]. \quad (4.33)$$

In light of (4.33), we have for such  $\theta, \phi \in \mathcal{C}_V(\hat{\delta})$ , and  $T \in [T^*, n^*T^*]$  that

$$\frac{\mathbb{E}_\phi V_{-\rho^*}^\theta(\mathbf{X}_T)}{V_{-\rho^*}^\theta(\phi)} = \exp \tilde{\eta}_{\phi,T}(\theta) \leq \exp \left( -\frac{1}{4}\kappa^*T\theta \right). \quad (4.34)$$

On the other hand, because of (3.7), we have for  $\phi \notin \mathcal{C}_V(\hat{\delta})$  but satisfying  $\phi \in \mathcal{C}_+^\circ \cap \mathcal{C}_{V,M}$  and  $T \in [T^*, n^*T^*]$  that

$$\mathbb{E}_\phi V_{-\rho^*}^\theta(\mathbf{X}_T) \leq \exp(\theta n^*T^*H) \sup_{\phi \notin \mathcal{C}_V(\hat{\delta}), \phi \in \mathcal{C}_{V,M}} \{V_{-\rho^*}^\theta(\phi)\} =: \tilde{K}_\theta < \infty. \quad (4.35)$$

Combining (4.34) and (4.35) completes the proof.  $\square$

**Theorem 4.1.** *Assume that Assumptions 2.1, 2.2, and 2.3 hold. There is a finite constant  $K^*$  such that*

$$\limsup_{t \rightarrow \infty} \mathbb{E}_\phi V_{-\rho^*}^\theta(\mathbf{X}_t) \leq K^* \text{ for all } \phi \in \mathcal{C}_+^\circ.$$

As a result, for any  $\varepsilon > 0$ , there exists an  $R_* = R_*(\varepsilon) > 0$  such that

$$\liminf_{t \rightarrow \infty} \mathbb{P}_\phi \{R_*^{-1} \leq |X_i(t)| \leq R_*\} \geq 1 - \varepsilon \text{ for all } i = 1, \dots, n. \quad (4.36)$$

*Proof.* Once we obtained Proposition 4.1, the proof is similar to [24, Theorem 4.1]. By virtue of (3.5), we have

$$\mathcal{L}V_{-\rho^*}^\theta(\varphi) \leq -\theta\gamma_0 V_{-\rho^*}^\theta(\varphi) \text{ if } \varphi \notin \mathcal{C}_{V,M}, \quad (4.37)$$

where  $\mathcal{C}_{V,M}$  is as in Theorem 3.1. Define the stopping time

$$\tau = \inf\{t \geq 0 : \mathbf{X}_t \in \mathcal{C}_{V,M}\}. \quad (4.38)$$

We obtain from Dynkin's formula and (4.37) that

$$\begin{aligned}
& \mathbb{E}_\phi \left[ \exp \{ \theta \gamma_0 (\tau \wedge n^* T^*) \} V_{-\rho^*}^\theta (\mathbf{X}_{\tau \wedge n^* T^*}) \right] \\
& \leq V_{-\rho^*}^\theta (\phi) + \mathbb{E}_\phi \int_0^{\tau \wedge n^* T^*} \exp \{ \theta \gamma_0 s \} \left[ \mathcal{L} V_{-\rho^*}^\theta (\mathbf{X}_s) + \theta \gamma_0 V_{-\rho^*}^\theta (\mathbf{X}_s) \right] ds \\
& \leq V_{-\rho^*}^\theta (\phi).
\end{aligned}$$

As a consequence,

$$\begin{aligned}
V_{-\rho^*}^\theta (\phi) & \geq \mathbb{E}_\phi \left[ \exp \{ \theta \gamma_0 (\tau \wedge n^* T^*) \} V_{-\rho^*}^\theta (\mathbf{X}_{\tau \wedge n^* T^*}) \right] \\
& = \mathbb{E}_\phi \left[ \mathbf{1}_{\{\tau \leq (n^*-1)T^*\}} \exp \{ \theta \gamma_0 (\tau \wedge n^* T^*) \} V_{-\rho^*}^\theta (\mathbf{X}_{\tau \wedge n^* T^*}) \right] \\
& \quad + \mathbb{E}_\phi \left[ \mathbf{1}_{\{(n^*-1)T^* < \tau < n^* T^*\}} \exp \{ \theta \gamma_0 (\tau \wedge n^* T^*) \} V_{-\rho^*}^\theta (\mathbf{X}_{\tau \wedge n^* T^*}) \right] \\
& \quad + \mathbb{E}_\phi \left[ \mathbf{1}_{\{\tau \geq n^* T^*\}} \exp \{ \theta \gamma_0 (\tau \wedge n^* T^*) \} V_{-\rho^*}^\theta (\mathbf{X}_{\tau \wedge n^* T^*}) \right] \\
& \geq \mathbb{E}_\phi \left[ \mathbf{1}_{\{\tau \leq (n^*-1)T^*\}} V_{-\rho^*}^\theta (\mathbf{X}_\tau) \right] \\
& \quad + \exp \{ \theta \gamma_0 ((n^* - 1)T^*) \} \mathbb{E}_\phi \left[ \mathbf{1}_{\{(n^*-1)T^* < \tau < n^* T^*\}} V_{-\rho^*}^\theta (\mathbf{X}_\tau) \right] \\
& \quad + \exp \{ \theta \gamma_0 n^* T^* \} \mathbb{E}_\phi \left[ \mathbf{1}_{\{\tau \geq n^* T^*\}} V_{-\rho^*}^\theta (\mathbf{X}_{n^* T^*}) \right].
\end{aligned} \tag{4.39}$$

Combining the Markov property of  $(\mathbf{X}_t)$  and Proposition 4.1 yields

$$\begin{aligned}
& \mathbb{E}_\phi \left[ \mathbf{1}_{\{\tau \leq (n^*-1)T^*\}} V_{-\rho^*}^\theta (\mathbf{X}_{n^* T^*}) \right] \\
& \leq \mathbb{E}_\phi \left[ \mathbf{1}_{\{\tau \leq (n^*-1)T^*\}} \left[ \tilde{K}_\theta + e^{-\frac{1}{4}\theta\kappa^*(n^* T^* - \tau)} V_{-\rho^*}^\theta (\mathbf{X}_\tau) \right] \right] \\
& \leq \tilde{K}_\theta + \exp \left( -\frac{1}{4}\theta\kappa^* T^* \right) \mathbb{E}_\phi \left[ \mathbf{1}_{\{\tau \leq (n^*-1)T^*\}} V_{-\rho^*}^\theta (\mathbf{X}_\tau) \right].
\end{aligned} \tag{4.40}$$

Using again the strong Markov property of  $(\mathbf{X}_t)$  and (3.7), we obtain that

$$\begin{aligned}
& \mathbb{E}_\phi \left[ \mathbf{1}_{\{(n^*-1)T^* < \tau < n^* T^*\}} V_{-\rho^*}^\theta (\mathbf{X}_{n^* T^*}) \right] \\
& \leq \mathbb{E}_\phi \left[ \mathbf{1}_{\{(n^*-1)T^* < \tau < n^* T^*\}} e^{\theta A_0 (n^* T^* - \tau)} V_{-\rho^*}^\theta (\mathbf{X}_\tau) \right] \\
& \leq \exp(\theta A_0 T^*) \mathbb{E}_\phi \left[ \mathbf{1}_{\{(n^*-1)T^* < \tau < n^* T^*\}} V_{-\rho^*}^\theta (\mathbf{X}_\tau) \right].
\end{aligned} \tag{4.41}$$

Applying (4.40) and (4.41) to (4.39) leads to

$$\begin{aligned}
V_{-\rho^*}^\theta (\phi) & \geq \exp \left( \frac{1}{4}\theta\kappa^* T^* \right) \mathbb{E}_\phi \left[ \mathbf{1}_{\{\tau \leq (n^*-1)T^*\}} V_{-\rho^*}^\theta (\mathbf{X}_{n^* T^*}) \right] - \exp \left( \frac{1}{4}\theta\kappa^* T^* \right) \tilde{K}_\theta \\
& \quad + \exp(-\theta A_0 T^*) \exp(\theta \gamma_0 (n^* - 1)T^*) \mathbb{E}_\phi \left[ \mathbf{1}_{\{(n^*-1)T^* < \tau < n^* T^*\}} V_{-\rho^*}^\theta (\mathbf{X}_{n^* T^*}) \right] \\
& \quad + \exp(\theta \gamma_0 n^* T^*) \mathbb{E}_\phi \left[ \mathbf{1}_{\{\tau \geq n^* T^*\}} V_{-\rho^*}^\theta (\mathbf{X}_{n^* T^*}) \right] \\
& \geq \exp(m\theta T^*) \mathbb{E}_\phi V_{-\rho^*}^\theta (\mathbf{X}_{n^* T^*}) - \tilde{K}_\theta \exp \left( \frac{1}{4}\theta\kappa^* T^* \right),
\end{aligned} \tag{4.42}$$

where  $m = \min \{ \frac{1}{4}\kappa^*, \gamma_0 n^*, \gamma_0 (n^* - 1) - A_0 \} > 0$  by (4.11). As a result,

$$\mathbb{E}_\phi V_{-\rho^*}^\theta (\mathbf{X}_{n^* T^*}) \leq \hat{q}_1 V_{-\rho^*}^\theta (\phi) + q_1^* \text{ for all } \phi \in \mathcal{C}_+^\circ,$$

for some  $0 < \hat{q}_1 < 1$ ,  $0 < q_1^* < \infty$ . Therefore, by the Markov property of  $\mathbf{X}_t$ , we have

$$\mathbb{E}_\phi V_{-\rho^*}^\theta(\mathbf{X}_{(k+1)n^*T^*}) \leq \hat{q}_1 \mathbb{E}_\phi V_{-\rho^*}^\theta(\mathbf{X}_{kn^*T^*}) + q_1^* \text{ for all } \phi \in \mathcal{C}_+^\circ,$$

Using this recursively, we obtain

$$\mathbb{E}_\phi V_{-\rho^*}^\theta(\mathbf{X}_{kn^*T^*}) \leq \hat{q}_1^k V_{-\rho^*}^\theta(\phi) + \frac{q_1^*(1 - \hat{q}_1^k)}{1 - \hat{q}_1}. \quad (4.43)$$

We obtain from (4.43) and (3.7) that

$$\mathbb{E}_\phi V_{-\rho^*}^\theta(\mathbf{X}_T) \leq \left[ \hat{q}_1^k V_{-\rho^*}^\theta(\phi) + \frac{q_1^*(1 - \hat{q}_1^k)}{1 - \hat{q}_1} \right] e^{A_0 \theta T^*} \text{ for all } T \in [kn^*T^*, kn^*T^* + T^*].$$

Hence, by letting  $k \rightarrow \infty$ , we obtain the existence of a finite constant  $K^*$  such that

$$\limsup_{t \rightarrow \infty} \mathbb{E}_\phi V_{-\rho^*}^\theta(\mathbf{X}_t) \leq K^* \text{ for all } \phi \in \mathcal{C}_+^\circ. \quad (4.44)$$

Finally, (4.36) is obtained by applying Markov's inequality to (4.44), and using  $V_{-\rho^*}(\varphi) \geq \frac{1 + \mathbf{c}^\top \mathbf{x}}{\prod_{i=1}^n x_i^{\rho_i^*}}$  and  $\sum_{i=1}^n \rho_i^* < 1$ ,  $\rho_i^* > 0$ .  $\square$

To proceed, we prove the uniqueness of the invariant measure under suitable assumptions. For  $\mathbf{x} \in \mathbb{R}_+^{n, \circ}$ ,  $\ln \mathbf{x}$  is understood as the component-wise logarithm of  $\mathbf{x}$ . By using the fact  $|\mathbf{x}^{(1)} - \mathbf{x}^{(2)}| \leq |\ln \mathbf{x}^{(1)} - \ln \mathbf{x}^{(2)}| |1 + \mathbf{x}^{(1)} + \mathbf{x}^{(2)}|$  and basic inequalities (Young's inequality and the Cauchy-Schwarz inequality), we obtain from (2.7) and Assumption 2.4 (ii) that there is some constant  $D_1$  depending only  $D_0, d_0$  satisfying

$$\begin{aligned} & \sum_i \left( \left| f_i(\varphi^{(1)}) - f_i(\varphi^{(2)}) \right| + \left| g_i(\varphi^{(1)}) - g_i(\varphi^{(2)}) \right| + \left| g_i^2(\varphi^{(1)}) - g_i^2(\varphi^{(2)}) \right| \right) \\ & \leq D_1 \left| \ln \mathbf{x}^{(1)} - \ln \mathbf{x}^{(2)} \right| \left| 1 + \mathbf{x}^{(1)} + \mathbf{x}^{(2)} \right|^{d_0+1} \\ & \quad + D_1 \int_{-r}^0 \left| \ln \varphi^{(1)}(s) - \ln \varphi^{(2)}(s) \right| \left| 1 + \varphi^{(1)}(s) + \varphi^{(2)}(s) \right|^{d_0+1} \mu(ds), \end{aligned} \quad (4.45)$$

and

$$\begin{aligned} & \sum_i \left( \left| f_i(\varphi^{(1)}) - f_i(\varphi^{(2)}) \right| + \left| g_i(\varphi^{(1)}) - g_i(\varphi^{(2)}) \right| + \left| g_i^2(\varphi^{(1)}) - g_i^2(\varphi^{(2)}) \right| \right) \left| \ln \mathbf{x}^{(1)} - \ln \mathbf{x}^{(2)} \right| \\ & \leq D_1 \left| \ln \mathbf{x}^{(1)} - \ln \mathbf{x}^{(2)} \right|^2 \left| 1 + \mathbf{x}^{(1)} + \mathbf{x}^{(2)} \right|^{d_0+1} \\ & \quad + D_1 \int_{-r}^0 \left| \ln \varphi^{(1)}(s) - \ln \varphi^{(2)}(s) \right|^2 \left| 1 + \varphi^{(1)}(s) + \varphi^{(2)}(s) \right|^{2d_0+2} \mu(ds), \end{aligned} \quad (4.46)$$

and

$$\begin{aligned} & \sum_i \left( \left| f_i(\varphi^{(1)}) - f_i(\varphi^{(2)}) \right| + \left| g_i(\varphi^{(1)}) - g_i(\varphi^{(2)}) \right| + \left| g_i^2(\varphi^{(1)}) - g_i^2(\varphi^{(2)}) \right| \right) \left| \ln \mathbf{x}^{(1)} - \ln \mathbf{x}^{(2)} \right|^3 \\ & \leq D_1 \left| \ln \mathbf{x}^{(1)} - \ln \mathbf{x}^{(2)} \right|^4 \left| 1 + \mathbf{x}^{(1)} + \mathbf{x}^{(2)} \right|^{4d_0+4} \\ & \quad + D_1 \int_{-r}^0 \left| \ln \varphi^{(1)}(s) - \ln \varphi^{(2)}(s) \right|^4 \left| 1 + \varphi^{(1)}(s) + \varphi^{(2)}(s) \right|^{4d_0+4} \mu(ds). \end{aligned} \quad (4.47)$$

To apply asymptotic couplings method as well as the theories and results in [23], we let  $Y_i(t) = \ln X_i(t)$  and consider the following equations

$$\begin{cases} dY_i(t) = \left[ f_i(\mathbf{X}_t) - \frac{g_i^2(\mathbf{X}_t)\sigma_{ii}^2}{2} \right] dt + g_i(\mathbf{X}_t)dE_i(t), \quad i = 1, \dots, n, \\ \mathbf{Y}_0 = \ln \phi, \quad \phi \in \mathcal{C}_+^\circ, \end{cases} \quad (4.48)$$

and

$$\begin{cases} d\tilde{Y}_i(t) = \left[ f_i(\tilde{\mathbf{X}}_t) - \frac{g_i^2(\tilde{\mathbf{X}}_t)\sigma_{ii}^2}{2} \right] dt + \tilde{\lambda} \left[ 1 + X_i(t) + \tilde{X}_i(t) \right]^{4d_0+4} \left[ Y_i(t) - \tilde{Y}_i(t) \right] dt \\ \quad + g_i(\tilde{\mathbf{X}}_t)dE_i(t), \quad i = 1, \dots, n \\ \tilde{\mathbf{Y}}_0 = \ln \tilde{\phi}, \quad \tilde{\phi} \neq \phi \in \mathcal{C}_+^\circ, \end{cases} \quad (4.49)$$

where  $\tilde{\lambda}$  is sufficiently large to be determined later; and  $\tilde{\mathbf{X}}(t) := (e^{\tilde{Y}_1(t)}, \dots, e^{\tilde{Y}_n(t)})$ . Put  $\mathbf{Z} := \mathbf{Y} - \tilde{\mathbf{Y}}$ . Combining the functional Itô formula and (4.45), (4.46), (4.47), one has

$$\begin{aligned} d|\mathbf{Z}(t)|^2 &\leq \left[ D_2 \int_{-r}^0 \left| 1 + \tilde{\mathbf{X}}(t+s) + \mathbf{X}(t+s) \right|^{4d_0+4} |\mathbf{Z}(t+s)|^2 \mu(ds) \right] dt \\ &\quad - \left( \tilde{\lambda} - D_2 \right) \left| 1 + \tilde{\mathbf{X}}(t) + \mathbf{X}(t) \right|^{4d_0+4} |\mathbf{Z}(t)|^2 dt \\ &\quad + 2 \sum_i \left( Y_i(t) - \tilde{Y}_i(t) \right) \left( g_i(\mathbf{X}_t) - g_i(\tilde{\mathbf{X}}_t) \right) dE_i(t), \end{aligned} \quad (4.50)$$

and

$$\begin{aligned} d|\mathbf{Z}(t)|^4 &\leq \left[ D_2 \int_{-r}^0 \left| 1 + \tilde{\mathbf{X}}(t+s) + \mathbf{X}(t+s) \right|^{4d_0+4} |\mathbf{Z}(t+s)|^4 \mu(ds) \right] dt \\ &\quad - \left( \tilde{\lambda} - D_2 \right) \left| 1 + \tilde{\mathbf{X}}(t) + \mathbf{X}(t) \right|^{4d_0+4} |\mathbf{Z}(t)|^4 dt \\ &\quad + 4 \sum_i \left( Y_i(t) - \tilde{Y}_i(t) \right)^3 \left( g_i(\mathbf{X}_t) - g_i(\tilde{\mathbf{X}}_t) \right) dE_i(t), \end{aligned} \quad (4.51)$$

for some constant  $D_2$  depending only  $D_0, d_0, \sigma_{ij}$  and independent of  $\mathbf{X}_t, \tilde{\mathbf{X}}_t$ . For  $\varphi^{(1)}, \varphi^{(2)} \in \mathcal{C}_+^\circ$ , define

$$\begin{aligned} \tilde{U}(\varphi^{(1)}, \varphi^{(2)}) &:= \left| \ln \mathbf{x}^{(1)} - \ln \mathbf{x}^{(2)} \right|^4 \\ &\quad + \frac{D_2 + 9n\sigma^*D_1}{\gamma} \int_{-r}^0 \mu(ds) \int_s^0 e^{\gamma(u-s)} \left| 1 + \varphi^{(1)}(s) + \varphi^{(2)}(s) \right|^{4d_0+4} \left| \varphi^{(1)}(s) - \varphi^{(2)}(s) \right|^4 du, \end{aligned}$$

and

$$\begin{aligned} U(\varphi^{(1)}, \varphi^{(2)}) &:= \left| \ln \mathbf{x}^{(1)} - \ln \mathbf{x}^{(2)} \right|^2 \\ &\quad + \frac{D_2 + 9n\sigma^*D_1}{\gamma} \int_{-r}^0 \mu(ds) \int_s^0 e^{\gamma(u-s)} \left| 1 + \varphi^{(1)}(s) + \varphi^{(2)}(s) \right|^{4d_0+4} \left| \varphi^{(1)}(s) - \varphi^{(2)}(s) \right|^2 du, \end{aligned}$$

where  $\sigma^* := \max\{\sigma_{ij} : 1 \leq i, j \leq n\}$ . Hence, by direct calculations using the functional Itô formula, [44, Remark 2.2] and then applying (4.50), (4.51), it is easily seen that we can choose  $\tilde{\lambda}$  being

sufficiently large such that

$$\begin{aligned}
d\tilde{U}(\mathbf{X}_t, \tilde{\mathbf{X}}_t) \leq & -2D_3 \left( \tilde{U}(\mathbf{X}_t, \tilde{\mathbf{X}}_t) + |1 + \mathbf{X}(t) + \tilde{\mathbf{X}}(t)|^{4d_0+4} |\mathbf{Z}(t)|^4 \right. \\
& \left. + \int_{-r}^0 |1 + \mathbf{X}(t+s) + \tilde{\mathbf{X}}(t+s)|^{4d_0+4} |\mathbf{Z}(t+s)|^4 \mu(ds) \right) dt \\
& + 4 \sum_i \left( Y_i(t) - \tilde{Y}_i(t) \right)^3 \left( g_i(\mathbf{X}_t) - g_i(\tilde{\mathbf{X}}_t) \right) dE_i(t),
\end{aligned} \tag{4.52}$$

and

$$\begin{aligned}
dU(\mathbf{X}_t, \tilde{\mathbf{X}}_t) \leq & -2D_3 \left( U(\mathbf{X}_t, \tilde{\mathbf{X}}_t) + |1 + \mathbf{X}(t) + \tilde{\mathbf{X}}(t)|^{4d_0+4} |\mathbf{Z}(t)|^2 \right. \\
& \left. + \int_{-r}^0 |1 + \mathbf{X}(t+s) + \tilde{\mathbf{X}}(t+s)|^{4d_0+4} |\mathbf{Z}(t+s)|^2 \mu(ds) \right) dt \\
& + 2 \sum_i \left( Y_i(t) - \tilde{Y}_i(t) \right) \left( g_i(\mathbf{X}_t) - g_i(\tilde{\mathbf{X}}_t) \right) dE_i(t),
\end{aligned} \tag{4.53}$$

for some positive constant  $D_3 > 8n\sigma^*D_1$ .

Similar to [23, Theorem 3.1], let

$$\mathbf{v}(t) = \tilde{\lambda} \left[ (g_i(\tilde{\mathbf{X}}_t)g_j(\tilde{\mathbf{X}}_t)\sigma_{ij})_{n \times n} \right]^{-1} \left[ 1 + |\mathbf{X}(t)| + |\tilde{\mathbf{X}}(t)| \right]^{4d_0+4} \left( \mathbf{Y}(t) - \tilde{\mathbf{Y}}(t) \right),$$

where  $[(g_i(\varphi)g_j(\varphi)\sigma_{ij})_{n \times n}]^{-1}$  is the inverse matrix of matrix  $[(g_i(\varphi)g_j(\varphi)\sigma_{ij})_{n \times n}]$  and for each  $\varepsilon > 0$

$$\tilde{\tau}_\varepsilon := \inf \left\{ t \geq 0 : \int_0^t |\mathbf{v}(s)|^2 ds \geq \varepsilon^{-1} \|\mathbf{Y}_0 - \tilde{\mathbf{Y}}_0\|^2 \right\}.$$

**Lemma 4.4.** *The following assertions hold:*

$$\lim_{\varepsilon \rightarrow 0} \mathbb{P}\{\tilde{\tau}_\varepsilon = \infty\} = 1, \tag{4.54}$$

and

$$\lim_{t \rightarrow \infty} |\mathbf{Y}(t) - \tilde{\mathbf{Y}}(t)| = 0 \text{ a.s.} \tag{4.55}$$

*Remark 4.* In the proof of the Lemma, our purpose is to prove  $\lim_{t \rightarrow \infty} U(\mathbf{X}_t, \tilde{\mathbf{X}}_t) = 0$  a.s. To handle the diffusion part of  $U(\mathbf{X}_t, \tilde{\mathbf{X}}_t)$ , we need some helps from  $\tilde{U}(\mathbf{X}_t, \tilde{\mathbf{X}}_t)$ . That is why we introduced both  $U(\mathbf{X}_t, \tilde{\mathbf{X}}_t)$  and  $\tilde{U}(\mathbf{X}_t, \tilde{\mathbf{X}}_t)$  in the above.

*Proof.* A consequence of (4.52) and (4.53) is that

$$d \left[ e^{D_3 t} \mathbb{E} U(\mathbf{X}_t, \tilde{\mathbf{X}}_t) \right] \leq -D_3 e^{D_3 t} \mathbb{E} U(\mathbf{X}_t, \tilde{\mathbf{X}}_t) dt,$$

and

$$d \left[ e^{D_3 t} \mathbb{E} \tilde{U}(\mathbf{X}_t, \tilde{\mathbf{X}}_t) \right] \leq -D_3 e^{D_3 t} \mathbb{E} \tilde{U}(\mathbf{X}_t, \tilde{\mathbf{X}}_t) dt,$$

which implies that

$$\lim_{t \rightarrow \infty} \mathbb{E} e^{D_3 t} U(\mathbf{X}_t, \tilde{\mathbf{X}}_t) = \lim_{t \rightarrow \infty} \mathbb{E} e^{D_3 t} \tilde{U}(\mathbf{X}_t, \tilde{\mathbf{X}}_t) = 0, \tag{4.56}$$

and

$$\mathbb{E} \int_0^\infty e^{D_3 t} U(\mathbf{X}_t, \tilde{\mathbf{X}}_t) dt \leq \frac{U(\phi, \tilde{\phi})}{D_3}. \quad (4.57)$$

An application of Markov's inequality and (4.57) imply that

$$\mathbb{P} \left( \int_0^\infty e^{D_3 t} U(\mathbf{X}_t, \tilde{\mathbf{X}}_t) dt \leq \frac{U(\phi, \tilde{\phi})}{D_3 \sqrt{\varepsilon}} \right) \geq 1 - \sqrt{\varepsilon}. \quad (4.58)$$

To proceed, similar to the proof of Proposition 5.1 part (i) (in particular, the process of getting (3.24)), we can obtain that for  $p < p_0$ ,

$$\mathbb{E}_\phi \sup_{t \in [0,1]} V_0^p(\mathbf{X}_t) \leq C_1 V_0^p(\mathbf{X}_0),$$

for some constant  $C_1$ . Then, combining with (3.8), we have

$$\mathbb{E}_\phi \sup_{t \in [k, k+1]} V_0^p(\mathbf{X}_t) \leq C_2(1 + V_0^p(\mathbf{X}_0)),$$

for some constant  $C_2$ . We have for any  $C > 0$ ,

$$\begin{aligned} \mathbb{P} \left\{ e^{-\frac{D_3}{2}t} V_0^{4d_0+4}(\mathbf{X}_t) \geq \frac{C}{\sqrt{\varepsilon}}, \text{ for some } t \in [k, k+1] \right\} &\leq \mathbb{P} \left\{ V_0^p(\mathbf{X}_t) \geq \frac{C e^{D_3 p k / (8d_0+8)}}{\varepsilon^{p/(8d_0+8)}}, \text{ for some } t \in [k, k+1] \right\} \\ &\leq C_2(1 + V_0^p(\mathbf{X}_0)) \frac{\varepsilon^{p/(8d_0+8)}}{C e^{D_3 p k / (8d_0+8)}}. \end{aligned}$$

Thus, one has

$$\begin{aligned} \mathbb{P} \left\{ e^{-\frac{D_3}{2}t} V_0^{4d_0+4}(\mathbf{X}_t) \leq \frac{C}{\sqrt{\varepsilon}} \forall t \geq 0 \right\} &\geq 1 - \sum_{k=0}^\infty C_2(1 + V_0^p(\mathbf{X}_0)) \frac{\varepsilon^{p/(8d_0+8)}}{C e^{D_3 p k / (8d_0+8)}} \\ &\geq 1 - K_{C, V_0^p(\mathbf{X}_0)} \varepsilon^{p/(8d_0+8)}, \end{aligned} \quad (4.59)$$

for some finite constant  $K_{C, V_0^p(\mathbf{X}_0)}$  depending on  $C, V_0^p(\mathbf{X}_0)$ . Combining (4.59) and (4.58), we can obtain

$$\mathbb{P} \left( \int_0^\infty \left[ 1 + |\mathbf{X}(t)| + |\tilde{\mathbf{X}}(t)| \right]^{4d_0+4} |\mathbf{Y}(t) - \tilde{\mathbf{Y}}(t)| dt \leq \frac{C}{\varepsilon} \right) \geq 1 - \sqrt{\varepsilon} - K_{C, V_0^p(\mathbf{X}_0)} \varepsilon^{\frac{p}{8d_0+8}}. \quad (4.60)$$

We obtain from the definition of  $\tilde{\tau}_\varepsilon$  that

$$\mathbb{P}\{\tilde{\tau}_\varepsilon = \infty\} \geq \mathbb{P} \left\{ \int_0^\infty |\mathbf{v}(s)|^2 ds < \varepsilon^{-1} \|\phi - \tilde{\phi}\|^2 \right\}. \quad (4.61)$$

Since (4.60), (4.61), definition of  $\mathbf{v}(\cdot)$ , and Assumption (2.4)(iii), we obtain (4.54) by letting  $\varepsilon \rightarrow 0$ .

On the other hand, applying the Burkholder-Davis-Gundy inequality and (4.45), we have

$$\begin{aligned} &\mathbb{E} \sup_{t \in [0,1]} \sum_i \left( Y_i(t) - \tilde{Y}_i(t) \right) \left( g_i(\mathbf{X}_t) - g_i(\tilde{\mathbf{X}}_t) \right) dE_i(t) \\ &\leq 4\mathbb{E} \left( n\sigma^* \sum_i^n \int_0^1 \left( Y_i(t) - \tilde{Y}_i(t) \right)^2 \left( g_i(\mathbf{X}_t) - g_i(\tilde{\mathbf{X}}_t) \right)^2 dt \right)^{\frac{1}{2}} \\ &\leq 4n\sqrt{\sigma^* D_1} \left( \mathbb{E} \int_0^1 \left( |1 + \mathbf{X}(t) + \tilde{\mathbf{X}}(t)|^{4d_0+4} |\mathbf{Z}(t)|^4 \right. \right. \\ &\quad \left. \left. + \int_{-r}^0 |1 + \mathbf{X}(t+s) + \tilde{\mathbf{X}}(t+s)|^{4d_0+4} |\mathbf{Z}(t+s)|^4 \mu(ds) \right) dt \right)^{\frac{1}{2}}. \end{aligned} \quad (4.62)$$

We obtain from (4.52) and the functional Itô formula that

$$\begin{aligned} & \mathbb{E} \int_0^1 \left( |1 + \mathbf{X}(t) + \tilde{\mathbf{X}}(t)|^{4d_0+4} |\mathbf{Z}(t)|^4 + \int_{-r}^0 |1 + \mathbf{X}(t+s) + \tilde{\mathbf{X}}(t+s)|^{4d_0+4} |\mathbf{Z}(t+s)|^4 \mu(ds) \right) dt \\ & \leq \frac{1}{2D_3} \mathbb{E} \tilde{U}(\phi, \tilde{\phi}). \end{aligned} \quad (4.63)$$

Applying (4.63) to (4.62) yields that

$$\mathbb{E} \sup_{t \in [0,1]} \sum_i \left( g_i(\mathbf{X}_t) - g_i(\tilde{\mathbf{X}}_t) \right) dE_i(t) \leq 4n \sqrt{\sigma^* D_1} \left( \frac{1}{2D_3} \mathbb{E} \tilde{U}(\phi, \tilde{\phi}) \right)^{\frac{1}{2}}. \quad (4.64)$$

Hence, combining (4.53) and (4.64), by a standard argument, we conclude that

$$\mathbb{E} \sup_{t \in [0,1]} U(\mathbf{X}_{t+t_0}, \tilde{\mathbf{X}}_{t+t_0}) \leq D_4 \left( \mathbb{E} U(\mathbf{X}_{t_0}, \tilde{\mathbf{X}}_{t_0}) + \left( \mathbb{E} \tilde{U}(\mathbf{X}_{t_0}, \tilde{\mathbf{X}}_{t_0}) \right)^{\frac{1}{2}} \right), \text{ for all } t_0 > 0, \quad (4.65)$$

for some constant  $D_4$ , independent of  $\mathbf{X}_t, \tilde{\mathbf{X}}_t$ . A consequence of Markov's inequality and (4.65) is that

$$\mathbb{P} \left\{ \sup_{t \in [n-1, n]} U(\mathbf{X}_t, \tilde{\mathbf{X}}_t) \geq e^{-\frac{D_3 n}{4}} \right\} \leq D_4 e^{\frac{D_3 n}{4}} \left( \mathbb{E} U(\mathbf{X}_{n-1}, \tilde{\mathbf{X}}_{n-1}) + \left( \mathbb{E} \tilde{U}(\mathbf{X}_{n-1}, \tilde{\mathbf{X}}_{n-1}) \right)^{\frac{1}{2}} \right). \quad (4.66)$$

We obtain from (4.56) and (4.66) that

$$\sum_{n=1}^{\infty} \mathbb{P} \left\{ \sup_{t \in [n-1, n]} U(\mathbf{X}_t, \tilde{\mathbf{X}}_t) \geq e^{-\frac{D_3 n}{4}} \right\} < \infty. \quad (4.67)$$

It follows from the Borel-Cantelli lemma and (4.67) that  $\lim_{t \rightarrow \infty} U(\mathbf{X}_t, \tilde{\mathbf{X}}_t) = 0$  a.s. and thus we get (4.55).  $\square$

Once we have Lemma 4.4, we can mimic the proof of [23, Theorem 3.1] to obtain the uniqueness of the invariant measure of (4.48), which is stated as in the following Proposition.

**Proposition 4.2.** *Under Assumptions 2.1 and 2.4, the solution process of (4.48) has at most one invariant measure, and moreover (2.1) has at most one invariant measure concentrated on  $\mathcal{C}_+^\circ$ .*

By the tightness (3.35) of the occupation measures and Theorem 4.1, the existence of invariant measure of (2.1) concentrated on  $\mathcal{C}_+^\circ$  is guaranteed. Combined with Proposition 4.2, we have the following Theorem to end this section.

**Theorem 4.2.** *Under Assumptions 2.1, 2.2, 2.3, and 2.4, system (2.1) has a unique invariant measure concentrated on  $\mathcal{C}_+^\circ$ .*

## 5 Applications

This section presents a number of applications of our main results to different models. We make use of Theorems 2.2 together with the following lemma whose proof can be found in [24], to characterize the persistence.

**Lemma 5.1.** *For any  $\pi \in \mathcal{M}$  and  $i \in I_\pi$ , we have  $\lambda_i(\pi) = 0$ .*

Moreover, it is worth noting that these sufficient conditions for persistence are sharp and are almost necessary in the sense that if they are not satisfied and critical cases are excluded, the extinction will take place, which will be seen in part (II).

## 5.1 Stochastic delay Lotka-Volterra competitive models

The Lotka-Volterra model, introduced in [38, 61], is one of the most popular models in mathematical biology and has been studied extensively in the literature. When two or more species live in proximity and share the same basic resources, they usually compete for food, habitat, territory, etc., we therefore have the Lotka-Volterra competitive model. To capture many complex properties in real life, other terms (white noises, Markov switching, delayed time, etc) are added to the original system. Stochastic delay Lotka-Volterra competitive models have also been widely studied; see, for example, [2, 34] and references therein. This kind model for two species has the form

$$\begin{cases} dX_1(t) = X_1(t) \left( a_1 - b_{11}X_1(t) - b_{12}X_2(t) - \widehat{b}_{11}X_1(t-r) - \widehat{b}_{12}X_2(t-r) \right) dt + X_1(t)dE_1(t), \\ dX_2(t) = X_2(t) \left( a_2 - b_{21}X_1(t) - b_{22}X_2(t) - \widehat{b}_{21}X_1(t-r) - \widehat{b}_{22}X_2(t-r) \right) dt + X_2(t)dE_2(t). \end{cases} \quad (5.1)$$

Note that in the above  $X_i(t)$  is the size of the species  $i$  at time  $t$ ;  $a_i > 0$  represents the growth rate of the species  $i$ ;  $b_{ii} > 0$  is the intra-specific competition of the  $i^{th}$  species;  $b_{ij} \geq 0$ , ( $i \neq j$ ) stands for the inter-specific competition;  $\widehat{b}_{ij} > -b_{ii}$  ( $i, j = 1, 2$ ) (i.e.,  $\widehat{b}_{ij}$  can be negative);  $r$  is the delay time;  $(E_1(t), E_2(t))^T = \Gamma^T \mathbf{B}(t)$  with  $\mathbf{B}(t) = (B_1(t), B_2(t))^T$  being a vector of independent standard Brownian motions and  $\Gamma$  being a  $2 \times 2$  matrix such that  $\Gamma^T \Gamma = (\sigma_{ij})_{2 \times 2}$  is a positive definite matrix.

Before applying our Theorems, let us verify our Assumptions. First, it is easy to see that there is a sufficiently large  $M_1$  such that

$$\frac{\sum_{i,j=1}^2 \sigma_{ij} x_i x_j}{(1 + x_1 + x_2)^2} \geq 2\sigma_* \text{ if } |\mathbf{x}| > M_1, \mathbf{x} := (x_1, x_2), \quad (5.2)$$

for some  $\sigma_* > 0$ . There exist  $0 < b_2^* < b_1^*$  and  $M_2 > 0$  satisfying

$$\frac{\sum_{i=1}^2 x_i \left( a_i - b_{i1}x_1 - b_{i2}x_2 - \widehat{b}_{i1}\varphi_1(-r) - \widehat{b}_{i2}\varphi_2(-r) \right)}{1 + x_1 + x_2} < -b_1^*(1 + |\mathbf{x}|) + b_2^*|\varphi(-r)|, \quad (5.3)$$

for all  $\varphi \in \mathcal{C}_+$  satisfying  $|\mathbf{x}| := |\varphi(0)| > M_2$ , and

$$\frac{\sum_{i=1}^2 x_i \left( a_i - b_{i1}x_1 - b_{i2}x_2 - \widehat{b}_{i1}\varphi_1(-r) - \widehat{b}_{i2}\varphi_2(-r) \right)}{1 + x_1 + x_2} < |\mathbf{x}| \sum_i a_i + b_2^*|\varphi(-r)|, \quad \forall \varphi \in \mathcal{C}_+. \quad (5.4)$$

Let  $M > \max\{M_1, M_2\}$ ,  $\mathbf{c} = (1, 1)$ ,

$$0 < \gamma_b < \min \left\{ \frac{b_1^*}{2 \sum_i a_i}, \frac{\sigma_*}{2}, \frac{b_1^* - b_2^*}{\sum_{i,j} (b_{ij} + |\widehat{b}_{ij}|)} \right\}, \quad 0 < \gamma_0 < \frac{b_1^*}{2} - \gamma_b \sum_i a_i,$$

$A_1, A_2$  be such that

$$0 < b_2^* + \gamma_b \sum_{i,j} |\widehat{b}_{ij}| < A_2 < A_1 < b_1^* - \gamma_b \sum_{i,j} b_{ij} \text{ and } A_1 - A_2 < \frac{b_1^*}{2},$$

and  $h(\mathbf{x}) := 1 + |\mathbf{x}|$ ,  $\mu$  is the Dirac delta measure (concentrated) at  $\{-r\}$ , and

$$\begin{aligned} A_0 := & \gamma_0 + A_1(1 + M) + \gamma_b \left( \sum_i a_i + M \sum_{i,j} b_{ij} + 2 \right) + M \sum_i a_i \\ & + \sup_{|\mathbf{x}| < M} \left\{ \frac{\sum_{i,j=1}^2 \sigma_{ij} x_i x_j}{(1 + x_1 + x_2)^2} + \frac{\sum_{i=1}^2 x_i (a_i - b_{i1}x_1 - b_{i2}x_2)}{1 + x_1 + x_2} \right\}. \end{aligned}$$

Combined with (5.2), (5.3), and (5.4), direct calculations lead to that (2.2) is satisfied and that Assumption 2.1 holds. Moreover, it is easy to confirm that Assumption 2.2 and Assumption 2.4 also hold.

Applying our Theorems in Section 2, we have that  $\lambda_i(\boldsymbol{\delta}^*) = a_i - \frac{\sigma_{ii}}{2}, i = 1, 2$ . Let  $\mathcal{C}_{1+}^\circ := \{(\varphi_1, 0) \in \mathcal{C}_+ : \varphi_1(s) > 0 \forall s \in [-r, 0]\}$  and  $\mathcal{C}_{2+}^\circ := \{(0, \varphi_2) \in \mathcal{C}_+ : \varphi_2(s) > 0 \forall s \in [-r, 0]\}$ . In view of Theorem 2.2, if  $\lambda_i(\boldsymbol{\delta}^*) > 0$ , there is a unique invariant probability measure  $\pi_i$  on  $\mathcal{C}_{i+}^\circ, i = 1, 2$ . By Lemma 5.1, we have

$$\lambda_i(\pi_i) = a_i - \frac{\sigma_{ii}}{2} - \int_{\mathcal{C}_{i+}^\circ} \left( b_{ii}\varphi_i(0) + \widehat{b}_{ii}\varphi_i(-r) \right) \pi_i(d\boldsymbol{\varphi}) = 0, \text{ where } \boldsymbol{\varphi} = (\varphi_1, \varphi_2),$$

which implies

$$\int_{\mathcal{C}_{i+}^\circ} \left( b_{ii}\varphi_i(0) + \widehat{b}_{ii}\varphi_i(-r) \right) \pi_i(d\boldsymbol{\varphi}) = a_i - \frac{\sigma_{ii}}{2}. \quad (5.5)$$

Since  $\pi_i$  is an invariant probability measure of  $\{\mathbf{X}_t\}$ , it is easy to see that

$$\int_{\mathcal{C}_{i+}^\circ} \varphi_i(0) \pi_i(d\boldsymbol{\varphi}) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T X_{i,t}(0) dt = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T X_i(t) dt, \quad (5.6)$$

where  $(X_{1,t}, X_{2,t}) = \mathbf{X}_t$ . Similarly,

$$\int_{\mathcal{C}_{i+}^\circ} \varphi_i(-r) \pi_i(d\boldsymbol{\varphi}) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T X_i(t-r) dt. \quad (5.7)$$

By virtue of (5.6) and (5.7), we can prove that

$$\int_{\mathcal{C}_{i+}^\circ} \varphi_i(0) \pi_i(d\boldsymbol{\varphi}) = \int_{\mathcal{C}_{i+}^\circ} \varphi_i(-r) \pi_i(d\boldsymbol{\varphi}). \quad (5.8)$$

Combining (5.5) and (5.8) yields that

$$\int_{\mathcal{C}_{i+}^\circ} \varphi_i(0) \pi_i(d\boldsymbol{\varphi}) = \int_{\mathcal{C}_{i+}^\circ} \varphi_i(-r) \pi_i(d\boldsymbol{\varphi}) = \frac{a_i - \frac{\sigma_{ii}}{2}}{b_{ii} + \widehat{b}_{ii}}.$$

Therefore, we have

$$\begin{aligned} \lambda_2(\pi_1) &= \int_{\mathcal{C}_{1+}^\circ} \left[ a_2 - \frac{\sigma_{22}}{2} - b_{21}\varphi_1(0) - \widehat{b}_{21}\varphi_1(-r) \right] \pi_1(d\boldsymbol{\varphi}) \\ &= a_2 - \frac{\sigma_{22}}{2} - \left( a_1 - \frac{\sigma_{11}}{2} \right) \cdot \frac{b_{21} + \widehat{b}_{21}}{b_{11} + \widehat{b}_{11}}, \end{aligned}$$

and

$$\begin{aligned} \lambda_1(\pi_2) &= \int_{\mathcal{C}_{2+}^\circ} \left[ a_1 - \frac{\sigma_{11}}{2} - b_{12}\varphi_2(0) - \widehat{b}_{12}\varphi_2(-r) \right] \pi_2(d\boldsymbol{\varphi}) \\ &= a_1 - \frac{\sigma_{11}}{2} - \left( a_2 - \frac{\sigma_{22}}{2} \right) \cdot \frac{b_{12} + \widehat{b}_{12}}{b_{22} + \widehat{b}_{22}}. \end{aligned}$$

If  $\lambda_1(\boldsymbol{\delta}^*) > 0, \lambda_2(\boldsymbol{\delta}^*) > 0$  and  $\lambda_1(\pi_2) > 0, \lambda_2(\pi_1) > 0$ , any invariant probability measure in  $\partial\mathcal{C}_+$  has the form  $\pi = q_0\boldsymbol{\delta}^* + q_1\pi_1 + q_2\pi_2$  with  $0 \leq q_0, q_1, q_2$  and  $q_0 + q_1 + q_2 = 1$ . Then, one has

$\max_{i=1,2} \{\lambda_i(\pi)\} > 0$  for any  $\pi$  having the form as above. As a consequence of Theorem 2.2, there is a unique invariant probability measure  $\pi^*$  on  $\mathcal{C}_+^\circ$ . This result generalizes the results of long-term properties in [34].

In the above, we considered a 2-dimension case to illustrate the idea as well as to simplify the explicit computation. For the stochastic delay Lotka-Volterra competitive model with  $n$ -species, our results can still be applied to characterize the long-term behavior of the solution.

## 5.2 Stochastic delay Lotka-Volterra predator-prey models

To continue our study of Lotka-Volterra competitive models, this section is devoted to applying our results to stochastic Lotka-Volterra predator-prey models with time delay. Such models are frequently used to describe the dynamics of biological systems in which two species interact, one as a predator and the other one as prey. In this section, we consider Lotka-Volterra predator-prey system with one prey and two competing predators as follows

$$\begin{cases} dX_1(t) = X_1(t) \left\{ a_1 - b_{11}X_1(t) - b_{12}X_2(t) - b_{13}X_3(t) \right. \\ \quad \left. - \widehat{b}_{11}X_1(t-r) - \widehat{b}_{12}X_2(t-r) - \widehat{b}_{13}X_3(t-r) \right\} dt + X_1(t)dE_1(t), \\ dX_2(t) = X_2(t) \left\{ -a_2 + b_{21}X_1(t) - b_{22}X_2(t) - b_{23}X_3(t) \right. \\ \quad \left. - \widehat{b}_{21}X_1(t-r) - \widehat{b}_{22}X_2(t-r) - \widehat{b}_{23}X_3(t-r) \right\} dt + X_2(t)dE_2(t), \\ dX_3(t) = X_3(t) \left\{ -a_3 + b_{31}X_1(t) - b_{32}X_2(t) - b_{33}X_3(t) \right. \\ \quad \left. - \widehat{b}_{31}X_1(t-r) - \widehat{b}_{32}X_2(t-r) - \widehat{b}_{33}X_3(t-r) \right\} dt + X_3(t)dE_3(t), \end{cases} \quad (5.9)$$

where  $X_1(t)$ ,  $X_2(t)$ , and  $X_3(t)$  are the densities at time  $t$  of the prey, and two predators, respectively;  $a_1 > 0$  is the growth rate;  $a_2, a_3 > 0$  are the death rate of  $X_2, X_3$ ;  $b_{ii} > 0, i = 1, 2, 3$  denote the intra-specific competition coefficient of  $X_i$ ;  $b_{ij} \geq 0, i \neq j = 1, 2, 3$ , in which  $b_{12}, b_{13}$  represent the capture rates,  $b_{21}, b_{31}$  represent the growth from food, and  $b_{23}$  and  $b_{32}$  signify the competitions between predators (species 2 and 3); for each  $i, j \in \{1, 2, 3\}$ ,  $\widehat{b}_{ij}$  is either positive or in  $(-b_{ii}, 0]$ ;  $r$  is the time delay;  $(E_1(t), E_2(t), E_3(t))^\top = \Gamma^\top \mathbf{B}(t)$  with  $\mathbf{B}(t) = (B_1(t), B_2(t), B_3(t))^\top$  being a vector of independent standard Brownian motions and  $\Gamma$  being a  $3 \times 3$  matrix such that  $\Gamma^\top \Gamma = (\sigma_{ij})_{3 \times 3}$  is a positive definite matrix.

The model in the current setup, was considered in [32]. However, by switching the sign of  $a_i$  or  $b_{ij}, i \neq j$ , we can obtain a stochastic time-delay Lotka-Volterra system with the prey and the mesopredator or intermediate predator. Note that the case involving a superpredator or top predator, was studied in [33, 64], and the stochastic time-delay Lotka-Volterra system with one predator and two preys was investigated in [22].

By a similar calculation as in Subsection 5.1, we can check that (2.2) is satisfied if we let  $\mathbf{c} = \left(1, \frac{b_{12}}{b_{21}}, \frac{b_{13}}{b_{31}}\right)$  and other parameters be similarly determined as in Section 5.1. Moreover, other assumptions also hold.

We consider the equation on the boundaries  $\mathcal{C}_{12+} := \{(\varphi_1, \varphi_2, 0) \in \mathcal{C}_+ : \varphi_1(s), \varphi_2(s) \geq 0 \forall s \in [-r, 0]\}$  and  $\mathcal{C}_{13+} := \{(\varphi_1, 0, \varphi_3) \in \mathcal{C}_+ : \varphi_1(s), \varphi_3(s) \geq 0, \forall s \in [-r, 0]\}$ . If  $\lambda_1(\delta^*) > 0$ , there is an invariant probability measure  $\pi_1$  on  $\mathcal{C}_{1+}^\circ := \{(\varphi_1, 0, 0) \in \mathcal{C}_+ : \varphi_1(s) > 0 \forall s \in [-r, 0]\}$ .

In view of Lemma 5.1, we obtain

$$\int_{\mathcal{C}_{1+}^\circ} \left( b_{11}\varphi_1(0) + \widehat{b}_{11}\varphi_1(-r) \right) \pi_1(d\varphi) = a_1 - \frac{\sigma_{11}}{2}. \quad (5.10)$$

Similar to the process of getting (5.8), we obtain from (5.10) that

$$\int_{\mathcal{C}_{1+}^{\circ}} \varphi_1(0) \pi_1(d\varphi) = \int_{\mathcal{C}_{1+}^{\circ}} \varphi_1(-r) \pi_1(d\varphi) = \frac{a_1 - \frac{\sigma_{11}}{2}}{b_{11} + \widehat{b}_{11}}.$$

Therefore,

$$\begin{aligned} \lambda_i(\pi_1) &= \int_{\mathcal{C}_{1+}^{\circ}} \left[ -a_i - \frac{\sigma_{ii}}{2} + b_{i1}\varphi_1(0) - \widehat{b}_{i1}\varphi_1(-r) \right] \pi_1(d\varphi) \\ &= -a_i - \frac{\sigma_{ii}}{2} + \left( a_1 - \frac{\sigma_{11}}{2} \right) \cdot \frac{b_{i1} - \widehat{b}_{i1}}{b_{11} + \widehat{b}_{11}}, \quad i = 2, 3. \end{aligned}$$

In case of  $\lambda_1(\delta^*) > 0$  and  $\lambda_2(\pi_1) > 0$ , Theorem 2.2 implies that there is an invariant probability measure  $\pi_{12}$  on  $\mathcal{C}_{12+}^{\circ}$ . In view of Lemma 5.1 and (5.8), we obtain

$$\begin{aligned} \int_{\mathcal{C}_{12+}^{\circ}} \varphi_1(0) \pi_{12}(d\varphi) &= \int_{\mathcal{C}_{12+}^{\circ}} \varphi_1(-r) \pi_{12}(d\varphi) = A_1, \\ \int_{\mathcal{C}_{12+}^{\circ}} \varphi_2(0) \pi_{12}(d\varphi) &= \int_{\mathcal{C}_{12+}^{\circ}} \varphi_2(-r) \pi_{12}(d\varphi) = A_2, \end{aligned}$$

where the pair  $(A_1, A_2)$  is the unique solution to

$$\begin{cases} a_1 - \frac{\sigma_{11}}{2} - (b_{11} + \widehat{b}_{11}) A_1 - (b_{12} + \widehat{b}_{12}) A_2 = 0, \\ -a_2 - \frac{\sigma_{22}}{2} + (b_{21} - \widehat{b}_{21}) A_1 - (b_{22} + \widehat{b}_{22}) A_2 = 0. \end{cases}$$

In this case,

$$\begin{aligned} \lambda_3(\pi_{12}) &= \int_{\mathcal{C}_{12+}^{\circ}} \left[ -a_3 - \frac{\sigma_{33}}{2} + (b_{31}\varphi_1(0) - \widehat{b}_{31}\varphi_1(-r)) - (b_{32}\varphi_2(0) + \widehat{b}_{32}\varphi_2(-r)) \right] \pi_{12}(d\varphi) \\ &= -a_3 - \frac{\sigma_{33}}{2} + (b_{31} - \widehat{b}_{31}) A_1 - (b_{32} + \widehat{b}_{32}) A_2. \end{aligned}$$

Similarly, if  $\lambda_1(\delta^*) > 0$  and  $\lambda_3(\pi_1) > 0$ , by Theorem 2.2, there is an invariant probability measure  $\pi_{13}$  on  $\mathcal{C}_{13+}^{\circ}$  and

$$\begin{aligned} \lambda_2(\pi_{13}) &= \int_{\mathcal{C}_{13+}^{\circ}} \left[ -a_2 - \frac{\sigma_{22}}{2} + (b_{21}\varphi_1(0) - \widehat{b}_{21}\varphi_1(-r)) - (b_{23}\varphi_3(0) + \widehat{b}_{23}\varphi_3(-r)) \right] \pi_{13}(d\varphi) \\ &= -a_2 - \frac{\sigma_{22}}{2} + (b_{21} - \widehat{b}_{21}) \widehat{A}_1 - (b_{23} + \widehat{b}_{23}) \widehat{A}_3, \end{aligned}$$

where  $(\widehat{A}_1, \widehat{A}_3)$  is the unique solution to

$$\begin{cases} a_1 - \frac{\sigma_{11}}{2} - (b_{11} + \widehat{b}_{11}) \widehat{A}_1 - (b_{13} + \widehat{b}_{13}) \widehat{A}_3 = 0, \\ -a_3 - \frac{\sigma_{33}}{2} + (b_{31} - \widehat{b}_{31}) \widehat{A}_1 - (b_{33} + \widehat{b}_{33}) \widehat{A}_3 = 0. \end{cases}$$

Because of the ergodic decomposition theorem, every invariant probability measure on  $\partial\mathcal{C}_+$  is a convex combination of  $\delta^*, \pi_1, \pi_{12}, \pi_{13}$  (when these measures exist). As a consequence, some computations for the Lyapunov exponents with respect to a convex combination of these ergodic measures together with an application of Theorem 2.2 yield that there exists a unique invariant probability measure in  $\mathcal{C}_+^{\circ}$  if one of the following conditions is satisfied:

- $\lambda_1(\delta^*) > 0$ ,  $\lambda_2(\pi_1) > 0$ ,  $\lambda_3(\pi_1) < 0$  and  $\lambda_3(\pi_{12}) > 0$ .
- $\lambda_1(\delta^*) > 0$ ,  $\lambda_2(\pi_1) < 0$ ,  $\lambda_3(\pi_1) > 0$  and  $\lambda_2(\pi_{13}) > 0$ .
- $\lambda_1(\delta^*) > 0$ ,  $\lambda_2(\pi_1) > 0$ ,  $\lambda_3(\pi_1) > 0$ ,  $\lambda_3(\pi_{12}) > 0$ , and  $\lambda_2(\pi_{13}) > 0$ .

The above assertions generalize the results in [32]. Moreover, if we switch the sign of  $a_i$  or  $b_{ij}$ ,  $i \neq j$ , we obtain another modifications of Lotka-Volterra prey-predator equation as we mentioned at the beginning of this section with modification of the above characterization, which improve the results in [22, 33, 64].

Confining our analysis to  $\mathcal{C}_{12+}$  (this describes the evolution of one predator and its prey), we get

$$\begin{cases} dX_1(t) = X_1(t) \left\{ a_1 - b_{11}X_1(t) - \widehat{b}_{11}X_1(t-r) - b_{12}X_2(t) - \widehat{b}_{12}X_2(t-r) \right\} dt + X_1(t)dE_1(t), \\ dX_2(t) = X_2(t) \left\{ -a_2 + b_{21}X_1(t) + \widehat{b}_{21}X_1(t-r) - b_{22}X_2(t) - \widehat{b}_{22}X_2(t-r) \right\} dt + X_2(t)dE_2(t). \end{cases} \quad (5.11)$$

This further leads to that if  $\lambda_1(\delta^*) > 0$ ,  $\lambda_2(\pi_1) > 0$ , there exists a unique invariant probability measure of (5.11) on  $\mathcal{C}_{12+}^0$ , which improves the results in [31].

### 5.3 Stochastic delay replicator equations

In evolutionary game theory, originally, a replicator equation is a deterministic monotone, nonlinear, and non-innovative game dynamic system. Such a deterministic system has been expanded to systems with stochastic perturbations. In this section, we consider the replicator dynamics for a game with  $n$  strategies, involving social-type time delay (see, e.g., [1] for details of such delays) and white noise perturbation. The system of interest can be expressed as

$$\begin{cases} dx_i(t) = x_i(t) \left( f_i(\mathbf{x}(t-r)) - \frac{1}{X} \sum_{j=1}^n x_j(t) f_j(\mathbf{x}(t-r)) \right) dt \\ \quad + x_i(t) \left( \sigma_i dB_i(t) - \frac{1}{X} \sum_{j=1}^n \sigma_j x_j dB_j(t) \right); \quad i = 1, \dots, n, \\ \mathbf{x}(s) = \mathbf{x}_0(s); \quad t \in [-r, 0], \end{cases} \quad (5.12)$$

where  $X$  is the size of the populations;  $x_i(t)$  is the portion of population that has selected the  $i^{th}$  strategy and the distribution of the whole population among the strategy; the fitness functions  $f_i(\cdot) : \mathbb{R}_+^n \rightarrow \mathbb{R}$ ,  $i = 1, \dots, n$  are the payoffs obtained by the individuals playing the  $i^{th}$  strategy;  $r$  is the time delay; and  $\mathbf{x}_0(s) \in \Delta_X := \{\mathbf{x} \in \mathbb{R}_+^n : \sum_{i=1}^n x_i = X\}$  for all  $s \in [-r, 0]$  is the initial value.

The replicator equation was introduced in 1978 by Taylor and Jonker in [56]. Since then significant contributions have been made in biology [25, 46], economics [62], and optimization and control for a variety of systems [7, 47, 49, 57]. Much attention has been devoted to studying their properties. For instance, when  $f_i(\cdot) : \mathbb{R}_+^n \rightarrow \mathbb{R}$ ,  $i = 1, \dots, n$  are linear mappings, equation (5.12) without time delay was studied in [26, 27]. Moreover, the deterministic version of equation (5.12) was studied in [1, 48].

By a similar argument as in [48, 62], we can show that  $\Delta_X$  remains invariant a.s. As a consequence, our assumptions are verified. Hence, our results in Theorem 2.2 holds for (5.12). To

demonstrate, for better visualization, we apply our results to some low-dimensional systems. First, we consider equation (5.12) in case of two dimensions. Define

$$\mathcal{C}_+^X := \{(\varphi_1, \varphi_2) : \varphi_1(s) + \varphi_2(s) = X \text{ and } \varphi_1(s), \varphi_2(s) \geq 0 \text{ for all } s \in [-r, 0]\},$$

$$\partial\mathcal{C}_+^X := \{(\varphi_1, \varphi_2) \in \mathcal{C}_+^X : \|\varphi_1\| = 0 \text{ or } \|\varphi_2\| = 0\},$$

$$\mathcal{C}_+^{X,\circ} := \{(\varphi_1, \varphi_2) \in \mathcal{C}_+^X : \varphi_1(s), \varphi_2(s) > 0 \text{ for all } s \in [-r, 0]\}.$$

In this case, it is clear that there are two invariant measures on the boundary  $\partial\mathcal{C}_+^X$ , which are  $\delta_1$  and  $\delta_2$  concentrating on  $(X, 0)$  and  $(0, X)$ , respectively, where  $0, X$  are understood to be constant functions. We have

$$\lambda_1(\delta_2) = f_1((0, X)) - f_2((0, X)) - \frac{\sigma_1^2 + \sigma_2^2}{2},$$

$$\lambda_2(\delta_1) = f_2((X, 0)) - f_1((X, 0)) - \frac{\sigma_1^2 + \sigma_2^2}{2}.$$

By Theorem 2.2, in case of (5.12) of 2-dimensional systems, if  $\lambda_1(\delta_2) > 0$  and  $\lambda_2(\delta_1) > 0$ , there is a unique invariant probability measure of (5.12) on  $\mathcal{C}_+^{X,\circ}$ .

Next, we consider (5.12) in three dimensions. Similarly, we also define the following set

$$\mathcal{C}_+^X := \{(\varphi_1, \varphi_2, \varphi_3) : \varphi_1(s) + \varphi_2(s) + \varphi_3(s) = X \text{ and } \varphi_1(s), \varphi_2(s), \varphi_3(s) \geq 0 \text{ for all } s \in [-r, 0]\},$$

$$\partial\mathcal{C}_+^X := \mathcal{C}_{12+}^X \cup \mathcal{C}_{23+}^X \cup \mathcal{C}_{13+}^X,$$

$$\mathcal{C}_{ij+}^X := \{(\varphi_1, \varphi_2, \varphi_3) \in \mathcal{C}_+^X : \|\varphi_k\| = 0, k \neq i, j\}, \text{ for } i \neq j \in \{1, 2, 3\},$$

$$\mathcal{C}_+^{X,\circ} := \{(\varphi_1, \varphi_2, \varphi_3) \in \mathcal{C}_+^X : \varphi_1(s), \varphi_2(s), \varphi_3(s) > 0 \text{ for all } s \in [-r, 0]\}.$$

Denote by  $\delta_1, \delta_2, \delta_3$  the invariant probability measures on the boundary  $\partial\mathcal{C}_+^X$  of (5.12), concentrating on  $(X, 0, 0)$ ,  $(0, X, 0)$ , and  $(0, 0, X)$ , respectively. We have

$$\lambda_i(\delta_1) = f_i((X, 0, 0)) - f_1((X, 0, 0)) - \frac{\sigma_1^2 + \sigma_i^2}{2}, i = 2, 3,$$

$$\lambda_j(\delta_2) = f_j((0, X, 0)) - f_2((0, X, 0)) - \frac{\sigma_2^2 + \sigma_j^2}{2}, j = 1, 3,$$

and

$$\lambda_k(\delta_3) = f_k((0, 0, X)) - f_3((0, 0, X)) - \frac{\sigma_3^2 + \sigma_k^2}{2}, k = 1, 2.$$

If  $\max_{j=1,3} \lambda_j(\delta_2) > 0$  and  $\max_{k=1,2} \lambda_k(\delta_3) > 0$ , there is a unique invariant measure on  $\mathcal{C}_{23+}^X$ , denoted by  $\pi_{23}$ . When  $\pi_{23}$  exists, we have

$$\lambda_1(\pi_{23}) = -\frac{\sigma_1^2}{2} + \int_{\mathcal{C}_{23+}^X} \left( f_1(\varphi) - \frac{2X\varphi_2(0)f_2(\varphi) + \sigma_2^2\varphi_2^2(0)}{X^2} - \frac{2X\varphi_3(0)f_3(\varphi) + \sigma_3^2\varphi_3^2(0)}{X^2} \right) \pi_{23}(d\varphi).$$

By Lemma 5.1 and  $\lambda_2(\pi_{23}) = \lambda_3(\pi_{23}) = 0$ , we have

$$\begin{aligned} & \int_{\mathcal{C}_{23+}^X} \left( \frac{2X\varphi_2(0)f_2(\varphi) + \sigma_2^2\varphi_2^2(0)}{2X^2} + \frac{2X\varphi_3(0)f_3(\varphi) + \sigma_3^2\varphi_3^2(0)}{2X^2} \right) \pi_{23}(d\varphi) \\ &= \frac{\sigma_2^2}{2} + \int_{\mathcal{C}_{23+}^X} f_2(\varphi) \pi_{23}(d\varphi) \\ &= \frac{\sigma_3^2}{2} + \int_{\mathcal{C}_{23+}^X} f_3(\varphi) \pi_{23}(d\varphi). \end{aligned}$$

As a result,

$$\begin{aligned}\lambda_1(\pi_{23}) &= -\frac{\sigma_1^2 + \sigma_2^2}{2} + \int_{\mathcal{C}_{23+}^X} (f_1(\varphi) - f_2(\varphi)) \pi_{23}(d\varphi) \\ &= -\frac{\sigma_1^2 + \sigma_3^2}{2} + \int_{\mathcal{C}_{23+}^X} (f_1(\varphi) - f_3(\varphi)) \pi_{23}(d\varphi).\end{aligned}$$

The conditions to guarantee the existence of the unique invariant probability measure  $\pi_{12}, \pi_{13}$  on the boundary  $\mathcal{C}_{12+}^X, \mathcal{C}_{13+}^X$  are similarly obtained and  $\lambda_2(\pi_{13}), \lambda_3(\pi_{12})$  can be computed similar to  $\lambda_1(\pi_{23})$ . Therefore, we have the following classification for the long-run solution of (5.12) in three dimensions as following. System (5.12) admits a unique invariant probability measure on  $\mathcal{C}_+^{X,\circ}$  if

- $\max_{i=2,3} \lambda_i(\delta_1) > 0$ ,  $\max_{j=1,3} \lambda_j(\delta_2) > 0$ ,  $\max_{k=1,2} \lambda_k(\delta_3) > 0$  and  $\lambda_1(\pi_{23}) > 0$ ,  $\lambda_2(\pi_{13}) > 0$ ,  $\lambda_3(\pi_{12}) > 0$ .

The (explicit) condition for persistence of (5.12) in  $n$ -dimensions is more complex. However, our results (Theorem 2.2) still holds and will be computable in practice under suitable conditions. Moreover, if  $r = 0$  (i.e., there is no time delay) and  $f_i(\cdot)$ ,  $i = 1, \dots, n$  are linear, the condition of the persistence of (5.12) in this section is equivalent to results in [26, 27].

## 5.4 Stochastic delay epidemic SIR models

The SIR model is one of the basic building blocks of compartmental models, from which many infectious disease models are derived. The model consists of three compartments, S for the number of susceptible, I for the number of infectious, and R for the number of recovered (or immune). First introduced by Kermack and McKendrick in [28, 29], the models are deemed effective to depict the spread of many common diseases with permanent immunity such as rubella, whooping cough, measles, and smallpox. A variety of modifications of original equation are introduced due to the complexity of environment. Much attention has been devoted to analyzing the behavior of these systems; for example, see [12, 15] and the references therein. In this section, we study the stochastic epidemic SIR models with time delay.

To start, we consider the equation with linear incidence rate of the following form

$$\begin{cases} dS(t) = (a - b_1 S(t) - c_1 I(t) S(t) - c_2 I(t) S(t-r)) dt + S(t) dE_1(t), \\ dI(t) = (-b_2 I(t) + c_1 I(t) S(t) + c_2 I(t) S(t-r)) dt + I(t) dE_2(t), \end{cases} \quad (5.13)$$

where  $S(t)$  is the density of susceptible individuals,  $I(t)$  is the density of infected individuals,  $a > 0$  is the recruitment rate of the population,  $b_i > 0$ ,  $i = 1, 2$  are the death rates,  $c_i > 0$ ,  $i = 1, 2$  are the incidence rates,  $r$  is the delayed time,  $(E_1(t), E_2(t))^\top = \Gamma^\top \mathbf{B}(t)$  with  $\mathbf{B}(t) = (B_1(t), B_2(t))^\top$  being a vector of independent standard Brownian motions, and  $\Gamma$  being a  $2 \times 2$  matrix such that  $\Gamma^\top \Gamma = (\sigma_{ij})_{2 \times 2}$  is a positive definite matrix. It is well-known that the dynamics of recovered individuals have no effect on the disease transmission dynamics and that is why we only consider the dynamics of  $S(t), I(t)$  in (5.13).

Although equation (5.13) does not have the exact form as in (2.1), we can use the same idea and the same method to obtain similar results. First, we consider the equation on the boundary  $\{(\varphi_1, 0) : \varphi_1(s) \geq 0 \forall s \in [-r, 0]\}$  and let  $\hat{S}(t)$  be the solution of the equation on this boundary as following

$$d\hat{S}(t) = (a - b_1 \hat{S}(t)) dt + \hat{S}(t) dE_1(t). \quad (5.14)$$

Since the drift coefficient of this equation is negative if  $\widehat{S}(t)$  is sufficiently large and positive, if  $\widehat{S}(t)$  is sufficiently small, we can show that there is a unique invariant measure  $\pi$  of (5.13) on  $\mathcal{C}_{1+}^\circ := \{(\varphi_1, 0) : \varphi_1(s) > 0 \forall s \in [-r, 0]\}$ . On the other hand, since  $\lambda_2(\delta^*) = -b_2 - \frac{\sigma_{22}}{2} < 0$ , there is no invariant measure in  $\mathcal{C}_{2+}^\circ := \{(0, \varphi_2) : \varphi_2(s) > 0; \forall s \in [-r, 0]\}$ .

Hence, we define the following threshold

$$\lambda(\pi) = -b_2 - \frac{\sigma_{22}}{2} + \int_{\mathcal{C}_{1+}^\circ} (c_1 \varphi_1(0) + c_2 \varphi_1(-r)) \pi(d\varphi), \quad (5.15)$$

whose sign will be able to characterize the permanence and extinction. As an application of Lemma 5.1, we get

$$\int_{\mathcal{C}_{1+}^\circ} \varphi_1(0) \pi(d\varphi) = \frac{a}{b_1}. \quad (5.16)$$

By (5.8), we have that

$$\int_{\mathcal{C}_{1+}^\circ} \varphi_1(-r) \pi(d\varphi) = \int_{\mathcal{C}_{1+}^\circ} \varphi_1(0) \pi(d\varphi) = \frac{a}{b_1}.$$

Therefore, under this condition, we obtain from (5.15) and (5.16) that

$$\lambda(\pi) = -b_2 - \frac{\sigma_{22}}{2} + \frac{a(c_1 + c_2)}{b_1}.$$

Using the same idea and techniques, it is possible to obtain similar results to Theorem 2.2 for equation (5.13). We have that if  $\lambda(\pi) > 0$ , (5.13) has a unique invariant probability measure in  $\mathcal{C}_+^\circ$ . This characterization is equivalent to the result in [35, 36].

In the above, we consider the linear incidence to make our computations be more explicit. The characterizations still hold for the following stochastic delay SIR epidemic model with general incidence rate

$$\begin{cases} dS(t) = (a - b_1 S(t) - I(t) f_1(S(t), S(t-r), I(t), I(t-r))) dt + S(t) dE_1(t), \\ dI(t) = (-b_2 I(t) + I(t) f_2(S(t), S(t-r), I(t), I(t-r))) dt + I(t) dE_2(t), \end{cases} \quad (5.17)$$

where  $f_i : \mathbb{R}^4 \rightarrow \mathbb{R}, i = 1, 2$  are the incidence functions satisfying

- $f_1(0, 0, i_1, i_2) = f_2(0, 0, i_1, i_2) = 0$ .
- there exists some  $\kappa \in (0, \infty)$  such that for all  $\varphi \in \mathcal{C}_+$

$$\begin{aligned} f_2(\varphi_1(0), \varphi_1(-r), \varphi_2(0), \varphi_2(-r)) &\leq \kappa f_1(\varphi_1(0), \varphi_1(-r), \varphi_2(0), \varphi_2(-r)) \\ &\leq \kappa^2 (1 + |\varphi(0)| + |\varphi(-r)|). \end{aligned}$$

- $f_2(s_1, s_2, i_1, i_2)$  is non-decreasing in  $s_1, s_2$  and is non-increasing in  $i_1, i_2$ .

Almost all incidence functions used in the literature, e.g., linear functional response, Holling type II functional response, Beddington-DeAngelis functional response, etc., satisfy the above conditions. In the general case, the system has a unique invariant probability measure in  $\mathcal{C}_+^\circ$  if  $\lambda(\pi) > 0$ , where  $\lambda(\pi)$  is defined as follows

$$\lambda(\pi) = -b_2 - \frac{\sigma_{22}}{2} + \int_{\mathcal{C}_{1+}^\circ} f_2(\varphi_1(0), \varphi_1(-r), \varphi_2(0), \varphi_2(-r)) \pi(d\varphi),$$

where  $\varphi = (\varphi_1, \varphi_2)$  and  $\pi$  is the invariant probability measure of (5.14). These results significantly generalize and improve that of [8, 14, 37, 39].

## 5.5 Stochastic delay chemostat models

A chemostat is a bio-reactor. In a chemostat, fresh medium is continuously added, and culture liquid containing left-over nutrients, metabolic end products, and microorganisms are continuously removed at the same rate to keep a constant culture volume. The chemostat model is based on a technique introduced by Novick and Szilard in [45] and plays an important role in microbiology, biotechnology, and population biology. This section is devoted to studying a model of  $n$ -microbial populations competing for a single nutrient in a chemostat with delay in uptake conversion and under effects of white noises. Precisely, the model is described by the following system of stochastic functional differential equations

$$\begin{cases} dS(t) = \left( 1 - S(t) + aS(t-r) - \sum_{i=1}^n x_i(t)p_i(S(t)) \right) dt + S(t)dE_0(t), \\ dx_i(t) = x_i(t) (p_i(S(t-r)) - 1) dt + x_i(t)dE_i(t), \quad i = 1, \dots, n, \end{cases} \quad (5.18)$$

where  $S(t)$  is the concentration of nutrient at time  $t$ ;  $0 \leq a < 1$  is a constant;  $x_i(t), i = 1, \dots, n$  are the concentrations of the competing microbial populations;  $p_i(S), i = 1, \dots, n$  are the density-dependent uptakes of nutrient by population  $x_i$ ;  $r$  is the delayed time; and  $(E_0(t), \dots, E_n(t))^\top = \Gamma^\top \mathbf{B}(t)$  with  $\mathbf{B}(t) = (B_0(t), \dots, B_n(t))^\top$  being a vector of independent standard Brownian motions and  $\Gamma$  being a  $(n+1) \times (n+1)$  matrix such that  $\Gamma^\top \Gamma = (\sigma_{ij})_{(n+1) \times (n+1)}$  is a positive definite matrix. Moreover, in this section,  $\mathcal{C} := \mathcal{C}([-r, 0], \mathbb{R}^{n+1})$  instead of  $\mathcal{C}([-r, 0], \mathbb{R}^n)$ . The deterministic version of (5.18) is studied and the long-time behavior is characterized in [18, 21, 63]. Recently, much attention is devoted to studying the related stochastic systems; see [54, 55, 65].

It is similar to Section 5.4, if we assume that  $p_i : \mathbb{R} \rightarrow \mathbb{R}, i = 1, \dots, n$  satisfying non-decreasing and bounded properties and  $p_i(0) = 0$ , then our Assumptions hold. Therefore, our results in this paper can be applied to (5.18).

Before obtaining the results in multi-dimensional systems, we consider  $n = 1, 2$ . If  $n = 1$ , there is only one population  $x_1$  together with the nutrient  $S(t)$ . Similar to Section 5.4, there is no invariant measure of  $(S_t, x_{1t})$  in  $\mathcal{C}_{1+}^\circ := \{(0, \varphi_1) \in \mathcal{C}_+ : \varphi_1(s) > 0, \forall s \in [-r, 0]\}$ , where  $x_{1t}$  is the memory segment function of  $x_1(t)$ . Moreover, there is a unique invariant measure  $\pi_0$  in  $\mathcal{C}_{0+}^\circ := \{(\varphi_0, 0) \in \mathcal{C}_+ : \varphi_0(s) > 0, \forall s \in [-r, 0]\}$ . Hence, it is easy to see that for any invariant measure  $\pi$  in  $\partial\mathcal{C}_+$ , we have

$$\lambda_1(\pi) = \lambda_1(\pi_0) = -1 - \frac{\sigma_{11}}{2} + \int_{\mathcal{C}_{0+}^\circ} p_1(\varphi_0(-r))\pi_0(d\varphi).$$

By applying our result, if  $\lambda_1(\pi_0) > 0$  then  $(S_t, x_{1t})$  admits a unique invariant probability measure in  $\mathcal{C}_{0+}^\circ$ .

We next reveal the characterization of the longtime behavior in the case  $n = 2$ , which is similar to the case of  $n = 1$ . There is no invariant measure in  $\mathcal{C}_{i+}^\circ := \{(0, \varphi_1, \varphi_2) \in \mathcal{C}_+ : \|\varphi_j\| = 0, j \neq i \text{ and } \varphi_i(s) > 0, \forall s \in [-r, 0]\}$ , and there is a unique measure  $\pi_0$  in  $\mathcal{C}_{0+}^\circ := \{(\varphi_0, 0, 0) \in \mathcal{C}_+ : \varphi_0(s) > 0, \forall s \in [-r, 0]\}$ . As characterized in the case  $n = 1$ , if  $\lambda_i(\pi_0) > 0$ , where

$$\lambda_i(\pi_0) = -1 - \frac{\sigma_{ii}}{2} + \int_{\mathcal{C}_{0+}^\circ} p_i(\varphi_0(-r))\pi_0(d\varphi), \quad i = 1, 2,$$

then there is a unique invariant measure  $\pi_{0i}$  in  $\mathcal{C}_{0i+}^\circ := \{(\varphi_0, \varphi_1, \varphi_2) \in \mathcal{C}_+ : \|\varphi_j\| = 0, j \neq i \text{ and } \varphi_0(s), \varphi_i(s) > 0, \forall s \in [-r, 0]\}$ . Hence, let

$$\lambda_j(\pi_{0i}) = -1 - \frac{\sigma_{jj}}{2} + \int_{\mathcal{C}_{0+}^\circ} p_j(\varphi_0(-r))\pi_{0i}(d\varphi), \quad j \neq i.$$

The persistence is classified as follows. The  $(S_t, x_{1t}, x_{2t})$  admits a unique invariant probability measure in  $\mathcal{C}_+^0$  if  $\lambda_1(\pi_0) > 0$ ,  $\lambda_2(\pi_0) > 0$ ,  $\lambda_1(\pi_{02}) > 0$ , and  $\lambda_2(\pi_{01}) > 0$ .

The two examples in low dimension ( $n = 1, 2$ ) provide a scheme to construct recursively the characterization of the longtime behavior of (5.18) in higher dimensions. It is difficult to show concretely in case of general functions  $p_i(\cdot)$ , but it is computable in certain examples. These classifications improve the results in [54, 65].

*Remark 5.* In fact, in all the examples in Sections 5.1-5.5, similar results can be obtained for multi-delays or distributed delays. We used a single delay in this Section for simplifying the notation and calculations so as to present the main ideas without notation complication. On the other hand, if  $r = 0$ , i.e., there is no time delay, the above results are consistent with and/or improve the existing results in the literature.

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