A NON-CHAINABLE PLANE CONTINUUM WITH SPAN ZERO

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Abstract. A plane continuum is constructed which has span zero but is not chainable.

1. Introduction

1.1. **Background.** The notion of the span of a continuum was introduced by Lelek in [8]. There he proved that chainable continua have span zero, and in 1971 ([9]) he asked whether the converse also holds. This is known as Lelek's problem, and has become a topic of much interest in continuum theory, in part because there are few other means presently available to decide whether a given continuum is chainable. An affirmative answer to Lelek's problem would have provided a useful tool with applications to other open problems in continuum theory; for instance, it would have completed the classification of planar homogeneous continua (see [20]).

Lelek's problem has been featured in a number of recent surveys, appearing as Problem 8 in [5], Problem 2 in [7], Problem 81 in [4], Conjecture 2 in [12], and in [15, p. 255].

There has been previous work toward finding a counterexample for Lelek's problem. Repovš et al. exhibit in [21] a sequence of trees in the plane with arbitrarily small (positive) spans, none of which has a chain cover of mesh < 1. In [1], Bartošová et al. consider generalizations of the notions of chainability and span zero to the class of Hausdorff (not necessarily metrizable) continua, and prove via a model-theoretic construction that a counterexample for Lelek's problem in that context would imply that there exists a metric counterexample.

Many positive partial results for Lelek's problem have been obtained in [13], [16], [17], and [20]. Notably, Minc proves in [13] that span zero is equivalent to chainability among those continua which are inverse limits of trees with simplicial bonding maps, and Oversteegen does the same in [16] for continua which are the image of a chainable continuum under an induced map.

A number of properties of chainable continua have been established for span zero continua. It is known that span zero continua are atriodic [8], and Oversteegen and Tymchatyn show in [19] that they are tree-like and weakly chainable. Further, Marsh proves in [11] that products of span zero continua have the fixed point property, and Bustamante et al. prove in [3] theorems about fixed point and universality properties in the hyperspace of subcontinua of a span zero continuum, generalizing corresponding theorems for chainable continua.

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In this paper, we give an example showing that in general span zero does not imply chainable, even among continua in the plane. This example also provides a negative answer to a question of Mohler (Problem 171 of [4] and Problem 7 of [10]), which asks whether every weakly chainable atriodic tree-like continuum is chainable.

The example presented here is a simple-triod-like continuum, which we will develop as a nested intersection of thickened simple triods in the plane. In Section 2, we introduce some terminology that is useful for describing these simple triods in a combinatorial way. We then show how to extract combinatorial information from a given chain cover of a graph described this way in Section 3 (see [16] for some related work). Section 4 contains the necessary combinatorial lemmas pertaining to our particular graphs, and in Section 5 we construct the example precisely and prove it has the stated properties.

1.2. **Definitions and notation.** A *continuum* is a compact connected metric space. We will always denote the metric by d.

Given a continuum X, the *span of* X is the supremum of all $\eta \geq 0$ for which there exists a subcontinuum Z of $X \times X$ such that: 1) $d(x,y) \geq \eta$ for each $(x,y) \in Z$; and 2) $\pi_1(Z) = \pi_2(Z)$, where $\pi_1, \pi_2 : X \times X \to X$ are the first and second coordinate projections, respectively.

The following facts are straightforward (see [8]):

- if X and Y are continua with $X \subseteq Y$, then $\mathrm{span}(X) \leq \mathrm{span}(Y)$;
- the arc $[0,1]$ has span zero; and
- if $\langle X_n \rangle_{n=1}^{\infty}$ is a sequence of continua in a given compact metric space, then

$$\limsup_{n\to\infty} \mathrm{span}(X_n) \leq \mathrm{span}(\limsup_{n\to\infty} X_n).$$

The third fact implies in particular that given any space X and any $\varepsilon > 0$, there is some $\delta > 0$ such that $\mathrm{span}(\overline{X_\delta}) < \mathrm{span}(X) + \varepsilon$, where X_δ denotes the δ-neighborhood of X.

A *chain cover* of a continuum X is a finite open cover $\mathcal{U} = \langle U_\ell \rangle_{\ell < L}$ which is enumerated in such a way that $U_{\ell_1} \cap U_{\ell_2} \neq \emptyset$ if and only if $|\ell_1 - \ell_2| \leq 1$. X is *chainable* if every open cover of X has a refinement which is a chain cover.

A *simple triod* is a continuum T which is the union of three arcs, A_1, A_2, A_3, which have a common endpoint o and are otherwise pairwise disjoint. A_1, A_2, A_3 are called the *legs* of T, and o is the *branch point* of T.

If $f : X \to Y$ is a function and $x_1, \ldots, x_n \in X$, we will often write

$$x_1 \cdots x_n \overset{f}{\mapsto} y_1 \cdots y_n$$

to mean $f(x_i) = y_i$ for each i.

Given a set S, a *total quasi-order on* S is a binary relation $\leq$ on S which is reflexive and transitive, and which satisfies the property that for every $s_1, s_2 \in S$, we have $s_1 \leq s_2$ or $s_2 \leq s_1$ (or both). If $\leq$ is a total quasi-order, we write $s_1 \simeq s_2$ to mean $s_1 \leq s_2$ and $s_2 \leq s_1$, and we write $s_1 < s_2$ to mean $s_1 \leq s_2$ and $s_2 \not\leq s_1$.

If S is finite and $\leq$ is a total quasi-order on S, then there is a function $f : S \to \mathbb{Z}$ which is order preserving (i.e. $f(s_1) \leq f(s_2)$ iff $s_1 \leq s_2$) whose range is a contiguous block of integers.

By a *graph*, we will mean an undirected connected graph without multiple edges or loops (i.e. edge from a vertex to itself). If G is a graph, $V(G)$ denotes the set of vertices. A pair of vertices $v_1, v_2 \in V(G)$ are *adjacent in* G provided there is an edge between them. A sequence of distinct vertices $v_1, \ldots, v_n \in V(G)$ are *consecutive in* G provided there is an edge between v_i and v_{i+1} for each $0 \leq i \leq n-1$.

A graph G will be considered as a topological space in the usual way, where the edges are realized by arcs. If $v_1, v_2 \in V(G)$ are adjacent in G, then we will use the notation $[v_1, v_2]$ to denote the arc joining v_1 and v_2.

If T is a tree and $a, b \in T$, then $[a, b]$ denotes the minimal arc $A \subseteq T$ with $a, b \in A$.

By a *word*, we will mean a finite sequence of symbols. If ω is a word, then $|\omega|$ denotes the length of ω. A word ω will be considered as a function on the set of integers $\{0, 1, \ldots, |\omega| - 1\}$. $\omega^{\leftarrow}$ denotes the reverse of ω, defined by $\omega^{\leftarrow}(j) = \omega(|\omega| - j - 1)$.

Given words ω_1, ω_2 such that the last symbol of ω_1 coincides with the first symbol of ω_2, define $\omega_1 \pitchfork \omega_2$ to be the word obtained by concatenating onto ω_1 all but the first symbol in ω_2. For example, $abc \pitchfork caba = abcaba$.

2. Graph-words

2.1. Sketches and the graph-word ρ_N.

Definition. A *graph-word in the alphabet* Γ is a pair $\rho = \langle G_\rho, w_\rho \rangle$ where G_ρ is a graph, and $w_\rho : V(G_\rho) \to \Gamma$ is a function.

Let us fix, for the rest of this paper, the alphabet $\Gamma := \{a, b, c\} \cup \{d_t : t \in [0, 1]\}$.

For each positive integer N, denote by α_N, β_N, γ_N the following three words:

$$(abc)^{2N+1}[\prod_{i=0}^{2N-1} ad_{i/2N}cd_{i/2N}a(cba)^{2N-i-1}cbc(abc)^{2N-i-1}]ad_1cd_1a(cba)^{2N+1}$$

$$(abc)^{2N+1}[\prod_{i=0}^{2N-1} ad_{i/2N}cd_{i/2N}a(cba)^{2N-i-1}cbabc(abc)^{2N-i-1}]ad_1cd_1a(cba)^{2N+1}cb$$

$$ac$$

For later use, we also define the word β_N^- to be identical to the word β_N except without the final b.

Define the graph-word ρ_N as follows. Let G_{ρ_N} be a simple triod, with vertex set $V(G_{\rho_N}) = \{o, p_1, \ldots, p_{|\alpha_N|-1}, q_1, \ldots, q_{|\beta_N|-1}, r\}$, where o is the branch point of the triod, $p_{|\alpha_N|-1}, q_{|\beta_N|-1}, r$ are the endpoints of G_{ρ_N}, the points p_j belong to the leg $[o, p_{|\alpha_N|-1}]$ with $p_j \in [o, p_{j+1}]$ for each j, and the points q_j belong to the leg $[o, q_{|\beta_N|-1}]$ with $q_j \in [o, q_{j+1}]$ for each j. Put $p_0 := o$ and $q_0 := o$. Define $w_{\rho_N} : V(G_{\rho_N}) \to \Gamma$ by $w_{\rho_N}(p_j) := \alpha_N(j)$, $w_{\rho_N}(q_j) := \beta_N(j)$, and $w_{\rho_N}(r) := \gamma_N(1) = c$.

To construct the example of a non-chainable continuum X with span zero, we will define a sequence of simple triods $\langle T_N \rangle_{N=0}^{\infty}$ such that T_N is contained in a small neighborhood of T_{N-1} for each $N > 0$; X will then be defined as the intersection of the nested sequence of neighborhoods of the triods T_N. The graph-word ρ_N will be used to describe the pattern with which we nest the simple triod T_N inside a small neighborhood of T_{N-1}. To carry this out precisely, we introduce the notion of a *sketch* below.

Remark. The space X may alternatively be described as an inverse limit of simple triods, as follows. Let T be a simple tirod with endpoints denoted as a, b, c and branch point o. Denote a point in the interior of the arc $[o,b]$ by d_0, and parameterize the arc $[d_0,b]$ by d_t for $t \in [0,1]$, so that $d_1 = b$ (as per the notion of a Γ-marking defined below). Then the N-th bonding map $b_N : T \to T$ takes o to a, is the identity on the segment $[d_0,b]$, and otherwise maps the legs $[o,a]$, $[o,b]$, $[o,c]$ in a piecewise linear way according to the patterns α_N, β_N, γ_N, respectively. Figures 1, 2, and 3, along with the proof of Proposition 1 below, provide some geometric intuition for how this looks.

Definition. Given a simple triod T with branch point o, a Γ-*marking of* T is a function $\iota : \Gamma \to T$ such that $\iota(a)$, $\iota(b)$, $\iota(c)$ are the endpoints of T and $\{\iota(d_t) : t \in [0,1]\} \subset [o,\iota(b)]$ are such that whenever $t < t'$, we have $\iota(d_t) \in [o,\iota(d_{t'})]$ and $\mathrm{diam}([\iota(d_t),\iota(d_{t'})]) = d(\iota(d_t),\iota(d_{t'})) = t' - t$.

Define the simple triod $T_0 := \{(x,0) : x \in [-1,1]\} \cup \{(0,y) : y \in [0,2]\} \subset \mathbb{R}^2$, and define a Γ-marking $\iota : \Gamma \to T_0$ by:

$$\begin{aligned}\iota(a) &:= (-1,0)\\ \iota(b) &:= (0,2)\\ \iota(c) &:= (1,0)\\ \iota(d_t) &:= (0,1+t) \text{ for } t \in [0,1].\end{aligned}$$

To simplify definitions and arguments in the following, we will restrict our attention to a special class of graph-words.

Definition. A *compliant graph-word* is a graph-word $\langle G,w\rangle$ in the alphabet Γ such that there is no pair of adjacent vertices v_1, v_2 in G with $w(v_1) \approx_\Gamma w(v_2)$.

Observe that ρ_N is a compliant graph-word for each N.

Definition. Suppose T is a simple triod with a Γ-marking $\iota : \Gamma \to T$, and let $\rho = \langle G,w\rangle$ be a compliant graph-word in the alphabet Γ. Then $\widehat{w} : G \to T$ is a ρ-*suggested bonding map* provided $\widehat{w}|_{V(G)} = \iota \circ w$, and for any adjacent $v_1, v_2 \in V(G)$, we have that $\widehat{w}|_{[v_1,v_2]}$ is a homeomorphism from $[v_1,v_2]$ to $[\iota(w(v_1)),\iota(w(v_2))]$.

Definition. Let $\langle \Omega, d\rangle$ be a metric space, let $T \subseteq \Omega$ be a Γ-marked simple triod, let $G \subseteq \Omega$ be a graph, and let $\varepsilon > 0$. Then $\rho = \langle G,w\rangle$ is a $\langle T,\varepsilon\rangle$-*sketch of* G *in* Ω if ρ is a compliant graph-word in the alphabet Γ, and there is a ρ-suggested bonding map $\widehat{w} : G \to T$ such that $d(x,\widehat{w}(x)) < \frac{\varepsilon}{2}$ for every $x \in G$.

The next proposition assures us that we may use the graph word ρ_N defined above to describe the pattern with which we embed one simple triod into a small neighborhood of another, in the plane.

We will need some additional notation when working with the graph-word ρ_N. For each $i \leq 2N$, define $n(i)$ and $m(i)$ to be the unique integers such that

$$(n(i)-1)\ n(i)\ (n(i)+1) \overset{\alpha_N}{\mapsto} d_{i/2N} c d_{i/2N}$$
$$(m(i)-1)\ m(i)\ (m(i)+1) \overset{\beta_N}{\mapsto} d_{i/2N} c d_{i/2N}.$$

For each $i < 2N$, define $\theta(i) := 6N - 3i + 1$ and $\phi(i) := 6N - 3i + 2$, so that

$$(n(i)+\theta(i)-1)\ (n(i)+\theta(i))\ (n(i)+\theta(i)+1) \overset{\alpha_N}{\mapsto} cbc$$
$$(m(i)+\phi(i)-2)\ (m(i)+\phi(i)-1) \cdots (m(i)+\phi(i)+2) \overset{\beta_N}{\mapsto} cbabc.$$

Note that $n(0) = m(0) = 6N + 5$, and that $n(i) + 2\theta(i) = n(i+1)$ and $m(i) + 2\phi(i) = m(i+1)$ for each $i < 2N$.

Proposition 1. *Suppose $T \subset \mathbb{R}^2$ is a simple triod and $\iota : \Gamma \to T$ is a Γ-marking. For each integer $N > 0$ and any $\varepsilon > 0$, there is an embedding of the simple triod graph G_{ρ_N} in $\mathbb{R}^2$ such that ρ_N is a $\langle T, \varepsilon\rangle$-sketch of G_{ρ_N} in $\mathbb{R}^2$. Moreover, the embedding can be chosen to satisfy $[q_{|\beta_N|-2}, q_{|\beta_N|-1}] = [\iota(c), \iota(b)]$.*

Observe that this proposition would be more or less immediate if we were to replace $\mathbb{R}^2$ by $\mathbb{R}^3$. Thus, the reader who is content with a non-planar counterexample for Lelek's problem may skip the details.

Proof. For simplicity, we will argue only the case $T = T_0$, with the Γ-marking ι as described above; the general case can be treated similarly.

First we will analytically define a different embedding of G_{ρ_N} in $\mathbb{R}^2$, then we will describe how to obtain the desired embedding from it.

Let $\eta > 0$ be significantly smaller than ε, say $\eta < \frac{\varepsilon}{20N^2}$. For $0 \leq i \leq 2N$, put

$$p_{n(i)} := (1 + \eta, (4i + \tfrac{3}{2})\eta),$$
$$q_{m(i)} := (1, (4i + \tfrac{3}{2})\eta).$$

For $0 \leq i < 2N$ and $1 \leq j < \theta(i)$, put

$$p_{n(i)+j} := (1 - j, (4i + 3)\eta),$$
$$p_{n(i+1)-j} := (1 - j, 4(i + 1)\eta),$$

and put $p_{n(i)+\theta(i)} := (1 - \theta(i), (4i + \frac{7}{2})\eta)$. For $0 \leq i < 2N$ and $1 \leq j < \phi(i)$, put

$$q_{m(i)+j} := (1 - j, (4i + 2)\eta),$$
$$q_{m(i+1)-j} := (1 - j, (4(i + 1) + 1)\eta),$$

and put $q_{m(i)+\phi(i)} := (1 - \phi(i), (4i + \frac{7}{2})\eta)$. Further, put

$$\begin{aligned}
p_{n(0)-j} &:= (1 - j, 0) && \text{for } 1 \leq j < 6N + 5,\\
q_{m(0)-j} &:= (1 - j, \eta) && \text{for } 1 \leq j < 6N + 5,\\
q_{m(2N)+j} &:= (1 - j, (8N + 2)\eta) && \text{for } 1 \leq j \leq 6N + 7,\\
p_{n(2N)+j} &:= (1 - j, (8N + 3)\eta) && \text{for } 1 \leq j \leq 6N + 5.
\end{aligned}$$

Finally, put $o := (-6N - 4, \frac{1}{2}\eta)$ and $r := (-6N - 5, \frac{1}{2}\eta)$. Join each pair of adjacent vertices in G_{ρ_N} by a straight line segment in $\mathbb{R}^2$. Denote the resultant embedding of G_{ρ_N} in $\mathbb{R}^2$ by G'. Figure 1 depicts the embedding G' for $N = 1$.

Observe that in G', for each integer $k \leq -1$, if v and v' are two vertices in the line $x = k$, then $w(v) = w(v')$. Also notice that each vertex v in the line $x = -1$ is already close to the point $\iota(w(v)) = \iota(a) = (-1, 0)$, and that each vertex u of the form $p_{n(i)}$ or $q_{m(i)}$ is already close to the point $\iota(w(u)) = \iota(c) = (1, 0)$. We now describe heuristically in two steps how to mold G' into the embedding we seek.

First, for each $i \leq 2N$, for each triple $\langle v_1, v_2, v_3\rangle$ of the form $\langle p_{n(i)-2}, p_{n(i)-1}, p_{n(i)}\rangle$, $\langle q_{m(i)-2}, q_{m(i)-1}, q_{m(i)}\rangle$, $\langle p_{n(i)+2}, p_{n(i)+1}, p_{n(i)}\rangle$, or $\langle q_{m(i)+2}, q_{m(i)+1}, q_{m(i)}\rangle$, move the vertex v_2 up to be close to the point $\iota(d_{i/2N})$, move the vertex v_3 down slightly, and shape the arcs joining v_1 to v_2 and v_2 to v_3 so that:

(1) there is a homeomorphism $\hat{w}_1 : [v_1, v_2] \to [\iota(a), \iota(d_{i/2N})] \subset T_0$ such that $\hat{w}_1(v_1) = \iota(a)$, $\hat{w}_1(v_2) = \iota(d_{i/2N})$, and $d(x, \hat{w}_1(x)) < \eta$ for each $x \in [v_1, v_2]$,

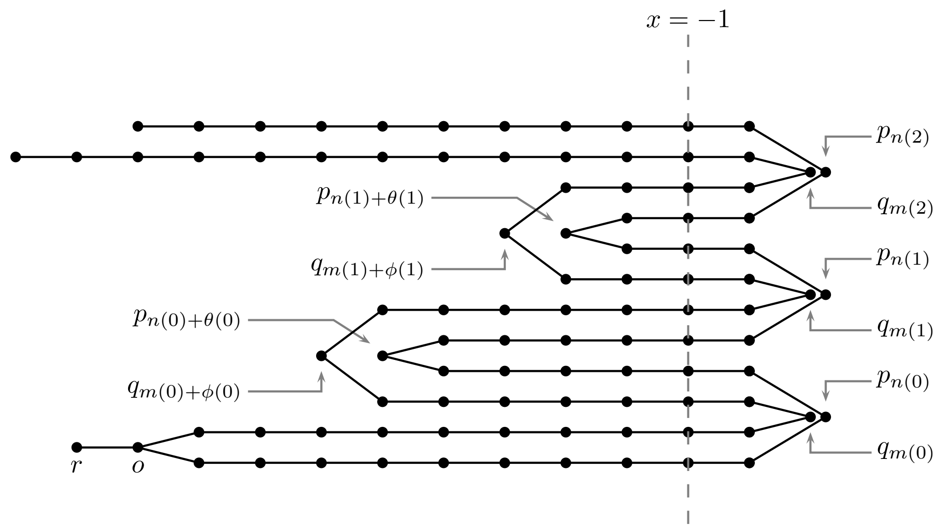


FIGURE 1. The intermediate stage G' for the embedding of G_{ρ_1} in $\mathbb{R}^2$.

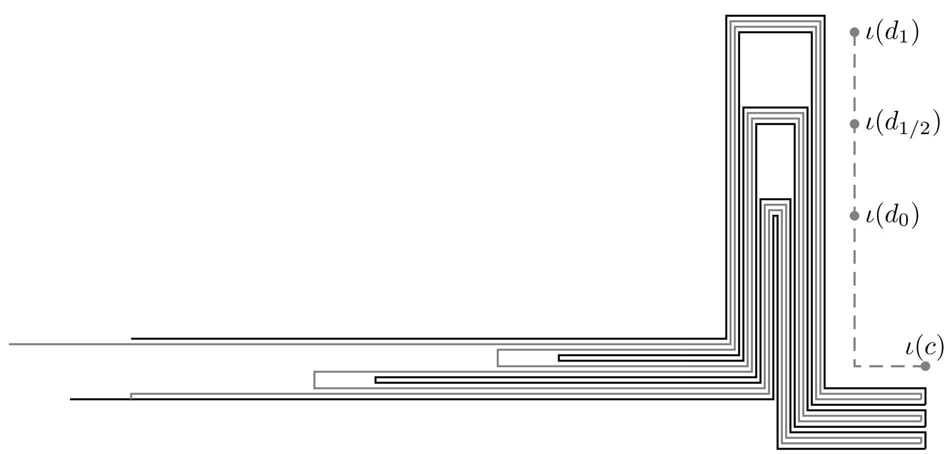
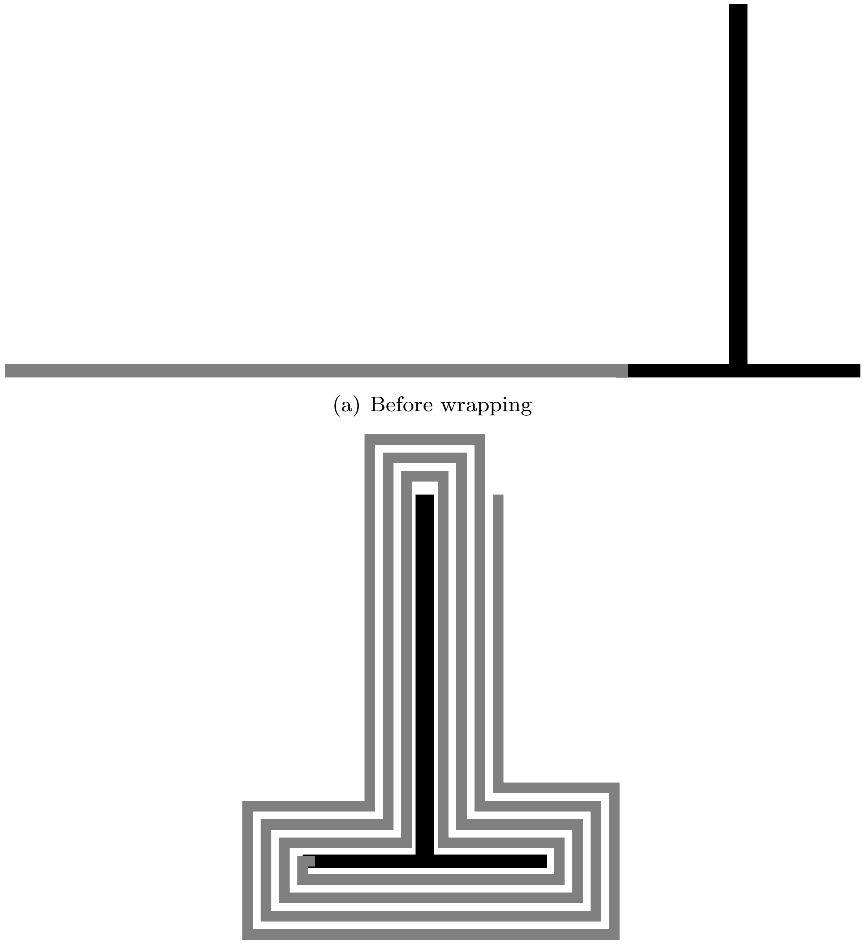


FIGURE 2. The second intermediate stage for the embedding of G_{ρ_1} in $\mathbb{R}^2$.

(2) there is a homeomorphism $\hat{w}_2 : [v_2, v_3] \to [\iota(d_{i/2N}), \iota(c)] \subset T_0$ such that $\hat{w}_2(v_2) = \iota(d_{i/2N})$, $\hat{w}_2(v_3) = \iota(c)$, and $d(x, \hat{w}_2(x)) < \eta$ for each $x \in [v_2, v_3]$, and
(3) $[v_1, v_2] \cup [v_2, v_3]$ misses the closed upper-right quandrant of the plane $\{(x, y) : x \geq 0, y \geq 0\}$,

and so that in the end no new intersections between those arcs have been introduced (i.e., so that the result is still an embedding of G_{ρ_N}). Figure 2 depicts the result for $N = 1$.



(a) Before wrapping

(b) After wrapping

FIGURE 3. Wrapping the strip counterclockwise around the simple triod to obtain the embedding of G_{ρ_N} in $\mathbb{R}^2$.

Next, take the strip $\{(x,y) : x \leq -1, 0 \leq y \leq (8N+3)\eta\}$ and stretch and wind it counter-clockwise $2N+2$ times around the outside of

$$\bigcup_{i=0}^{2N} \left([p_{n(i)-2}, p_{n(i)+2}] \cup [q_{m(i)-2}, q_{m(i)+2}]\right),$$

so that for each integer $k \leq -1$, all the vertices v in the line $x = k$ end up near the point $\iota(w(v)) \in T_0$, taking care to make sure $[q_{|\beta_N|-2}, q_{|\beta_N|-1}] = [\iota(c), \iota(b)]$. Figure 3 depicts roughly how this wrapping looks.

The resulting embedding satisfies the desired properties. □

2.2. **Span and ρ_N.** In this section we prove that the span of a simple triod described by ρ_N converges to 0 as $N \to \infty$. This ensures that we will obtain a continuum with span zero when we take the nested intersection of neighborhoods of triods described by the ρ_N's.

Lemma 2. *Let T be a simple triod with legs A_1, A_2, A_3 and branch point o. For each i let p_i be the endpoint of leg A_i other than o. Suppose $\delta > 0$ and $W \subset A_1 \times A_2$ is an arc such that $(o, o) \in W$, W meets $(\{p_1\} \times A_2) \cup (A_1 \times \{p_2\})$, and $d(x_1, x_2) \leq \delta$ for each $(x_1, x_2) \in W$. Then the span of T is $\leq \delta$.*

Proof. Suppose $Z \subset T \times T$ is a subcontinuum with $\pi_1(Z) = \pi_2(Z)$. If $\pi_1(Z)$ is an arc, then it is easy to see that Z meets the diagonal $\Delta T = \{(x, x) : x \in T\}$, as arcs have span zero.

If $\pi_1(Z)$ is a subtriod T' of T, then we may assume $T = T'$ by replacing the arc W by the component of $W \cap (T' \times T')$ that contains (o, o). Let K_1 and K_2 be disjoint clopen subsets of $(A_1 \times A_2) \smallsetminus W$ such that $(A_1 \times \{o\}) \smallsetminus W \subset K_1$, $(\{o\} \times A_2) \smallsetminus W \subset K_2$, and $K_1 \cup K_2 = (A_1 \times A_2) \smallsetminus W$.

For each $i \in \{1, 2, 3\}$ let U_i and V_i be the two components of $(A_i \times A_i) \smallsetminus \Delta T$, where $(A_i \smallsetminus \{o\}) \times \{o\} \subset U_i$ and $\{o\} \times (A_i \smallsetminus \{o\}) \subset V_i$. It can then be seen that the set

$$Y := \big(U_1 \cup U_2 \cup V_3 \cup (A_1 \times A_3) \cup (A_2 \times A_3) \cup K_1 \cup K_2^{-1}\big) \smallsetminus W$$

is clopen in $(T \times T) \smallsetminus (W \cup W^{-1} \cup \Delta T)$ (see Proposition 5.1 of [6]).

Observe that $p_3 \notin \pi_1(Y)$ and $p_3 \notin \pi_2((T \times T) \smallsetminus Y)$, hence $Z \not\subseteq Y$ and $Z \not\subseteq (T \times T) \smallsetminus Y$. Since Z is connected, it follows that Z must meet $W \cup W^{-1} \cup \Delta T$.

Thus in either case, there is some $(x_1, x_2) \in Z$ with $d(x_1, x_2) \leq \delta$. Therefore T has span $\leq \delta$. □

Proposition 3. *Suppose $T \subset \mathbb{R}^2$ is Γ-marked. If the triod graph G_{ρ_N} is embedded in $\mathbb{R}^2$ such that ρ_N is a $\langle T, \varepsilon \rangle$-sketch of G_{ρ_N} in $\mathbb{R}^2$, then the span of G_{ρ_N} is less than $\frac{1}{2N} + \varepsilon$.*

Proof. In order to apply Lemma 2, we will produce an arc $W \subset [o, p_{|\alpha_N|-1}] \times [o, q_{|\beta_N|-1}]$. Intuitively, one may think of W as a pair of points travelling simultaneously, one on the leg $[o, p_{|\alpha_N|-1}]$ and the other on $[o, q_{|\beta_N|-1}]$, starting with both at o, ending with one at the end of its leg, and at every moment staying within distance $\frac{1}{2N} + \varepsilon$ from one another. With this in mind, and referring to Figure 1, one should be easily convinced that such a W may be defined which passes through the following pairs, in order: (o, o), $(p_{n(0)}, q_{m(0)})$, $(p_{n(0)-\phi(0)}, q_{m(0)+\phi(0)})$, $(p_{n(0)}, q_{m(1)})$, $(p_{n(0)+\theta(0)}, q_{m(1)-\theta(0)})$, $(p_{n(1)}, q_{m(1)})$, $\ldots$, $(p_{n(2N)}, q_{m(2N)})$, $(p_{|\alpha_N|-1}, q_{m(2N)+6N+5})$. The precise definition of this arc W follows.

Suppose that n, n' and m, m' are two pairs of adjacent integers. Let $S^{n,n'}_{m,m'}$ denote the square $[p_n, p_{n'}] \times [q_m, q_{m'}]$. Suppose one of the following occurs:

(1) $w(p_n) = w(q_m)$, $w(p_{n'}) = w(q_{m'})$;
(2) $w(p_n) = w(q_m)$, $w(p_{n'}) = d_{i/2N}$, $w(q_{m'}) = d_{(i+1)/2N}$ for some i; or
(3) $w(p_{n'}) = w(q_{m'})$, $w(p_n) = d_{i/2N}$, $w(q_m) = d_{(i+1)/2N}$ for some i.

Then let $W^{n,n'}_{m,m'} \subset S^{n,n'}_{m,m'}$ be an arc such that $d(x_1, x_2) < \frac{1}{2N} + \varepsilon$ for each $(x_1, x_2) \in W^{n,n'}_{m,m'}$, and $W^{n,n'}_{m,m'} \cap \partial S^{n,n'}_{m,m'} = \{(p_n, q_m), (p_{n'}, q_{m'})\}$.

Define the arc $W \subset [o, p_{|\alpha_N|-1}] \times [o, q_{|\beta_N|-1}]$ as follows. It will be helpful to refer to Figure 1 when reading this formula.

$$\begin{aligned} W := \bigcup_{j=0}^{n(0)-1} W^{j,j+1}_{j,j+1} \cup \bigcup_{i=0}^{2N-1} \Bigg(& \bigcup_{j=0}^{\phi(i)-1} W^{n(i)-j,\ n(i)-j-1}_{m(i)+j,\ m(i)+j+1} \cup \\ & \bigcup_{j=0}^{\phi(i)-1} W^{n(i)-\phi(i)+j,\ n(i)-\phi(i)+j+1}_{m(i)+\phi(i)+j,\ m(i)+\phi(i)+j+1} \cup \\ & \bigcup_{j=0}^{\theta(i)-1} W^{n(i)+j,\ n(i)+j+1}_{m(i+1)-j,\ m(i+1)-j-1} \cup \\ & \bigcup_{j=0}^{\theta(i)-1} W^{n(i)+\theta(i)+j,\ n(i)+\theta(i)+j+1}_{m(i+1)-\theta(i)+j,\ m(i+1)-\theta(i)+j+1} \Bigg) \cup \\ & \bigcup_{j=0}^{6N+4} W^{n(2N)+j,\ n(2N)+j+1}_{m(2N)+j,\ m(2N)+j+1}. \end{aligned}$$

Then W contains (o, o) and meets $\{p_{|\alpha_N|-1}\} \times [o, q_{|\beta_N|-1}]$, and $d(x_1, x_2) < \frac{1}{2N} + \varepsilon$ for each $(x_1, x_2) \in W$, hence the claim follows by Lemma 2. □

3. Combinatorics from chain covers

3.1. **Chain quasi-orders.**

Definition. Define the equivalence relation $\approx_\Gamma$ on Γ by $\sigma \approx_\Gamma \tau$ if and only if $\sigma = \tau$ or $\sigma, \tau \in \{b\} \cup \{d_t : t \in [0, 1]\}$.

The relation $\approx_\Gamma$ partitions Γ into three equivalence classes. If ι is a Γ-marking of a triod T, then $\sigma \approx_\Gamma \tau$ if and only if $\iota(\sigma)$ and $\iota(\tau)$ belong to the same leg of T.

The following definition is closely related to the notion of a chain word reduction from [14]. It should be thought of as follows: if $\langle G, w\rangle$ is a $\langle T, \varepsilon\rangle$-sketch of G and we have a chain cover of G of small mesh, then $v_1 \leq v_2$ means roughly that the chain "covers v_1 before, or at around the same time as, v_2" (see Proposition 5).

Definition. Suppose $\langle G, w\rangle$ is a compliant graph-word. A *chain quasi-order of* $\langle G, w\rangle$ is a total quasi-order $\leq$ on $V(G)$ satisfying:

- **(C1)** if $v_1 \simeq v_2$, then $w(v_1) \approx_\Gamma w(v_2)$;
- **(C2)** if $v_1, v_2 \in V(G)$ are adjacent in G, then v_1 and v_2 are $\leq$-adjacent; and
- **(C3)** if $v_1, v_2, v_3 \in V(G)$ are consecutive in G, $v \in V(G)$, and if $\sigma, \tau \in \{a, c\}$ and $t, t' \in [0, 1]$ are such that $t' \geq t$, $v_1 v_2 v_3 \overset{w}{\mapsto} \sigma d_t \tau$, $w(v) = d_{t'}$, and $v_1 < v_2 \simeq v < v_3$, then $t' - t < \frac{1}{2}$.

Notice that if $\leq$ is a chain quasi-order, then the reverse order of $\leq$ (defined by $v_1 \leq^* v_2$ iff $v_2 \leq v_1$) is also a chain quasi-order.

The following simple lemma will be useful later on.

Lemma 4. *Let $\leq$ be a chain quasi-order of $\langle G, w\rangle$. Suppose $v_1, s_1, \ldots, s_\kappa, v_2$ are consecutive in G and $v \in V(G)$ is such that $v_1 < v < v_2$. Then there is some $i \in \{1, \ldots, \kappa\}$ such that $v \simeq s_i$.*

Proof. Put $s_0 := v_1$, $s_{\kappa+1} := v_2$, and let i be the largest integer in $\{0, \ldots, \kappa\}$ such that $s_i \leq v$. Then $s_{i+1} > v$, so since s_i and s_{i+1} are $\leq$-adjacent by property (C2), we must have $s_i \geq v$. Thus $s_i \simeq v$. □

3.2. Chain covers and the triod T_0.

Proposition 5. *Suppose $\langle G, w\rangle$ is a compliant graph-word which is a $\langle T_0, \varepsilon\rangle$-sketch of a graph G in $\mathbb{R}^2$. If there is a chain cover for G of mesh $< \frac{1}{2} - \varepsilon$, then there is a chain quasi-order of $\langle G, w\rangle$.*

Proof. Let $\mathcal{U} = \langle U_\ell : \ell < L\rangle$ be a chain cover for G of mesh $< \frac{1}{2} - \varepsilon$, ordered so that $U_\ell \cap U_{\ell'} \neq \emptyset$ iff $|\ell - \ell'| \leq 1$. For each $v \in V(G)$, let $\ell(v)$ be such that $v \in U_{\ell(v)}$ (for each v there are either one or two choices for $\ell(v)$).

Observe that if $v_1, v_2 \in V(G)$ and $\ell(v_1) = \ell(v_2)$, then $w(v_1) \approx_\Gamma w(v_2)$, since otherwise $d(\iota(w(v_1)), \iota(w(v_2))) \geq \sqrt{2} > \frac{1}{2}$, hence $d(v_1, v_2) > \frac{1}{2} - \varepsilon$, contradicting the fact that the diameter of $U_{\ell(v_1)} = U_{\ell(v_2)}$ is $< \frac{1}{2} - \varepsilon$.

Define the relation $\leq$ on $V(G)$ by setting $v_1 \leq v_2$ if and only if for every $v \in V(G)$ satisfying $\ell(v_2) \leq \ell(v) \leq \ell(v_1)$ we have $w(v) \approx_\Gamma w(v_1)$.

The following facts follow directly from the definition of $\leq$:

Facts.
1. If $\ell(v_1) \leq \ell(v_2)$, then $v_1 \leq v_2$.
2. If $v_1 \leq v_2$ and $w(v_1) \not\approx_\Gamma w(v_2)$, then $\ell(v_1) < \ell(v_2)$.
3. If $v_1, v_2 \in V(G)$ are $\leq$-adjacent, then $w(v_1) \not\approx_\Gamma w(v_2)$.

It is straightforward to check using the definition and these facts that $\leq$ is a total quasi-order.

We now check that $\leq$ satisfies properties (C1), (C2), and (C3) of the definition of a chain quasi-order.

(C1): Suppose $v_1, v_2 \in V(G)$ with $v_1 \simeq v_2$. Assume without loss of generality that $\ell(v_2) \leq \ell(v_1)$. It then follows immediately from the definition of $\leq$ and the assumption $v_1 \leq v_2$ that $w(v_1) \approx_\Gamma w(v_2)$ (take $v = v_2$).

(C2): Suppose $v_1, v_2 \in V(G)$ are adjacent in G. Since $\langle G, w\rangle$ is compliant, we know that $w(v_1) \not\approx_\Gamma w(v_2)$. Let $\sigma := w(v_1)$ and $\tau := w(v_2)$. Assume without loss of generality that $\ell(v_1) < \ell(v_2)$, which implies that $v_1 < v_2$.

If $v \in V(G)$ were such that $w(v) \not\approx_\Gamma \sigma, \tau$ and $v_1 < v < v_2$, then $\ell(v_1) < \ell(v) < \ell(v_2)$. This would imply that the link $U_{l(v)}$ contains the point v and meets the arc $[v_1, v_2]$. Since $\langle G, w\rangle$ is compliant, the only possible cases are:

$$\{\sigma, \tau\} = \{a, b\} \text{ and } w(v) = c$$
$$\{\sigma, \tau\} = \{a, c\} \text{ and } w(v) \in \{b\} \cup \{d_t : t \in [0, 1]\}$$
$$\{\sigma, \tau\} = \{b, c\} \text{ and } w(v) = a$$
$$\{\sigma, \tau\} = \{a, d_t\} \text{ and } w(v) = c \qquad \text{(for some } t \in [0, 1])$$
$$\{\sigma, \tau\} = \{c, d_t\} \text{ and } w(v) = a \qquad \text{(for some } t \in [0, 1])$$

In each case, we have $d(\iota(w(v)), [\iota(\sigma), \iota(\tau)]) \geq 1 > \frac{1}{2}$. But this yields a contradiction, since $\mathcal{U}$ has mesh $< \frac{1}{2} - \varepsilon$.

Suppose for a contradiction that v_1, v_2 are not adjacent in the $\leq$ order. Let v, v' be such that $v_1 < v < v'$, and v_1, v are $\leq$-adjacent and v, v' are $\leq$-adjacent. By the above, we have that $w(v), w(v')$ are each $\approx_\Gamma$ to either σ or τ, hence by

Fact (3) the only possibility is $w(v) \approx_\Gamma \tau$, $w(v') \approx_\Gamma \sigma$. Fact (2) then implies that $\ell(v_1) < \ell(v) < \ell(v')$.

Define the arc $A \subset T_0$ according to the value of σ as follows:

$$A := \begin{cases} [\iota(a), o] & \text{if } \sigma = a \\ [\iota(c), o] & \text{if } \sigma = c \\ [\iota(b), o] & \text{if } \sigma \in \{b\} \cup \{d_t : t \in [0,1]\} \end{cases}.$$

In each case, observe that $d(\iota(w(v)), A) \geq 1 > \frac{1}{2}$, and also $B_{\frac{1}{2}}(\iota(\sigma)) \subset A$ and $B_{\frac{1}{2}}(\iota(w(v'))) \subset A$.

Applying the homeomorphism $\widehat{w}|_{[v_1,v_2]}$ yields the chain cover $\langle \widehat{w}(U_\ell \cap [v_1, v_2]) : \ell' \leq \ell \leq \ell'' \rangle$ of the arc $[\iota(\sigma), \iota(\tau)]$ in T_0, where $\ell' := \min\{\ell : U_\ell \cap [v_1, v_2] \neq \emptyset\}$ and $\ell'' := \max\{\ell : U_\ell \cap [v_1, v_2] \neq \emptyset\}$.

Notice that $\widehat{w}(U_{\ell(v_1)})$ and $\widehat{w}(U_{\ell(v')})$ are sets of diameter $< \frac{1}{2}$ containing $\iota(\sigma)$ and $\iota(w(v'))$, respectively, hence are subsets of A. It follows in particular that the links $\widehat{w}(U_{\ell(v_1)} \cap [v_1, v_2])$ and $\widehat{w}(U_{\ell(v')} \cap [v_1, v_2])$ both meet the arc $A \cap [\iota(\sigma), \iota(\tau)]$, which implies each link $\widehat{w}(U_\ell \cap [v_1, v_2])$, $\ell(v_1) < \ell < \ell(v')$, must meet A as well. But $\widehat{w}(U_{\ell(v)})$ has diameter $< \frac{1}{2}$ and contains $\iota(w(v))$, hence misses A by the above. This is a contradiction, therefore we must have that v_1 and v_2 are $\leq$-adjacent.

(C3): Suppose $v \in V(G)$, v_1, v_2, v_3 are consecutive in G, and that $\sigma, \tau \in \{a, c\}$ and $t, t' \in [0,1]$ are such that $t' \geq t$, $w(v) = d_{t'}$, $v_1 v_2 v_3 \overset{w}{\mapsto} \sigma d_t \tau$, and $v_1 < v_2 \simeq v < v_3$.

From Fact (2) we know that $\ell(v)$ is between $\ell(v_1)$ and $\ell(v_3)$, hence the link $U_{\ell(v)}$ contains v and meets the arc $[v_1, v_2] \cup [v_2, v_3]$. Since $d(\iota(d_{t'}), [\iota(\sigma), \iota(d_t)] \cup [\iota(d_t), \iota(\tau)]) = d(\iota(d_{t'}), \iota(d_t)) = t' - t$ and $\mathcal{U}$ has mesh $< \frac{1}{2} - \varepsilon$, it follows that $t' - t < \frac{1}{2}$. □

4. Combinatorics of the graph-word ρ_N

4.1. Chain quasi-orders and ρ_N. Throughout this subsection assume that $\langle G, w \rangle$ is a compliant graph-word, and that $\leq$ is a chain quasi-order of $\langle G, w \rangle$.

Let $f : V(G) \to \mathbb{Z}$ be an order preserving function whose range is a contiguous block of integers.

Lemma 6. *Suppose $v_1, \ldots, v_n$ are consecutive in G, and that for each $1 < j < n$ we have $w(v_{j-1}) \not\approx_\Gamma w(v_{j+1})$. Then $f(v_1), \ldots f(v_n)$ are consecutive integers, i.e. either $f(v_{j+1}) = f(v_j) + 1$ for each $1 \leq j < n$, or $f(v_{j+1}) = f(v_j) - 1$ for each $1 \leq j < n$.*

Proof. This follows immediately from properties (C1) and (C2) of the chain quasi-order $\leq$. □

As an application of Lemma 6, we make the following observation.

Lemma 7. *Suppose for some $i < 2N$ that $v_0, v_1, \ldots, v_{2\theta(i)} \in V(G)$ are consecutive in G with $v_0 \cdots v_{2\theta(i)} \overset{w}{\mapsto} \alpha_N(n(i)) \cdots \alpha_N(n(i+1))$. Let $k := f(v_0)$. Then we have one of the following four cases:*

(1) *$v_0 \cdots v_{2\theta(i)} \overset{f}{\mapsto} k \cdots (k + 2\theta(i))$;*
(2) *$v_0 \cdots v_{\theta(i)} \overset{f}{\mapsto} k \cdots (k + \theta(i))$, $v_{\theta(i)} \cdots v_{2\theta(i)} \overset{f}{\mapsto} (k + \theta(i)) \cdots k$;*
(3) *$v_0 \cdots v_{\theta(i)} \overset{f}{\mapsto} k \cdots (k - \theta(i))$, $v_{\theta(i)} \cdots v_{2\theta(i)} \overset{f}{\mapsto} (k - \theta(i)) \cdots k$; or*

(4) $v_0 \cdots v_{2\theta(i)} \overset{f}{\mapsto} k \cdots (k - 2\theta(i))$.

Moreover, the analogous statement holds for the word β_N^- (where we replace n with m and θ with ϕ).

Proof. This is a simple consequence of Lemma 6. □

Lemma 8. *Suppose that for each $i \in \{0, N, 2N\}$, there are $v_1^{(i)}, v_2^{(i)}, v_3^{(i)} \in V(G)$ which are consecutive in G with $v_1^{(i)} v_2^{(i)} v_3^{(i)} \overset{w}{\mapsto} ad_{i/2N}c$. Then it cannot be the case that $v_3^{(0)} \simeq v_3^{(N)} \simeq v_3^{(2N)}$.*

Proof. Suppose for a contradiction that $f(v_3^{(0)}) = f(v_3^{(N)}) = f(v_3^{(2N)}) = k$. By Lemma 6, for each $i \in \{0, N, 2N\}$ we have either

$$v_1^{(i)} v_2^{(i)} v_3^{(i)} \overset{f}{\mapsto} (k-2)\,(k-1)\,k$$

or

$$v_1^{(i)} v_2^{(i)} v_3^{(i)} \overset{f}{\mapsto} (k+2)\,(k+1)\,k.$$

It then follows from the pidgeonhole principle that $f(v_2^{(i)}) = f(v_2^{(j)})$ for distinct $i, j \in \{0, N, 2N\}$. But this contradicts property (C3) of the chain quasi-order $\leq$. □

Lemma 9. *Suppose $v_0, \ldots, v_{|\alpha_N|-1} \in V(G)$ are consecutive in G and $v'_0, \ldots, v'_{|\beta_N|-2} \in V(G)$ are consecutive in G with $v_0 \cdots v_{|\alpha_N|-1} \overset{w}{\mapsto} \alpha_N$ and $v'_0 \cdots v'_{|\beta_N|-2} \overset{w}{\mapsto} \beta_N^-$. Suppose further that $v_0 \simeq v'_0$. Then $v_1 \not\simeq v'_1$.*

Proof. Assume without loss of generality that $v_0 \leq v_1$. Suppose for a contradiction that $v_1 \simeq v'_1$.

We know that $f(v_0) \leq f(v_1)$ and that $f(v_0) = f(v'_0)$, $f(v_1) = f(v'_1)$. Put $k := f(v_{n(0)})$, and recall that $n(0) = 6N+5 = m(0)$. It follows from Lemma 6 that

$$v_0 \cdots v_{n(0)} \overset{f}{\mapsto} (k - 6N - 5) \cdots k, \text{ and}$$

$$v'_0 \cdots v'_{m(0)} \overset{f}{\mapsto} (k - 6N - 5) \cdots k.$$

Claim 9.1. Let $i < 2N$. If $f(v_{n(i)}) = k$ and $f(v_{n(i)+\theta(i)}) < k$, then $f(v_{n(i+1)}) = k$. Similarly, if $f(v'_{m(i)}) = k$ and $f(v'_{m(i)+\phi(i)}) < k$, then $f(v'_{m(i+1)}) = k$.

Proof of Claim 9.1. Suppose $f(v_{n(i)}) = k > f(v_{n(i)+\theta(i)})$. If

$$v_{n(i)} \cdots v_{n(i+1)} \overset{f}{\mapsto} k \cdots (k - 2\theta(i)),$$

then in particular $f(v_{n(i)+\theta(i)+1}) = k - \theta(i) - 1$. Also, $f(v_{n(0)-\theta(i)-1}) = k - \theta(i) - 1$. But $w(v_{n(i)+\theta(i)+1}) = c \not\approx_\Gamma a = w(v_{n(0)-\theta(i)-1})$, so this contradicts property (C1) of the chain quasi-order $\leq$. Therefore by Lemma 7 we must have $f(v_{n(i+1)}) = k$.

Similarly, suppose $f(v'_{m(i)}) = k > f(v'_{m(i)+\phi(i)})$. If

$$v'_{m(i)} \cdots v'_{m(i+1)} \overset{f}{\mapsto} k \cdots (k - 2\phi(i)),$$

then in particular $f(v'_{m(i)+\phi(i)+1}) = k - \phi(i) - 1$. Also, $f(v'_{m(0)-\phi(i)-1}) = k - \phi(i) - 1$. But $w(v_{m(i)'+\phi(i)+1}) = b \not\approx_\Gamma c = w(v'_{m(0)-\phi(i)-1})$, so this contradicts property (C1) of the chain quasi-order $\leq$. Therefore by Lemma 7 we must have $f(v'_{m(i+1)}) = k$. □(Claim 9.1)

Claim 9.2. Either $f(v_{n(i)}) = k$ for each $i \leq 2N$ or $f(v'_{m(i)}) = k$ for each $i \leq 2N$.

Proof of Claim 9.2. If $f(v_{n(i)+\theta(i)}) < k$ and $f(v'_{m(i)+\phi(i)}) < k$ for each $i < 2N$, then this follows immediately from Claim 9.1 and induction. Hence assume this is not the case, and let i^* be the smallest i for which $f(v_{n(i)+\theta(i)}) > k$ or $f(v'_{m(i)+\phi(i)}) > k$.

Observe that by Claim 9.1 and induction, we have $f(v_{n(i)}) = f(v'_{m(i)}) = k$ for each $i \leq i^*$.

Case 1. $f(v'_{m(i^*)+\phi(i^*)}) > k$.

It follows from Lemma 6 that

$$v'_{m(i^*)} \cdots v'_{m(i^*)+\phi(i')} \overset{f}{\mapsto} k \cdots (k + \phi(i^*)).$$

Suppose $i^* \leq i < 2N$, and that $f(v_{n(i)}) = k$. If $f(v_{n(i)+\theta(i)}) < k$, then we have by Claim 9.1 that $f(v_{n(i+1)}) = k$.

If $f(v_{n(i)+\theta(i)}) > k$, suppose for a contradiction that

$$v_{n(i)} \cdots v_{n(i+1)} \overset{f}{\mapsto} k \cdots (k + 2\theta(i)).$$

In particular, this means $f(v_{n(i)+\theta(i)+1}) = k + \theta(i) + 1$. Also, since $\phi(i^*) > \theta(i)$, we have $f(v'_{m(i^*)+\theta(i)+1}) = k + \theta(i) + 1$. But $w(v_{n(i)+\theta(i)+1}) = c \not\approx_\Gamma a = w(v'_{m(i^*)+\theta(i)+1})$, so this contradicts property (C1) of the chain quasi-order $\leq$. Therefore by Lemma 7 we must have $f(v_{n(i+1)}) = k$.

Thus by induction, we have $f(v_{n(i)}) = k$ for each $i \leq 2N$.

Case 2. $f(v'_{m(i^*)+\phi(i^*)}) < k$ and $f(v_{n(i^*)+\theta(i^*)}) > k$.

Here we have by Claim 9.1 that $f(v'_{m(i^*+1)}) = k$.

It follows from Lemma 6 that

$$v_{n(i^*)} \cdots v_{n(i^*)+\theta(i^*)} \overset{f}{\mapsto} k \cdots (k + \theta(i^*)).$$

Suppose $i^* + 1 \leq i < 2N$, and that $f(v'_{m(i)}) = k$. If $f(v'_{m(i)+\phi(i)}) < k$, then we have by Claim 9.1 that $f(v'_{m(i+1)}) = k$.

If $f(v'_{m(i)+\phi(i)}) > k$, suppose for a contradiction that

$$v'_{m(i)} \cdots v'_{m(i+1)} \overset{f}{\mapsto} k \cdots (k + 2\phi(i)).$$

In particular, this means $f(v'_{m(i)+\phi(i)+1}) = k + \phi(i) + 1$. Also, since $\theta(i^*) > \phi(i)$, we have $f(v_{n(i^*)+\phi(i)+1}) = k + \phi(i) + 1$. But $w(v'_{m(i)+\phi(i)+1}) = b \not\approx_\Gamma c = w(v_{n(i^*)+\phi(i)+1})$, so this contradicts property (C1) of the chain quasi-order $\leq$. Therefore by Lemma 7 we must have $f(v'_{m(i+1)}) = k$.

Thus by induction, we have $f(v'_{m(i)}) = k$ for each $i \leq 2N$. □(Claim 9.2)

It remains only to notice that Claim 9.2 contradicts Lemma 8. So we must have $v_1 \not\approx v'_1$. □

For convenience in later statements and arguments, we will use the following notation:

Definition. Given $\sigma \in \Gamma$, define the word $\zeta_N(\sigma)$ by

$$\zeta_N(\sigma) := \begin{cases} \alpha_N & \text{if } \sigma = a \\ \beta_N & \text{if } \sigma = b \\ \gamma_N & \text{if } \sigma = c \\ \beta_N^- d_t & \text{if } \sigma = d_t \text{ (for some } t \in [0,1]). \end{cases}$$

Lemma 10. *Suppose $\sigma, \tau \in \Gamma$, $v_0, \ldots, v_\kappa \in V(G)$ are consecutive in G, and $v_0', \ldots, v_\lambda' \in V(G)$ are consecutive in G, with*

$$v_0 \cdots v_\kappa \overset{w}{\mapsto} \zeta_N(\sigma) \quad \textit{and} \quad v_0' \cdots v_\lambda' \overset{w}{\mapsto} \zeta_N(\tau).$$

Suppose further that $v_0 \simeq v_0'$ and $v_1 \simeq v_1'$. Then $\sigma \approx_\Gamma \tau$.

Proof. Suppose for a contradiction that $\sigma \not\approx_\Gamma \tau$. If $\sigma = a$ and $\tau \in \{b, d_t : t \in [0,1]\}$, or vice versa, then this contradicts Lemma 9. If one of them is c, say σ, then $w(v_1) = c$ while $w(v_1') = b \not\approx_\Gamma c$, so this contradicts property (C1) of the chain quasi-order $\leq$. □

Proposition 11. *There is no chain quasi-order for ρ_N, for any N.*

Proof. Suppose for a contradiction that $\leq$ is a chain quasi-order for ρ_N. Observe that since $r, p_1, q_1 \in V(G_{\rho_N})$ are all adjacent to o in G_{ρ_N}, we have that these three vertices are also adjacent to o in the $\leq$ order. Hence by the pidgeonhole principle, some pair of them are $\simeq$. But this is a contradiction by Lemma 10. □

Oversteegen & Tymchatyn exhibit in [17] for each $\delta > 0$ a 2-dimensional plane strip with span $< \delta$ which has no chain cover of mesh < 1. Repovš et al. modify this example in [21] to construct for each $\delta > 0$ a tree in the plane with span $< \delta$ which has no chain cover of mesh < 1. In both examples, the diameters of the continua converge to ∞ as $\delta \to 0$. We pause to point out that we have now obtained a bounded family of such examples.

Corollary 12. *There is a uniformly bounded sequence $\langle T_N \rangle_{N=1}^\infty$ of simple triods in $\mathbb{R}^2$ such that for each N, $\mathrm{span}(T_N) < \frac{1}{N}$ and T_N has no chain cover of mesh $< \frac{1}{4}$.*

Proof. This is simply a combination of Propositions 1 (using T_0 and taking $\varepsilon \leq \frac{1}{2N}$), 3, 5, and 11. □

We are working to prove a stronger result: that there is a continuum in $\mathbb{R}^2$ which has span zero and cannot be covered by a chain of mesh less than some positive constant. To this end we will need some further technical combinatorial lemmas.

Lemma 13. *Suppose $\sigma, \tau \in \Gamma$ with $\sigma \approx_\Gamma \tau$, and that $v_0, \ldots, v_\kappa \in V(G)$ are consecutive in G and $v_0', \ldots, v_\kappa' \in V(G)$ are consecutive in G with*

$$v_0 \cdots v_\kappa \overset{w}{\mapsto} \zeta_N(\sigma) \quad \textit{and} \quad v_0' \cdots v_\kappa' \overset{w}{\mapsto} \zeta_N(\tau).$$

Then:

(i) *if $v_0 < v_1$, then $v_0 < v_j < v_\kappa$ for each $0 < j < \kappa$;*
(ii) *if $v_{\kappa-1} < v_\kappa$, then $v_0 < v_j < v_\kappa$ for each $0 < j < \kappa$;*
(iii) *if $v_0 \simeq v_0'$ and $v_1 \simeq v_1'$, then $v_\kappa \simeq v_\kappa'$; and*
(iv) *if $v_\kappa \simeq v_\kappa'$ and $v_{\kappa-1} \simeq v_{\kappa-1}'$, then $v_0 \simeq v_0'$.*

Proof. Each of these statements is trivial if $\sigma = \tau = c$. We will prove the Lemma for $\sigma = \tau = a$; the case $\sigma \approx_\Gamma \tau \approx_\Gamma b$ proceeds analogously.

(i) Suppose $v_0 < v_1$.

Claim 13.1. $v_0 \cdots v_{n(0)} \overset{f}{\mapsto} (f(v_{n(0)}) - 6N - 5) \cdots f(v_{n(0)})$.

Proof of Claim 13.1. This is immediate from Lemma 6. □(Claim 13.1)

Claim 13.2. For each $i < 2N$, $v_{n(i)} \leq v_{n(i+1)}$.

Proof of Claim 13.2. We proceed by induction on $i < 2N$. Suppose the claim is true for each i' with $i' < i$. Put $k := f(v_{n(i)})$. Suppose for a contradiction that $f(v_{n(i)}) > f(v_{n(i+1)})$. By Lemma 7, this means

$$v_{n(i)} \cdots v_{n(i+1)} \overset{f}{\mapsto} k \cdots (k - 2\theta(i)).$$

In particular, we have $f(v_{n(i)+\theta(i)+1}) = k - \theta(i) - 1$.

Let j^* be the smallest $j \leq i$ such that $f(v_{n(j)}) = k$.

If $j^* = 0$, then since $n(0) > \theta(i)$, we have $f(v_{n(0)-\theta(i)-1}) = k - \theta(i) - 1$. But also $w(v_{n(i)+\theta(i)+1}) = c \not\approx_\Gamma a = w(v_{n(0)-\theta(i)-1})$, so this contradicts property (C1) of the chain quasi-order $\leq$.

If $j^* > 0$, then we know by Lemma 7 that

$$v_{n(j^*-1)} \cdots v_{n(j^*)} \overset{f}{\mapsto} (k - 2\theta(j^* - 1)) \cdots k.$$

Then similarly observe that since $\theta(j^*-1) > \theta(i)$, we have $f(v_{n(j^*)-\theta(i)-1}) = k - \theta(i) - 1$. But also $w(v_{n(i)+\theta(i)+1}) = c \not\approx_\Gamma a = w(v_{n(j^*)-\theta(i)-1})$, so this contradicts property (C1) of the chain quasi-order $\leq$. □(Claim 13.2)

Claim 13.3. $v_{n(2N)} \cdots v_\kappa \overset{f}{\mapsto} f(v_{n(2N)}) \cdots (f(v_{n(2N)}) + 6N + 5)$.

Proof of Claim 13.3. By Lemma 8 and Claim 13.2, we must have $v_{n(i-1)} < v_{n(i)}$ for some $0 < i \leq 2N$; let i^* be the largest such i, so that $f(v_{n(2N)}) = f(v_{n(i^*)})$.

Suppose for a contradiction that

$$v_{n(2N)} \cdots v_\kappa \overset{f}{\mapsto} f(v_{n(2N)}) \cdots (f(v_{n(2N)}) - 6N - 5).$$

Then in particular, since $6N+5 > \theta(i^*-1)$, we have $f(v_{n(2N)+\theta(i^*-1)+1}) = f(v_{n(2N)}) - \theta(i^* - 1) - 1$. But also $f(v_{n(i^*)-\theta(i^*-1)-1}) = f(v_{n(2N)}) - \theta(i^* - 1) - 1$ and $w(v_{n(i^*)-\theta(i^*-1)-1}) = c \not\approx_\Gamma a = w(v_{n(2N)+\theta(i^*-1)+1})$, so this contradicts property (C1) of the chain quasi-order $\leq$. Therefore by Lemma 6, we must have

$$v_{n(2N)} \cdots v_\kappa \overset{f}{\mapsto} f(v_{n(2N)}) \cdots (f(v_{n(2N)}) + 6N + 5).$$

□(Claim 13.3)

It is now easy to check that $f(v_0) = f(v_{n(0)}) - 6N - 5 < f(v_j) < f(v_{n(2N)}) + 6N + 5 = f(v_\kappa)$ for any $0 < j < \kappa$.

(ii) Observe that if we consider the reverse order of $\leq$, part (i) gives that if $v_0 > v_1$, then $v_0 > v_j > v_\kappa$ for each $0 < j < \kappa$. In particular, this would mean $v_{\kappa-1} > v_\kappa$. Therefore if $v_{\kappa-1} < v_\kappa$ then $v_0 < v_1$, hence the conclusion follows from part (i).

(iii) Suppose $v_0 \simeq v_0'$, $v_1 \simeq v_1'$, and assume without loss of generality that $v_0 < v_1$. This means Claims 13.1, 13.2, and 13.3 hold for the v_j's and the v_j''s. By Claim 13.1, we have

$$v_0 \cdots v_{n(0)} \stackrel{f}{\mapsto} (f(v_{n(0)}) - 6N - 5) \cdots f(v_{n(0)})$$

and

$$v_0' \cdots v_{n(0)}' \stackrel{f}{\mapsto} (f(v_{n(0)}) - 6N - 5) \cdots f(v_{n(0)}).$$

Claim 13.4. For each $i \leq 2N$, $v_{n(i)} \simeq v_{n(i)}'$.

Proof of Claim 13.4. Suppose not, and let i^* be the smallest $i < 2N$ such that $v_{n(i+1)} \not\simeq v_{n(i+1)}'$. Put $k := f(v_{n(i^*)}) = f(v_{n(i^*)}')$. It follows from Lemma 7 and Claim 13.2 that either $f(v_{n(i^*+1)}) = k$ and $f(v_{n(i^*+1)}') > k$, or $f(v_{n(i^*+1)}) > k$ and $f(v_{n(i^*+1)}') = k$; assume the former. This implies by Lemma 7 that

$$v_{n(i^*)}' \cdots v_{n(i^*+1)}' \stackrel{f}{\mapsto} k \cdots (k + 2\theta(i^*)).$$

We claim that $f(v_{n(i)}) = k$ for each $i \geq i^*$. Indeed, given $i > i^*$, suppose for a contradiction that

$$v_{n(i)} \cdots v_{n(i+1)} \stackrel{f}{\mapsto} k \cdots (k + 2\theta(i)).$$

This means in particular that $f(v_{n(i)+\theta(i)+1}) = k + \theta(i) + 1$. Since $\theta(i) < \theta(i^*)$, we have $f(v_{n(i^*)+\theta(i)+1}') = k + \theta(i) + 1$. But $w(v_{n(i)+\theta(i)+1}) = c \not\approx_\Gamma a = w(v_{n(i^*)+\theta(i)+1}')$, so this contradicts property (C1) of the chain quasi-order $\leq$. Therefore by Lemma 7 and Claim 13.2, we must have $f(v_{n(i+1)}) = k$. Hence, by induction, $f(v_{n(i)}) = k$ for each $i \geq i^*$.

In particular, $f(v_{n(2N)}) = k$. By Claim 13.3, we have

$$v_{n(2N)} \cdots v_\kappa \stackrel{f}{\mapsto} k \cdots (k + 6N + 5).$$

Since $6N + 5 > \theta(i^*)$, this means that $f(v_{n(2N)+\theta(i^*)+1}) = k + \theta(i^*) + 1$. Note $f(v_{n(i^*)+\theta(i^*)+1}') = k + \theta(i^*) + 1$ as well. But $w(v_{n(i^*)+\theta(i^*)+1}') = c \not\approx_\Gamma a = w(v_{n(2N)+\theta(i^*)+1})$, so this contradicts property (C1) of the chain quasi-order $\leq$. □(Claim 13.4)

Claim 13.4 implies in particular that $f(v_{n(2N)}) = f(v_{n(2N)}')$. Then by Claim 13.3, we have

$$v_{n(2N)} \cdots v_\kappa \stackrel{f}{\mapsto} f(v_{n(2N)}) \cdots (f(v_{n(2N)}) + 6N + 5),$$

and

$$v_{n(2N)}' \cdots v_\kappa' \stackrel{f}{\mapsto} f(v_{n(2N)}) \cdots (f(v_{n(2N)}) + 6N + 5).$$

This establishes part (iii).

(iv) Suppose $v_\kappa \simeq v_\kappa'$, $v_{\kappa-1} \simeq v_{\kappa-1}'$, and assume without loss of generality that $v_{\kappa-1} < v_\kappa$. By part (ii) this implies $v_0 < v_1$ and $v_0' < v_1'$, so again Claims 13.1, 13.2, and 13.3 hold for the v_j's and the v_j''s. By Claim 13.3, we have

$$v_\kappa \cdots v_{n(2N)} \stackrel{f}{\mapsto} (f(v_{n(2N)}) + 6N + 5) \cdots f(v_{n(2N)})$$

and

$$v_\kappa' \cdots v_{n(2N)}' \stackrel{f}{\mapsto} (f(v_{n(2N)}) + 6N + 5) \cdots f(v_{n(2N)}).$$

Claim 13.5. For each $i \leq 2N$, $v_{n(i)} \simeq v'_{n(i)}$.

Proof of Claim 13.5. Suppose not, and let i^* be the largest $i < 2N$ such that $v_{n(i)} \not\simeq v'_{n(i)}$. Put $k := f(v_{n(i^*+1)}) = f(v'_{n(i^*+1)})$. It follows from Lemma 7 and Claim 13.2 that either $f(v_{n(i^*)}) = k$ and $f(v'_{n(i^*)}) < k$, or $f(v_{n(i^*)}) < k$ and $f(v'_{n(i^*)}) = k$; assume the former. This implies by Lemma 7 that

$$v'_{n(i^*+1)} \cdots v'_{n(i^*)} \stackrel{f}{\mapsto} k \cdots (k - 2\theta(i^*)).$$

We claim that $f(v_{n(i)}) = k$ for each $i \leq i^*$. Indeed, given $i < i^*$, suppose for a contradiction that

$$v_{n(i+1)} \cdots v_{n(i)} \stackrel{f}{\mapsto} k \cdots (k - 2\theta(i)).$$

Since $\theta(i^*) < \theta(i)$, this means in particular that $f(v_{n(i+1)-\theta(i^*)-1}) = k - \theta(i^*) - 1$. Note $f(v'_{n(i^*+1)-\theta(i^*)-1}) = k - \theta(i^*) - 1$ as well. But $w(v_{n(i^*+1)-\theta(i^*)-1}) = c \not\approx_\Gamma a = w(v'_{n(i+1)-\theta(i^*)-1})$, so this contradicts property (C1) of the chain quasi-order $\leq$. Therefore by Lemma 7 and Claim 13.2 we must have $f(v_{n(i)}) = k$. Hence, by induction, $f(v_{n(i)}) = k$ for each $i \leq i^*$.

In particular, $f(v_{n(0)}) = k$. By Claim 13.1, we have

$$v_{n(0)} \cdots v_0 \stackrel{f}{\mapsto} k \cdots (k - 6N - 5).$$

Since $6N + 5 > \theta(i^*)$, this means that $f(v_{n(0)-\theta(i^*)-1}) = k - \theta(i^*) - 1$. Note $f(v'_{n(i^*+1)-\theta(i^*)-1}) = k - \theta(i^*) - 1$ as well. But $w(v'_{n(i^*+1)-\theta(i^*)-1}) = c \not\approx_\Gamma a = w(v_{n(0)-\theta(i^*)-1})$, so this contradicts property (C1) of the chain quasi-order $\leq$. □(Claim 13.5)

Claim 13.5 implies in particular that $f(v_{n(0)}) = f(v'_{n(0)})$. Then by Claim 13.1, we have

$$v_{n(0)} \cdots v_0 \stackrel{f}{\mapsto} f(v_{n(0)}) \cdots (f(v_{n(0)}) - 6N - 5)$$

and

$$v'_{n(0)} \cdots v'_0 \stackrel{f}{\mapsto} f(v_{n(0)}) \cdots (f(v_{n(0)}) - 6N - 5).$$

This establishes part (iv).

□

4.2. **Iterated sketches.** If $\iota_T : \Gamma \to T$ is a Γ-marking of the simple triod T and ρ_N is a $\langle T, \varepsilon \rangle$-sketch of the simple triod graph $T' := G_{\rho_N}$ such that $[q_{|\beta_N|-2}, q_{|\beta_N|-1}] = [\iota_T(c), \iota_T(b)]$ (as in Proposition 1), then one can define an induced Γ-marking $\iota_{T'} : \Gamma \to T'$ on T' as follows: define $\iota_{T'}(a) := p_{|\alpha_N|-1}$, $\iota_{T'}(b) := q_{|\beta_N|-1} = \iota_T(b)$, $\iota_{T'}(c) := r$, and for each $t \in [0,1]$ put $\iota_{T'}(d_t) := \iota_T(d_t) \in [q_{|\beta_N|-2}, q_{|\beta_N|-1}] = [\iota_T(c), \iota_T(b)]$.

Now let T_0 be as before, and suppose T_1 and T_2 are simple triods such that ρ_1 is a $\langle T_0, \varepsilon_0 \rangle$-sketch of T_1, and ρ_2 is a $\langle T_1, \varepsilon_1 \rangle$-sketch of T_2 (using the induced Γ-marking on T_1). Evidently we should be able to find a $\langle T_0, \varepsilon_0 + \varepsilon_1 \rangle$-sketch of T_2, and indeed this is necessary if we want to apply Proposition 5 to argue that T_2 has no chain cover of small mesh. This is the motivation for the next definition (see Proposition 14).

Definition. Suppose $\langle G, w\rangle$ is a compliant graph-word, and $N > 0$. A graph-word $\langle G^+, w^+\rangle$ is a *ρ_N-expansion of* $\langle G, w\rangle$ if:

- G^+ is identical to G as a topological space, but the vertex set of G^+ is finer: for any adjacent pair of vertices $v_1, v_2 \in V(G)$, there are distinct degree 2 vertices $s_j^{v_1v_2}$, $j = 1, \ldots, \kappa_{v_1v_2}$ where $\kappa_{v_1v_2} = |\zeta_N(w(v_1))| + |\zeta_N(w(v_2))| - 3$, inserted into the edge joining v_1, v_2 so that $v_1, s_1^{v_1v_2}, \ldots, s_{\kappa_{v_1v_2}}^{v_1v_2}, v_2$ are consecutive in G^+; and
- w^+ is defined by
$$v_1 s_1^{v_1v_2} \cdots s_{\kappa_{v_1v_2}}^{v_1v_2} v_2 \overset{w^+}{\mapsto} \zeta_N(w(v_1))^{\leftarrow} \pitchfork \zeta_N(w(v_2))$$
when $v_1, v_2 \in V(G)$ are adjacent in G.

Remarks. (1) Notice that $w^+|_{V(G)} = w$, and that $\langle G^+, w^+\rangle$ is also a compliant graph-word.

(2) Combinatorially, there is only one ρ_N expansion of a given graph-word $\langle G, w\rangle$; however, geometrically they may differ according to where along the edges of G the extra vertices are inserted (though their order on the edge is determined uniquely by the definition).

Proposition 14. *Suppose T is a Γ-marked simple triod, and ρ_N is a $\langle T, \varepsilon_1\rangle$-sketch of $T' := G_{\rho_N}$. Endow T' with a Γ-marking as above. If $\rho = \langle G, w\rangle$ is a compliant graph-word which is a $\langle T', \varepsilon_2\rangle$-sketch of G, then there is a ρ_N-expansion of $\langle G, w\rangle$ which is a $\langle T, \varepsilon_1 + \varepsilon_2\rangle$-sketch of G.*

Proof. Let $\widehat{w_{\rho_N}} : T' \to T$ be a ρ_N-suggested bonding map such that $d(x, \widehat{w_{\rho_N}}(x)) < \frac{\varepsilon_1}{2}$ for each $x \in T'$, and let $\widehat{w} : G \to T'$ be ρ-suggested bonding map such that $d(x, \widehat{w}(x)) < \frac{\varepsilon_2}{2}$ for each $x \in G$.

Consider any adjacent $v_1, v_2 \in V(G)$. Define
$$s_i^{v_1v_2} := \begin{cases} \widehat{w}^{-1}(p_{|\alpha_N|-1-i}) & \text{if } w(v_1) = a \\ \widehat{w}^{-1}(q_{|\beta_N|-1-i}) & \text{if } w(v_1) \approx_\Gamma b \end{cases}$$
for $1 \leq i \leq |\zeta_N(w(v_1))|$, and
$$s_{\kappa_{v_1v_2}-i}^{v_1v_2} := \begin{cases} \widehat{w}^{-1}(p_{|\alpha_N|-1-i}) & \text{if } w(v_2) = a \\ \widehat{w}^{-1}(q_{|\beta_N|-1-i}) & \text{if } w(v_2) \approx_\Gamma b \end{cases}$$
for $1 \leq i \leq |\zeta_N(w(v_2))|$.

Let $V(G^+)$ be equal to $V(G)$ together with all these new vertices, and let w^+ be defined as in the definition of a ρ_N-expansion. Observe that $w^+ = w_{\rho_N} \circ (\widehat{w}|_{V(G^+)})$. Put $\rho^+ := \langle G^+, w^+\rangle$, where G^+ is equal to G as a topological space, with vertex set $V(G^+)$.

It is now straightforward to see that $\widehat{w_{\rho_N}} \circ \widehat{w}$ is a ρ^+-suggested bonding map, and clearly $d(x, (\widehat{w_{\rho_N}} \circ \widehat{w})(x)) < \frac{\varepsilon_1+\varepsilon_2}{2}$ for each $x \in G$. □

Lemma 15. *Suppose $\langle G, w\rangle$ is a compliant graph-word, let $\langle G^+, w^+\rangle$ be a ρ_N-expansion of $\langle G, w\rangle$, and suppose $\leq^+$ is a chain quasi-order of $\langle G^+, w^+\rangle$.*

(i) *Let $v_1, v_2 \in V(G)$ be adjacent in G, and let $s_1, \ldots, s_\kappa \in V(G^+) \setminus V(G)$ be such that $v_1, s_1, \ldots, s_\kappa, v_2$ are consecutive in G^+. Then the following are equivalent:*

(1) *$v_1 <^+ v_2$;*

(2) $v_1 <^+ s_j <^+ v_2$ *for each* $j \in \{1, \ldots, \kappa\}$;
(3) $v_1 <^+ s_j <^+ v_2$ *for some* $j \in \{1, \ldots, \kappa\}$.

(ii) *If* $v_1, v_2 \in V(G)$ *are adjacent in* G *and* $v_1', v_2' \in V(G)$ *are adjacent in* G *with* $v_1 \simeq^+ v_1'$, $v_1 <^+ v_2$, *and* $v_1' <^+ v_2'$, *then* $v_2 \simeq^+ v_2'$.

Proof. (i) The implications (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1) are trivial. For (1) $\Rightarrow$ (2) we will prove that $v_1 <^+ s_1$ implies that $v_1 <^+ s_j <^+ v_2$ for each $j \in \{1, \ldots, \kappa\}$. Then by considering the reverse order of $\leq^+$, it follows that $v_1 <^+ v_2$ implies $v_1 <^+ s_1$, hence $v_1 <^+ s_j <^+ v_2$ for each $j \in \{1, \ldots, \kappa\}$.

Suppose $v_1 <^+ s_1$. Let $i \in \{1, \ldots, \kappa\}$ be such that

$$s_i \cdots s_1 v_1 \overset{w^+}{\mapsto} \zeta_N(w(v_1)) \quad \text{and} \quad s_i \cdots s_\kappa v_2 \overset{w^+}{\mapsto} \zeta_N(w(v_2)).$$

By Lemma 13 (ii), we have $v_1 <^+ s_j <^+ s_i$ for each $j \in \{1, \ldots, i-1\}$. Because G is compliant, we can deduce using Lemma 10 that $s_i <^+ s_{i+1}$. Then by Lemma 13 (i) we have $s_i <^+ s_j <^+ v_2$ for each $j \in \{i+1, \ldots, \kappa\}$.

(ii) Suppose $v_1, v_2 \in V(G)$ are adjacent in G and $v_1', v_2' \in V(G)$ are adjacent in G with $v_1 \simeq^+ v_1'$, $v_1 <^+ v_2$, and $v_1' <^+ v_2'$. Let $s_1, \ldots, s_\kappa$ and i be as in part (i), and let $s_1', \ldots, s_\lambda' \in V(G^+) \smallsetminus V(G)$ be such that $v_1' s_1', \ldots, s_\lambda', v_2'$ are consecutive in G^+ and

$$v_1' s_1' \cdots s_\lambda' v_2' \overset{w^+}{\mapsto} \zeta_N(w(v_1'))^{\leftarrow} \pitchfork \zeta_N(w(v_2')).$$

By property (C1) of the chain quasi-order $\leq^+$, $w(v_1) \approx_\Gamma w(v_1')$, hence $|\zeta_N(v_1)| = |\zeta_N(v_1')|$, and so

$$s_i' \cdots s_1' v_1' \overset{w^+}{\mapsto} \zeta_N(w(v_1')) \quad \text{and} \quad s_i' \cdots s_\lambda' v_2' \overset{w^+}{\mapsto} \zeta_N(w(v_2')).$$

By Lemma 13 (iv), we have $s_i \simeq^+ s_i'$, and as in part (i) we know that $s_{i+1}' >^+ s_i'$. By Lemma 10, this implies $w(v_2) \approx_\Gamma w(v_2')$, hence $\kappa = \lambda$. Then by Lemma 13 (iii), we conclude that $v_2 \simeq^+ v_2'$.

□

Proposition 16. *Suppose* $\langle G, w \rangle$ *is a compliant graph-word. If a (any)* ρ_N*-expansion of* $\langle G, w \rangle$ *has a chain quasi-order, then* $\langle G, w \rangle$ *also has a chain quasi-order.*

Proof. Let $\langle G^+, w^+ \rangle$ be a ρ_N-expansion of $\langle G, w \rangle$, and let $\leq^+$ be a chain quasi-order of $\langle G^+, w^+ \rangle$.

Define $\leq$ on $V(G)$ by $\leq \,:=\, \leq^+|_{V(G)}$. Clearly $\leq$ is a total quasi-order since $\leq^+$ is. We must check that $\leq$ satisfies properties (C1), (C2), and (C3) of the definition of a chain quasi-order.

(C1): This is immediate since $\leq^+$ satisfies this property.

(C2): We will need the following claim:

Claim 16.1. In $\langle G^+, w^+ \rangle$, if $v \in V(G)$ and $v' \in V(G^+)$ are such that $v \simeq^+ v'$, then in fact $v' \in V(G)$.

Proof of Claim 16.1. We proceed by induction on the number of vertices in G.

If $|V(G)| = 1$, then there is nothing to prove.

Assume the claim holds for all such graph-words whose graph has n or fewer vertices, and assume $|V(G)| = n+1$. Let $u \in V(G)$ be such that the subgraph G^- obtained by removing the vertex u (and all edges emanating from u) is connected. There is a ρ_N-expansion of $\langle G^-, w|_{V(G) \smallsetminus \{u\}} \rangle$ which is a sub-graph-word of $\langle G^+, w^+ \rangle$

(it has vertex set $V(G^+) \cap G^-$), and the restriction of $\leq^+$ to this sub-graph-word is a chain quasi-order. By induction, the claim holds for G^-.

Let $u' \in V(G) \smallsetminus \{u\}$ be adjacent to u in G. Let $s_1 \ldots, s_\kappa \in V(G^+) \smallsetminus V(G)$ be such that $u', s_1, \ldots, s_\kappa, u$ are consecutive in G^+ and

$$u' s_1 \cdots s_\kappa u \overset{w^+}{\mapsto} \zeta_N(w(u')) \pitchfork \zeta_N(w(u)).$$

Assume $u' <^+ u$ (the other case proceeds similarly), which implies by Lemma 15 (i) that $u' <^+ s_j <^+ u$ for each $j \in \{1, \ldots, \kappa\}$.

We have four things to check:

(1) for each $y \in V(G) \smallsetminus \{u\}$ and each $s \in V(G^+) \smallsetminus V(G)$ in the ρ_N-expansion of G^-, $y \not\simeq^+ s$;
(2) for each $y \in V(G) \smallsetminus \{u\}$ and each $j \in \{1, \ldots, \kappa\}$, $y \not\simeq^+ s_j$;
(3) for each $s \in V(G^+) \smallsetminus V(G)$ in the ρ_N-expansion of G^-, $u \not\simeq^+ s$; and
(4) for each $j \in \{1, \ldots, \kappa\}$, $u \not\simeq^+ s_j$.

Observe that (1) holds by induction, and (4) is immediate from the fact that $u' <^+ s_j <^+ u$ for each $j \in \{1, \ldots, \kappa\}$. For (2) and (3), we consider two cases.

Case 1. For every $y \in V(G) \smallsetminus \{u\}$, $y \leq^+ u'$.

Since $u' <^+ s_j <^+ u$ for each $j \in \{1, \ldots, \kappa\}$, we have immediately that $y \not\simeq^+ s_j$ for any $y \in V(G) \smallsetminus \{u\}$.

Also, from Lemma 15 (i) it follows that for every $s \in V(G^+) \smallsetminus V(G)$ in the ρ_N-expansion of G^-, $s <^+ u'$. Therefore $u \not\simeq^+ s$ for any such s.

Case 2. There exists some $y \in V(G) \smallsetminus \{u\}$ such that $u' <^+ y$.

Let $\mathcal{P}$ be a path of vertices in G^- starting at u' and ending at y. Let y_1 be the latest vertex y' in $\mathcal{P}$ with $y' \leq^+ u'$, and let y_2 be the next vertex in $\mathcal{P}$ after y_1, so that y_1 and y_2 are adjacent in G and $y_1 \leq^+ u' <^+ y_2$.

Suppose for a contradiction that $y_1 <^+ u'$. Let $z_1, \ldots, z_\lambda \in V(G^+) \smallsetminus V(G)$ be such that $y_1, z_1, \ldots, z_\lambda, y_2$ are consecutive in G^+. Then by Lemma 4 there is some $i \in \{1, \ldots, \lambda\}$ such that $u' \simeq^+ z_i$. But this contradicts the fact that the claim holds for G^- by induction. Therefore we must have $u' \simeq^+ y_1$.

Then from Lemma 15 (ii) we know that $u \simeq^+ y_2$. It follows immediately that $u \not\simeq^+ s$ for each $s \in V(G^+) \smallsetminus V(G)$ in the ρ_N-expansion of G^-, because $y_2 \not\simeq^+ s$ for every such s by induction.

Moreover, for each $j \in \{1, \ldots, \kappa\}$, since $y_1 \simeq^+ u' <^+ s_j <^+ u \simeq^+ y_2$, we know from Lemma 4 that there is some $s \in V(G^+) \smallsetminus V(G)$ inserted between y_1 and y_2 such that $s_j \simeq^+ s$. It follows that $y \not\simeq^+ s_j$ for any $y \in V(G) \smallsetminus \{u\}$, because $y \not\simeq^+ s$ for every such y by induction. $\square$(Claim 16.1)

Now suppose $v_1, v_2 \in V(G)$ are adjacent in G, and assume $v_1 \leq v_2$. Let $s_1, \ldots, s_\kappa \in V(G^+) \smallsetminus V(G)$ be such that $v_1, s_1, \ldots, s_\kappa, v_2$ are consecutive in $V(G^+)$. If $v \in V(G)$ were such that $v_1 < v < v_2$, then $v_1 <^+ v <^+ v_2$ as well, so by Lemma 4 there would be some $i \in \{1, \ldots, \kappa\}$ such that $v \simeq^+ s_i$. But this contradicts Claim 16.1.

(C3): Suppose $v \in V(G)$, v_1, v_2, v_3 are consecutive in G, and that $\sigma, \tau \in \{a, c\}$ and $t, t' \in [0,1]$ are such that $t' \geq t$, $w(v) = d_{t'}$, $v_1 v_2 v_3 \overset{w}{\mapsto} \sigma d_t \tau$, and $v_1 < v_2 \simeq v < v_3$.

Let $s_1, \ldots, s_\kappa, s'_1, \ldots, s'_\lambda \in V(G^+) \setminus V(G)$ be such that $v_1, s_1, \ldots, s_\kappa, v_2, s'_\lambda, \ldots, s'_1, v_3$ are consecutive in G^+, and

$$v_1 s_1 \cdots s_\kappa v_2 s'_\lambda \cdots s'_1 v_3 \overset{w^+}{\mapsto} \zeta_N(\sigma)^\leftarrow \pitchfork \beta_N^- d_t(\beta_N^-)^\leftarrow \pitchfork \zeta_N(\tau).$$

Observe that $w^+(s_\kappa) = w^+(s'_\lambda) = c$. Since $v_1 <^+ v_2$, by Lemma 15 (i) we must have $s_\kappa <^+ v_2$. Likewise, we have $v_2 <^+ s'_\lambda$. It now follows from property (C3) of the chain quasi-order $\leq^+$ that $t' - t < \frac{1}{2}$. □

5. The example

Example. There exists a continuum $X \subset \mathbb{R}^2$ which is non-chainable and has span zero.

Proof. First we define by recursion a sequence $\langle T_N \rangle_{N=0}^\infty$ of simple triods in $\mathbb{R}^2$ and a sequence $\langle \varepsilon_N \rangle_{N=0}^\infty$ of positive reals as follows.

Let $T_0 \subset \mathbb{R}^2$ be as defined above, and put $\varepsilon_0 := \frac{1}{8}$.

Suppose T_N, ε_N have been defined. Apply Proposition 1 to obtain an embedding T_{N+1} of the simple triod graph $G_{\rho_{N+1}}$ in $\mathbb{R}^2$ such that ρ_{N+1} is a $\langle T_N, \varepsilon_N \rangle$-sketch of T_{N+1}. Endow T_{N+1} with a Γ-marking as above. Notice that $T_{N+1} \subset (T_N)_{\varepsilon_N}$, where Y_ε denotes the ε-neighborhood of the space Y. By Proposition 3, the span of T_{N+1} is $< \frac{1}{2(N+1)} + \varepsilon_N$. Let $0 < \varepsilon_{N+1} < 2^{-N-4}$ be small enough so that $\overline{(T_{N+1})_{\varepsilon_{N+1}}} \subseteq \overline{(T_N)_{\varepsilon_N}}$, and so that $\operatorname{span}(\overline{(T_{N+1})_{\varepsilon_{N+1}}}) < \frac{1}{2(N+1)} + 2\varepsilon_N$.

Put $X := \bigcap_{N=0}^\infty \overline{(T_N)_{\varepsilon_N}}$.

Observe that for any N, we have $X \subseteq \overline{(T_{N+1})_{\varepsilon_{N+1}}}$, hence

$$\operatorname{span}(X) \leq \operatorname{span}(\overline{(T_{N+1})_{\varepsilon_{N+1}}}) < \frac{1}{2(N+1)} + 2\varepsilon_N.$$

Since ε_N converges to 0 as $N \to \infty$, it follows that X has span zero.

Suppose for a contradiction that X has a chain cover of mesh $< \frac{1}{4}$. Then there is some $N > 0$ for which T_N has a chain cover of mesh $< \frac{1}{4}$.

Define by recursion the graph-words $\langle G_i, w_i \rangle$, $0 \leq i \leq N-1$, as follows: $\langle G_{N-1}, w_{N-1} \rangle := \rho_N$, and for $i < N-1$, $\langle G_i, w_i \rangle$ is the ρ_{i+1}-expansion of $\langle G_{i+1}, w_{i+1} \rangle$ provided by Proposition 14 which is a $\langle T_i, \sum_{j=i}^{N-1} \varepsilon_j \rangle$-sketch of T_N. In particular, $\langle G_0, w_0 \rangle$ is a $\langle T_0, \sum_{j=0}^{N-1} \varepsilon_j \rangle$-sketch of T_N.

Since $\sum_{j=0}^{N-1} \varepsilon_j < \sum_{j=0}^{N-1} 2^{-j-3} < \frac{1}{4}$, by Proposition 5 we have that $\langle G_0, w_0 \rangle$ has a chain quasi-order. Then by Proposition 16 and induction, we obtain a chain quasi-order for each graph-word $\langle G_i, w_i \rangle$. In particular, $\langle G_{N-1}, w_{N-1} \rangle$ has a chain quasi-order. But $\langle G_{N-1}, w_{N-1} \rangle$ is ρ_N, so this contradicts Proposition 11. □

6. Questions

The construction presented in this paper can be carried out so that every proper subcontinuum of the resulting space is an arc; hence, in particular, it is far from being hereditarily indecomposable. On the other hand, it follows from results of [17] that if there exists a non-degenerate homogeneous continuum in the plane which is not homeomorphic to the circle, the pseudo-arc, or the circle of pseudo-arcs, then there would be one which is hereditarily indecomposable and with span zero. Given that the pseudo-arc is the only hereditarily indecomposable chainable continuum, this raises the following question:

Question 1 (See Problem 9 of [18])**.** *Is there a hereditarily indecomposable non-chainable continuum with span zero?*

If such an example exists, then by [19, Corollary 6] it would be a continuous image of the pseudo-arc. Since any map to a hereditarily indecomposable continuum is confluent [22, Lemma 15], it would also be a counterexample to Problem 84 of [4], which asks whether every confluent image of a chainable continuum is chainable.

Regarding the planarity of the example in this paper, while every chainable continuum can be embedded in the plane [2], the same is not known to be true of continua with span zero.

Question 2. *Can every continuum with span zero be embedded in* $\mathbb{R}^2$*?*

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