# Pick's Theorem

Computing Area by Counting Points

**Professor Pi says:**

Today we will learn a formula that feels almost impossible at first:

Area of a polygon = something you can find just by counting grid points.

No base times height. No complicated measurements. Just counting.

## 1. Lattice Points and Lattice Polygons

A **lattice point** is a point whose coordinates are both integers.

Examples:

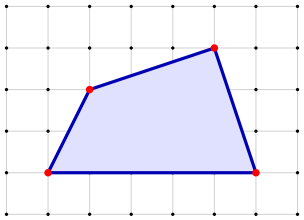
$$(0,0), \quad (3,5), \quad (-2,7)$$

are lattice points, while

$$\left(\frac{1}{2}, 3\right), \quad (2.7, 1)$$

are not lattice points.

A **lattice polygon** is a polygon whose vertices are lattice points.



## 2. The Two Numbers We Count

For a lattice polygon $P$, we define:

$$I = \text{the number of lattice points strictly inside } P,$$

$$B = \text{the number of lattice points on the boundary of } P.$$

**Professor Pi says:**

The letter $I$ stands for **Interior**. The letter $B$ stands for **Boundary**.

## 3. Pick's Theorem

**Theorem**

Let $P$ be a lattice polygon. If $A$ is its area, $I$ is the number of interior lattice points, and $B$ is the number of boundary lattice points, then

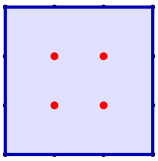
$$\boxed{A = I + \frac{B}{2} - 1.}$$

This is surprising because it says that area can be computed just by counting points.

## 4. Example 1: A Square

Consider the square with vertices

$$(0,0),\ (3,0),\ (3,3),\ (0,3).$$



There are

$$I = 4$$

interior lattice points, and

$$B = 12$$

boundary lattice points.
Pick's theorem gives

$$A = I + \frac{B}{2} - 1 = 4 + \frac{12}{2} - 1 = 4 + 6 - 1 = 9.$$

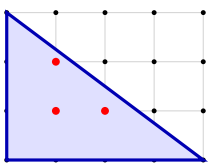
This agrees with the usual area:

$$3 \cdot 3 = 9.$$

## 5. Example 2: A Right Triangle

Consider the triangle with vertices

$$(0,0),\ (4,0),\ (0,3).$$



Here,

$$I = 3.$$

The boundary lattice points are:

$$(0,0), (1,0), (2,0), (3,0), (4,0), (0,1), (0,2), (0,3),$$

so

$$B = 8.$$

Thus,

$$A = I + \frac{B}{2} - 1 = 3 + \frac{8}{2} - 1 = 3 + 4 - 1 = 6.$$

This agrees with the usual triangle formula:

$$A = \frac{1}{2} \cdot 4 \cdot 3 = 6.$$

## 6. Practice: Level 1

For each polygon, compute $I$, $B$, and $A$.

**1.** Square with vertices
$$(0,0), (2,0), (2,2), (0,2).$$

**2.** Rectangle with vertices
$$(0,0), (5,0), (5,2), (0,2).$$

**3.** Right triangle with vertices
$$(0,0), (6,0), (0,2).$$

**4.** Right triangle with vertices
$$(0,0), (5,0), (0,5).$$

## 7. Practice: Level 2

Now try polygons that are less standard.

**1.** Quadrilateral with vertices
$$(0,0), (4,0), (5,2), (1,3).$$

**2.** Pentagon with vertices
$$(0,0), (4,0), (5,2), (3,4), (0,3).$$

**3.** Concave polygon with vertices
$$(0,0), (5,0), (5,3), (3,3), (3,1), (0,1).$$

**4.** Hexagon with vertices
$$(0,0), (3,0), (5,2), (4,5), (1,5), (-1,2).$$

**Professor Pi says:**

A concave polygon has a "dent." Pick's theorem still works!

## 8. Reverse Problems

Now instead of being given the polygon, you are given information about $I$, $B$, or $A$.

1. Find the area of a lattice polygon with
$$I = 5, \qquad B = 10.$$
2. Find the area of a lattice polygon with
$$I = 0, \qquad B = 12.$$
3. Find all possible values of $B$ if
$$I = 3, \qquad A = 8.$$
4. Find all possible values of $I$ if
$$B = 14, \qquad A = 12.$$
5. Can there be a lattice polygon with
$$I = 2, \qquad B = 5?$$
Explain.

## 9. Construction Challenges

**Challenge**

Draw examples of lattice polygons satisfying the requested conditions.

1. Construct a lattice polygon with
$$I = 0, \qquad B = 4.$$
2. Construct a lattice polygon with
$$I = 1, \qquad B = 4.$$
3. Construct a lattice polygon with
$$I = 2, \qquad B = 6.$$
4. Construct a lattice polygon with area 7.
5. Construct two different lattice polygons with the same values of $I$ and $B$.
6. Construct two different lattice polygons with the same area but different values of $I$ and $B$.

## 10. A Useful Boundary Formula

Suppose a line segment goes from

$$(x_1, y_1)$$

to

$$(x_2, y_2).$$

Let

$$\Delta x = |x_2 - x_1|, \qquad \Delta y = |y_2 - y_1|.$$

Then the number of lattice points on this segment, including both endpoints, is

$$\gcd(\Delta x, \Delta y) + 1.$$

So the number of *steps* contributed by this edge to the boundary count is

$$\gcd(\Delta x, \Delta y).$$

Therefore, for a lattice polygon,

$$B = \sum_{\text{edges}} \gcd(|\Delta x|, |\Delta y|).$$

**Professor Pi says:**

This is one of the best tricks for counting boundary points quickly. Do not double-count the vertices!

### Example

For the segment from $(1, 2)$ to $(7, 5)$,

$$\Delta x = 6, \qquad \Delta y = 3.$$

So

$$\gcd(6, 3) = 3.$$

The segment contains

$$3 + 1 = 4$$

lattice points.

## 11. Boundary Counting and Lines

**Professor Pi says:**

There is a deep connection between lattice points and lines.
To understand how many lattice points lie on a segment, it is often helpful to write the equation of the line.

Suppose a segment goes from

$$(x_1, y_1) \text{ to } (x_2, y_2).$$

Let

$$\Delta x = |x_2 - x_1|, \qquad \Delta y = |y_2 - y_1|.$$

Then the number of lattice points on the segment (including endpoints) is:

$$\gcd(\Delta x, \Delta y) + 1.$$

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**Example**

Consider the segment from $(1, 2)$ to $(7, 5)$.

**Step 1: Compute differences**

$$\Delta x = 6, \qquad \Delta y = 3.$$

**Step 2: Count lattice points**

$$\gcd(6, 3) = 3 \Rightarrow \text{there are } 4 \text{ lattice points.}$$

**Step 3: Find the equation of the line**

Slope:

$$m = \frac{5-2}{7-1} = \frac{3}{6} = \frac{1}{2}.$$

Equation:

$$y - 2 = \frac{1}{2}(x - 1) \quad \Rightarrow \quad y = \frac{1}{2}x + \frac{3}{2}.$$

**Step 4: Find all lattice points**

We need $x$ such that $y$ is an integer.

$$y = \frac{1}{2}x + \frac{3}{2} = \frac{x+3}{2}.$$

So $x$ must be odd.
Checking values between 1 and 7:

$$x = 1, 3, 5, 7$$

We get:

$$(1, 2),\ (3, 3),\ (5, 4),\ (7, 5).$$

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## 12. Practice: Lines and Lattice Points

For each segment:

1. From $(0, 0)$ to $(6, 0)$
   - Find the equation of the line.
   - List all lattice points.
   - Verify using the gcd formula.
2. From $(0, 0)$ to $(6, 3)$
   - Find the equation.
   - List all lattice points.
   - Compare with gcd.
3. From $(1, 1)$ to $(7, 5)$
   - Find the slope.
   - Write the equation.
   - Find all lattice points.
4. From $(2, 3)$ to $(10, 7)$
   - Find the equation.
   - How many lattice points are there?
5. From $(-1, 4)$ to $(8, 1)$
   - Find the equation.
   - Determine all lattice points.

## 13. Lines Inside Polygons

**Professor Pi says:**

Every edge of a lattice polygon lies on a line.
Understanding these lines helps us compute $B$ efficiently.

### Example

Consider the edge from $(0, 0)$ to $(6, 3)$.

Equation:

$$y = \frac{1}{2}x.$$

Lattice points occur when $x$ is even:

$$(0, 0), (2, 1), (4, 2), (6, 3).$$

There are 4 points, consistent with:

$$\gcd(6, 3) + 1 = 4.$$

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**Practice**

1. For the edge $(0,0)$ to $(8,4)$:
   - Find the equation.
   - List all lattice points.
   - Compute gcd.
2. For the edge $(1,2)$ to $(9,6)$:
   - Find the equation.
   - Determine all lattice points.
3. Explain why lattice points occur at equal spacing along a line.

## 14. Deeper Insight

**Professor Pi says:**

Why does $\gcd(\Delta x, \Delta y)$ appear?
Because lattice points occur at regular steps along the line.
If $\Delta x = 6$ and $\Delta y = 3$, then each step is:

$$(2,1),$$

so the points are evenly spaced.

## 15. Challenge: Use Pick's Theorem Backwards

For each polygon, compute $B$ using the gcd formula, compute area using geometry or the shoelace formula, and then find $I$.

1. Triangle:
   $$(0,0), (8,0), (2,6).$$
2. Triangle:
   $$(1,1), (7,2), (4,8).$$
3. Quadrilateral:
   $$(0,0), (6,1), (5,5), (1,4).$$
4. Pentagon:
   $$(0,0), (5,0), (7,3), (3,6), (-1,3).$$

## 16. Final Problems

1. Is it possible to have a lattice polygon with
   $$I = 0, \qquad B = 3?$$
   If yes, what is its area?

**2.** Is it possible to have a lattice polygon with

$$I = 0, \qquad B = 100?$$

If yes, what is its area?

**3.** Suppose a lattice polygon has area 20 and $B = 14$. Find $I$.

**4.** Suppose a lattice polygon has area 15.5 and $I = 8$. Find $B$.

**5.** Prove that the area of every lattice polygon is either an integer or a half-integer.

**6.** Prove that if a lattice polygon has no interior lattice points, then

$$A = \frac{B}{2} - 1.$$

**7.** Find a lattice polygon with no interior lattice points and area 10.

**8.** Find a lattice polygon with exactly one interior lattice point and area 10.

**9.** Can a lattice polygon have area 3.7? Explain.

**10.** Can a lattice polygon have area 100 and $B = 3$? Explain.

**Professor Pi says:**

Final thought: Pick's theorem is a perfect example of how geometry and number theory secretly talk to each other.