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ARTICLE

ERNST CASSIRER'S NEO-KANTIAN PHILOSOPHY OF GEOMETRY[1]

Jeremy Heis

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One of the most important philosophical topics in the early twentieth century – and a topic that was seminal in the emergence of analytic philosophy – was the relationship between Kantian philosophy and modern geometry. This paper discusses how this question was tackled by the Neo-Kantian trained philosopher Ernst Cassirer. Surprisingly, Cassirer does not affirm the theses that contemporary philosophers often associate with Kantian philosophy of mathematics. He does not defend the necessary truth of Euclidean geometry but instead develops a kind of logicism modeled on Richard Dedekind's foundations of arithmetic. Further, because he shared with other Neo-Kantians an appreciation of the developmental and historical nature of mathematics, Cassirer developed a philosophical account of the unity and methodology of mathematics over time. With its impressive attention to the detail of contemporary mathematics and its exploration of philosophical questions to which other philosophers paid scant attention, Cassirer's philosophy of mathematics surely deserves a place among the classic works of twentieth century philosophy of mathematics. Though focused on Cassirer's philosophy of geometry, this paper also addresses both Cassirer's general philosophical orientation and his reading of Kant.

KEYWORDS: Ernst Cassirer; geometry; Immanuel Kant; Neo-Kantianism

A common thread in some recent work on the emergence and early development of analytic philosophy has been its relationship to Kant. Alberto Coffa has urged us to see the development of philosophy from Kant to Carnap as 'the stages through which it came to be recognized that [Kant's] pure intuition must be excluded from the a priori sciences and that

[1]This paper greatly benefited from comments and conversations with Jeremy Avigad, Scott Edgar, Anil Gupta, Penelope Maddy, Kenneth Manders, Erich Reck, Thomas Ricketts, Geoffrey Sayre-McCord, Mark Wilson, and many, many others. I would also like to especially acknowledge the extremely helpful and truly supererogatory assistance I received from an anonymous referee.

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consequently the Kantian picture of mathematics and geometry must be replaced by some other'.[2] There are good reasons to try to understand the origins of analytic philosophy against the backdrop of Kant's theoretical philosophy – as Coffa puts it, 'for better and worse, almost every philosophical development of significance since 1800 has been a response to Kant'.[3] But the success of the 'Kant and early analytic philosophy' research programme depends on an accurate picture of how Kant's writings were interpreted and treated at the end of the nineteenth and beginning of the twentieth century. How was Kant understood by the Neo-Kantians that dominated philosophy during analytic philosophy's early years? What doctrines of Kant were inspirational for philosophers trying to understand the radical changes that had taken place in the sciences in the nineteenth century? What doctrines were thought dispensable or hopeless?

In this paper, I consider how one of the most important philosophical topics of the period – the relationship of Kantian philosophy to modern geometry – was tackled by one of the most important Neo-Kantian philosophers of the period, Ernst Cassirer.[4] Cassirer was educated at Marburg University under Hermann Cohen and Paul Natorp, the founders of one of the most significant philosophical schools in turn of the century Germany, the Marburg or 'logical idealist' Neo-Kantians.[5] One principal

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[2]Coffa, *The Semantic Tradition from Kant to Carnap*, 2.

[3]*The Semantic Tradition*, 7.

[4]For a nice recent introduction to Cassirer's thought as a whole, see J. Krois, *Cassirer: Symbolic Forms and History*; M. Friedman, *A Parting of the Ways: Carnap, Cassirer, and Heidegger*; and especially Massimo Ferrari, *Ernst Cassirer: Stationen einer philosophischen Biographie*. More specific work has been done on his philosophy of exact science by Thomas Ryckman; see his very helpful paper '*Condition Sine Qua Non? Zuordnung* in the Early Epistemologies of Cassirer and Schlick'; and Ryckman, *The Reign of Relativity: Philosophy in Physics, 1915–1925*. On Cassirer's philosophy of science, with special attention to his interpretation of quantum mechanics, see Stephen French's paper, 'Symmetry, Structure, and the Constitution of Objects', given at Oxford in January 2001. On Cassirer's philosophy of science in general, see Friedman, 'Ernst Cassirer and the Philosophy of Science'. Less attention has been devoted to Cassirer's philosophy of mathematics. A good recent monograph that discusses his philosophy of geometry in detail is Karl-Norbert Ihmig, *Cassirers Invariantentheorie der Erfahrung und seine Rezeption des 'Erlangers Programms'*. Thomas Mormann's 'Idealization in Cassirer's Philosophy of Mathematics', which unfortunately appeared after this paper was completed, largely complements the discussion in this paper. Mormann focuses on the issues that I discuss in section 4, and he also helpfully considers Cassirer's view of the role of idealization in natural science. As a confirmation of Cassirer's approach, Mormann very nicely considers more recent instances of 'ideal' elements in mathematics. Given the different focus of his paper, he does not address the issues I raise in sections 3 and 5, and he does not explicitly consider Cassirer's relation to Kant and early analytic philosophy. Heis, '"Critical Philosophy Begins at the Very Point where Logistic Leaves Off": Cassirer's Response to Frege and Russell' (forthcoming), considers Cassirer's reception of Frege's and Russell's logic and logicism.

[5]Michael Friedman (*Reconsidering Logical Positivism*, Chapter 6; and *A Parting of the Ways*) Alan Richardson (*Carnap's Construction of the World: The Aufbau and the Emergence of Logical Empiricism*), and A. W. Carus (*Carnap and Twentieth-Century Thought*) have each argued that the Marburg Neo-Kantians significantly influenced Carnap.

conclusion of this paper is that Cassirer does not affirm the theses that contemporary philosophers often associate with Kantian philosophy of mathematics. He does not defend the necessary truth of Euclidean geometry nor does he assert that we have *a priori* knowledge of the curvature of physical space. Not only does he deny that arithmetic depends on a pure intuition of time, but he actually defends Richard Dedekind's logicism about arithmetic. So inasmuch as Cassirer is representative of Neo-Kantianism, it will simply not do to characterize (as Coffa does) the fundamental divergence between Neo-Kantianism and Frege, Russell, and the logical empiricists as a debate about pure intuition.

Though the divergence between Cassirer and the philosophers in the analytic tradition cannot be located in the debate over Kantian pure intuition, this does not mean that there are no interesting philosophical questions left for a Neo-Kantian like Cassirer to address. Far from it. Though Cassirer shared with Russell and Frege a kind of 'logicism', his logicism differed from theirs in being modelled on Richard Dedekind's foundations of arithmetic. Moreover, Cassirer, because he shared with other Neo-Kantians an appreciation of the *developmental* and *historical* nature of mathematics, was attentive to questions in the philosophy of mathematics to which Frege, Russell, and the logical empiricists paid scant attention. He wanted to know how mathematics remains a unified science over time despite the conceptual and ontological revolutions that it has undergone, and he wanted to understand the methodological pressures within pure mathematics that drove mathematicians to expand and modify their ontologies. The primary goal of the paper, then, is this: I hope to demonstrate that Cassirer's philosophy of mathematics, though largely unknown today, is subtle, original, and mathematically informed enough to earn a place among the better-known philosophies of the early twentieth century – a period that is rightly thought to be a golden period in the philosophical study of mathematics.

Though Cassirer, from the time of his dissertation in 1899 up to about 1920, was a leading representative of the Marburg Neo-Kantian school, he is better known today by historians of philosophy for other reasons. First, Cassirer, at least in Paul Guyer's estimation, is 'the greatest of all modern historians of philosophy'.[6] His works, which cover all of the major philosophers and many of the minor philosophers from the Renaissance to the nineteenth century, are still widely read. Second, contemporary readers know Cassirer as a philosophy of culture, whose three-volume *Philosophy of Symbolic Forms* (1923–9) defends the autonomy of the methodologies of the various special sciences by arguing that language, myth, and art are each 'symbolic forms' by which we come to represent a structured world of objects.

[6]Guyer, *Kant*, 14.

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Historians concerned with Cassirer's relation to early analytic philosophy, though, have understandably focused more on his extensive writings on physics, logic, and mathematics. As we will see, these topics were hardly of peripheral interest to Cassirer. Moreover, by narrowing our view to Cassirer's writings on mathematics, we won't lose sight of these other better-known aspects of Cassirer's writings. As will be clear in what follows, Cassirer's historical work – especially his reading of Kant – was never separate from his systematic philosophical work. And (as I discuss in section 5 of this paper) his turn from classical Marburg Neo-Kantianism to the Philosophy of Symbolic Forms had an interesting impact on his philosophy of mathematics as well. Thus, although this paper concerns Cassirer's philosophy of geometry, this topic was central enough for Cassirer that the discussion can serve as an introduction to his thought as a whole.

The paper is divided into five sections. In section 1, I introduce the philosophical challenge raised for Kantian philosophy by nineteenth-century geometry, emphasizing the problems – not well known today – raised by the use of ideal or imaginary elements. In section 2, I show that Cassirer and the other Marburg Neo-Kantians committed themselves to explaining modern mathematics and its historical development without condemning any part of it, not even the imaginary or ideal elements, as false or meaningless. This philosophical commitment then constrained Cassirer to ask for a philosophical justification for the more abstract elements that had become central to nineteenth-century geometry. What's more, Cassirer's study of the history of mathematics led him to ask how geometry has remained a unified science over time despite the deep changes in its fundamental concepts and ontology.

I give Cassirer's answers to these two questions in sections 3 and 4. In section 3, I show how Cassirer saw in the mathematician Richard Dedekind's foundations of arithmetic a model for a modern version of Kant's thesis that mathematics is rational cognition from the 'construction of concepts'. Thus, since modern mathematics is – on this Dedekindian picture – the study of abstract relational structures, three-dimensional Euclidean space and n-dimensional complex projective space are ontologically on the same footing. In section 4, I show that Cassirer takes on the conviction – widespread among mathematicians – that newer systems of mathematical concepts make even elementary mathematical objects more intelligible. There are then important methodological considerations internal to pure mathematics that necessitated the expansion of the ontology of geometry; indeed, it is these methodological considerations that make mathematics what it is, and that explain how geometry can be a unified science over time despite the evolution in its ontology and fundamental concepts.

In the last section, I explore an opposing Kantian element in Cassirer and the other Marburg Neo-Kantians' philosophy of geometry – an element that stood in a creative tension with the view I ascribe to Cassirer in sections 3

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and 4. For Cassirer, Kant's philosophy of geometry rightly avoided formalism and Platonism by maintaining that mathematical judgments are objectively valid only because mathematics is essentially applicable in natural science. It is, however, hard to respect this good idea in the context of abstract mathematical theories that are not applied in natural science. Cassirer's eventual solution to this problem is twofold: on the one hand, mathematics avoids being a game inasmuch as it is employed in empirical science; on the other hand, mathematics is such a unity that removing those portions of mathematics that are not applied would leave the rest of mathematics simply unintelligible.

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1. TWO GEOMETRICAL REVOLUTIONS, TWO PHILOSOPHICAL PROJECTS

One of the geometrical revolutions that took place in the nineteenth century was the discovery of non-Euclidean geometries equiconsistent with Euclidean geometry. From a certain kind of Kantian point of view, for instance, that of Bertrand Russell's early *Essay on the Foundations of Geometry*, this revolution demanded new study of 'the bearing of [non-Euclidean geometry] on the argument of the Transcendental Aesthetic'.[7] For philosophers like Russell and the early Carnap, this kind of study would culminate in isolating an *a priori* theory of space weaker than Euclid.[8]

Geometry underwent a second kind of revolution in the nineteenth century. Starting in earnest in the 1820s, mathematicians working in 'projective' or 'higher' or 'modern' geometry radically simplified and expanded familiar geometrical figures by adding new elements: first, they added to each line one and only one point, its point at infinity; second, they added to the plane imaginary points. This allowed geometers to embed elementary figures in wider settings. The founder of this way of thinking, Poncelet, discovered the so-called circular points, the unique pair of infinitely distant and imaginary points that every circle intersects (though of course we cannot see these points when we draw the figure). Describing this point of view, the historian Steven Kleiman writes, 'one senses that many classical geometers had a platonic view of figures like conics. There are ideal conics of which we see only shadows or aspects like their point sets and their envelopes of tangent lines.'[9] The late-nineteenth-century geometer Arthur Cayley emphasized the great mathematical value and remoteness from

[7] *An Essay on the Foundations of Geometry*, 55. The book first appeared in 1897.
[8] In his *Essay*, Russell argued that consciousness of a world of mutually external things requires that things have the spatial properties captured by projective geometry and those metrical properties common to all finite dimensional geometries of constant curvature. Rudolf Carnap, in his dissertation, *Der Raum. Ein Beitrag zur Wissenschaftslehre*, claims that experience is constituted rather by n-dimensional topological space.
[9] Kleiman, 'Chasles's Enumerative Theory of Conics', 133.

spatial visualization of these new elements by calling them the '*ontōs onta*' compared to which the 'real' elements are just shadows in the cave.[10] For example, in 1859, Cayley had constructed a distance function for pairs of points in the plane in terms of a certain projective relationship (the cross-ratio) between that pair and two fixed, infinitely distant points (the 'circular points') whose coordinates require the square root of -1.[11]

Earlier in the century, some geometers were unwilling to allow imaginary elements into geometry: the great synthetic geometer Jakob Steiner, for instance, called the imaginary points 'ghosts' or 'the shadow land of geometry'.[12] Cayley himself recognized that the procedures of modern mathematics, including the use of imaginary and infinitely distant points in projective geometry, posed a distinctly philosophical problem that required a distinctly philosophical solution. He deplored the fact that no philosophers had worried about this problem in anything like the way it deserved:

> [T]he notion of a negative magnitude has become quite a familiar one ... But it is far otherwise with the notion which is really the fundamental one (and I cannot too strongly emphasize the assertion) underlying and pervading the whole of modern analysis and geometry, that of imaginary magnitude in analysis and of imaginary space (or space as a *locus in quo* of imaginary points and figures) in geometry ... This has not been, so far as I am aware, a subject of philosophical discussion or enquiry. As regards the older metaphysical writers this would be quite accounted for by saying that they knew nothing, and were not bound to know anything, about it; but at present, and, considering the prominent position which this notion occupies – say even that the conclusion were that the notion belongs to mere technical mathematics, or has reference to nonentities in regard to which no science is possible, still it seems to me that (as a subject of philosophical discussion) the notion ought not to be thus ignored; it should at least be shown that there is a right to ignore it.[13]

The philosophical task posed by this problem is different from that posed by Euclidean geometry. In his *Essay*, Russell argues that since each of the various metrical geometries of constant curvature are logically possible descriptions of empirical space, only empirical observation can determine which set of axioms is true of space. But observation being necessarily imprecise, we can never be sure that space is exactly flat, and we can never

[10]'Presidential Address to the British Association', in *Mathematical Papers*, vol. 9, 433. Reprinted in *From Kant to Hilbert*, vol. I, edited by Ewald, 546.

[11]Cayley, 'Sixth Memoir on Quantics', in *Mathematical Papers*, vol. 2, 592. There is a good summary account in Wussing, *The Genesis of the Abstract Group Concept*, 167–78. In fact, it was this reduction of metrical to projective geometries that first allowed Felix Klein to prove the equiconsistency of Euclidean and the classical non-Euclidean geometries.

[12]As quoted by Klein, *Development of Mathematics in the Nineteenth Century*, 118.

[13]Cayley, 'Presidential Address', 547 in reprint; original edition, 434.

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rule out any of the non-Euclidean alternatives. The same kind of argument will not justify the free use of imaginary points. When we consider the work of geometers like Cayley in light of our 'perception of space' or try to give it a spatial interpretation, we expose it as a nest of absurdities: 'everyone can *see*', Russell says about the circular points, 'that a circle, being a closed curve, cannot get to infinity'.[14] He concludes that, for all of the fruitful use of imaginaries in geometry, they are merely a convenient fiction, possessing no spatial correlate, and that they are useable only as technical instruments for manipulating algebraic expressions describing actual space.

The contrast between Russell and Cayley shows that the lines one takes on the two philosophical challenges posed by the two geometrical revolutions can be independent of one another: one could be a liberal about non-Euclidean geometries and a conservative about the imaginary elements, as Russell was; or one could be a liberal about the imaginary elements and a conservative about non-Euclidean geometries, as Cayley was.[15] Now suppose that you wanted to understand how Kant's philosophy of mathematics would have to change to fit with *both* of the revolutions that moved geometry away from Kant's ancient synthetic model. And suppose further that you wanted to answer Cayley's challenge without condemning seventy-five years of revolutionary work in geometry into a formalist or fictionalist limbo. What would a Kantian position look like?

2. SOME HISTORICAL BACKGROUND: THE MARBURG NEO-KANTIAN SCHOOL AND THE FACT OF MODERN MATHEMATICS

Cassirer thinks it a condition of the adequacy of any philosophical theory of even elementary objects (like Euclidean plane figures) that they not be given a different ontological status than the more recently introduced 'imaginary' elements (like figures in the complex plane). I describe in sections 3–4 the theory that Cassirer defends. In this section, however, I lay out first the philosophical background to Cassirer's philosophy of geometry. For him, carrying out his philosophical project will require re-reading Kant in the way characteristic of the 'Marburg' or 'logical idealist' Neo-Kantian school. In particular, I consider the fundamental idea that Cassirer and his teachers thought they found in Kant: a new conception of philosophy as 'transcendental logic'.[16]

[14] *Essay*, 45, emphasis added.

[15] Cayley in fact thought that the axiom of parallels is indemonstrable and part of our notion of space ('Presidential Address', 547 in reprint; original edition, 434–5).

[16] I discuss in some more detail this conception of 'transcendental logic', and I consider its relation both to Neo-Kantian philosophy of logic and Cassirer's famous distinction between 'substance concepts' and 'function concepts', in Heis, 'Cassirer's Response to Frege and Russell'.

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Paul Natorp, who, along with Hermann Cohen, began in the 1880s the philosophical movement centred around Marburg University, characterized the ‘leading thought’ and ‘fixed starting point’ of the school as an anti-metaphysical and anti-psychologistic method – the ‘transcendental method’ or the method of ‘transcendental logic’.[17] According to this approach, philosophy begins not with *a priori* metaphysical speculation, nor with observation of the psychology of individual human subjects, but with the concrete mathematical sciences of nature. These sciences are our paradigms of knowledge and their historical achievements constitute the ‘fact’ whose preconditions it is the task of philosophy to study. Kant himself first isolated this method and applied it to the Newtonian science of his day. But since mathematical natural science has changed since Newton, there is no reason to think that the particular doctrines of the *Critique* will stay fixed for all time.

> For the ‘fact’ of science according to its nature is and remains obviously a historically developing fact. If this insight is not unambiguously clear in Kant, if the categories appear to him still as *fixed* ‘stem-concepts of the understanding’ in their number and content, the modern development [*Fortbildung*] of critical and idealistic logic has brought clarity on this point.[18]

Because philosophy has no independent grip on the human understanding or on the nature of objects of experience, there is no leverage by means of which philosophy could condemn an established result of the sciences, in particular, an established result of mathematics, as false or meaningless. Any adequate philosophical study of mathematics will have to recognize that ‘the modern development of mathematics has thus created a new “fact”, which the critical philosophy, which does not seek to dominate [*meistern*] the sciences but to understand them, can no longer overlook’.[19]

Students of the history of analytic philosophy will likely note a resemblance between Cassirer’s attempt to *historicize* Kant’s transcendental logic and the better-known attempt by Hans Reichenbach to *relativize* Kant’s theory of synthetic *a priori* judgments.[20] Indeed, many interpreters have claimed that Reichenbach’s theory of the relativized *a priori* was anticipated by Cassirer, from whom Reichenbach had in fact taken classes

[17]Natorp, ‘Kant und die Marburger Schule’, 196. See also Cassirer, *Das Erkenntnisproblem in der Philosophie und Wissenschaft der neuren Zeit*, vol. 1, 3rd edn, 14–18. (I abbreviate the volumes of *Erkenntnisproblem* as ‘*EP*’.) A recent discussion is Alan Richardson’s ‘“The Fact of Science” and Critique of Knowledge: Exact Science as Problem and Resource in Marburg Neo-Kantianism’.

[18]Cassirer, *EP* I, 18.

[19]Cassirer, ‘Kant und die moderne Mathematik’, 31. (I abbreviate this paper as ‘KMM’.)

[20]Reichenbach distinguishes between a priority as necessary and unrevisable validity, and a priority as ‘constitutive of the object of knowledge’. See *Relativitätstheorie und Erkenntnis apriori*, Chapters V–VII. Reichenbach rejects the first conception and retains the second.

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when he was a student in Berlin.[21] However, the Marburg transcendental project differed in a subtle but important way from Reichenbach's early Kantianism, and getting clear on this difference is an excellent route to understanding Cassirer's view more clearly.

Reichenbach thought that there need to be non-empirical principles that, though revisable in periods of conceptual revolution, are nevertheless constitutive of the framework of empirical investigation at a given stage in the history of science, and thus make possible objective scientific experience. Compare this with the Marburg school's different historicized understanding of Kant's project.

> The task, which is posed to philosophy in every single phase of its development, consists always anew in this, to single out in a concrete, historical sum total of determinate scientific concepts and principles the general logical functions of cognition in general. This sum total can change and has changed since Newton: but there remains the question whether or not in the new content [*Gehalt*] that has emerged there are some maximally general relations, on which alone the critical analysis directs its gaze, and that now present themselves under a different form [*Gestalt*] and covering. The concept of the *history of science* itself already contains in itself the thought of the *maintenance of a general logical structure* in the entire sequence of special conceptual systems.[22]

Like a relativized *a priori* programme, the Marburg school's transcendental method requires determining which concepts and laws play the role of Kant's categories and principles at some given stage in the history of science. Just as Kant had given a defence of the non-empirical truth of both Euclidean geometry and the principle of causality by showing that they are preconditions of the possibility of Newtonian science, so too Cassirer claims that Riemannian differential geometry and Einstein's principle of general covariance are conditions of the possibility of general relativity.[23]

But in the second and third quoted sentences, we find a further demand: we need to be able to substantiate the claim that the history of science is the history of one subject and forms one series of events. When Cassirer is taking on the responsibility to substantiate this claim, he is doing more than simply 'relativizing' Kant's critical project. Indeed, this further demand is meant to neutralize the threat that the relativity of constitutive concepts and principles will force a more troubling relativism. A fundamental thought of Kant's Copernican revolution is that the '*a priori* conditions of a possible

[21]Richardson, *Carnap's Construction*, Chapter 5; Ryckman, *Reign*, Chapter 2; Padovani, 'Relativizing the Relativized A Priori: Reichenbach's Axioms of Coordination Divided'.

[22]Cassirer, *EP* I, 16.

[23]*Zur Einstein'schen Relativitätstheorie*, translated by Swabey and Swabey in *Substance and Function & Einstein's Theory of Relativity*, 415. I abbreviate this book as '*ETR*' and cite from the English translation.

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experience in general are at the same time conditions of the possibility of the objects of experience'[24]; it follows that the constitutive concepts and principles make possible not only experience – and so, science – but also the objects of experience – and so, nature. Now Cassirer thinks that it is obvious that twentieth-century scientists were trying to understand the same world as Newton. But relativizing Kant's transcendental logic seems to make this obvious fact puzzling. Moreover, Kant's Copernican revolution prevents us from solving this puzzle by trying to explain this unity of science over time in terms of a common subject, the world-spirit, or a common subject matter, the world.

Cassirer's solution is that there has to be 'a general logical structure' – some common set of principles and concepts – present in all phases of the history of science.[25] What constitutes this 'general logical structure' in the case of natural science is not the topic of this paper, which is not concerned with Cassirer's philosophy of science. When we move to Cassirer's philosophy of mathematics, however, we can draw the following conclusions. Though a properly Neo-Kantian philosophy of mathematics will appreciate that mathematics itself has undergone fundamental conceptual changes throughout its history, such a philosophy will also have to substantiate the claim that the various stages in the historical development of mathematics constitute *one history*. Now, Kant's Copernican revolution excludes an explanation of the 'unity' of mathematics – the fact that the various activities performed by mathematicians over time all belong together as part of one history – in terms of a common domain of objects. It is not false to say that there are geometrical objects, and that geometers from Euclid to Cayley were all discovering further properties of these objects. But the fact that both nineteenth-century projective geometers and ancient geometers were studying space does not *explain* why Euclid and Cayley belong together as participants in one *history*. As Cassirer puts it, 'the unity of mathematics no longer lies in its object – whether it be the study of magnitude and number, the study of extension, as the general theory of manifolds, the theory of motion, or equally as the theory of forces.'[26] Rather the explanation has to go in the opposite direction: we can say that they were studying the same objects only because we can say that they are parts of the same history.[27]

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[24]Kant, *Critique of Pure Reason*, A111. I refer to passages from Kant's *Critique of Pure Reason* (translated by Guyer and Wood) using the standard pagination from the first ('A') and second ('B') editions.

[25]See also *Substanzbegriff und Funktionsbegriff. Untersuchungen über die Grundfragen der Erkenntniskritik*, translated by Swabey and Swabey in *Substance and Function & Einstein's Theory of Relativity*, 321–2. (I abbreviate this book as '*SF*' and cite from the English translation.) On this point, see Friedman, 'Ernst Cassirer and the Philosophy of Science'.

[26]KMM, 31. I return to the various senses in which one might speak of the 'unity' of mathematics in section 4.

[27]The Marburg Neo-Kantians called this general approach 'logical idealism'. It is idealism not because the activities of scientists create or produce the world; rather, our *explanation* of the existence of a common subject matter for our science *depends* on our being able to explain the

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What does explain the unity of mathematics will then have to be what Cassirer calls a 'general logical structure' present throughout the history of mathematics. Cassirer expresses this point by saying that the unity of mathematics lies in its *method*.[28] As I hope to make clear, Cassirer's account of the mathematical method is twofold. On the one hand, he argues that mathematicians proceed by introducing abstract relational structures. On the other hand, he argues that there are real *mathematical* reasons that explain why mathematicians study particular structures in particular contexts, and he wants to uncover these *methodological* principles that guide mathematical progress over time. These two aspects of the mathematical method will provide the 'logical structure' for the historical development of mathematics, and reflection on the 'logic' of mathematical progress will make it clear why the so-called 'imaginary elements' are not ontologically any more suspect than so-called 'real elements'. The first aspect will be the topic of the next section; the second aspect the topic of the section following it.

3. KANT'S MATHEMATICAL METHOD AND CASSIRER'S DEDEKINDIAN PARADIGM

Cassirer thinks that Kant's theory of the mathematical method – his view of the distinctive kind of concepts and concept use that characterizes mathematics – will help us make sense of the radical conceptual innovations one finds in nineteenth-century mathematics. Surprisingly, Cassirer singles out Dedekind's foundations of arithmetic – which Cassirer takes as a model for the methodology of all modern mathematics – as a modern version of Kant's idea that mathematics is cognition from the 'construction of concepts'. This is a surprising move, since Dedekind thought himself to be defending a kind of logicism – a position usually thought to be strongly opposed to Kantianism. I'll begin this section by discussing Kant's position, and then I'll show how Cassirer thinks he finds this position expressed in Dedekind's foundations of arithmetic.

For Kant, what makes mathematics distinct from other disciplines is its form: the particular way in which concepts are employed in it and thus also the particular kind of concepts that it makes use of. We can think of this form of concept deployment, 'rational cognition from the construction of

unity of our scientific activities. As Cassirer puts it (*Philosophie der Symbolischen Formen. Erster Teil: Die Sprache*, translated by Hendel and Manheim as *Philosophy of Symbolic Forms. Volume 1: Language*, 78–9): 'The 'revolution in method' which Kant brought to theoretical philosophy rests on the fundamental idea that the relation between cognition and its object, generally accepted until then, must be radically modified ... [T]he crucial question always remains whether we seek to understand the function by the structure or the structure by the function, which one we choose to ''base'' upon the other ... [T]he fundamental principle of critical thinking [is] the principle of the ''primacy'' of the function over the object'.

[28]KMM, 31.

concepts' (A713/B741), as made up of two parts. We say that a concept F is mathematical, or is being used mathematically, when (1) an object *a* contained under F can be exhibited *a priori*,[29] and (2) I can reason to new facts about all objects contained under F merely by reasoning about this *a*.[30] We can illustrate this picture using Kant's Euclidean paradigm. I possess the concept <circle>[31] when I have grasped the Euclidean postulate that enables me to construct, without the aid of experience, a circle with a given centre and radius.[32] Once I've drawn this circle, I can prove something new about it: for instance, that this circle *a*, together with a circle *b* whose centre lies on *a* and that intersects the centre of *a*, forms an equilateral triangle whose three vertices are the two centres and an intersection point of *a* and *b*. And further, this fact about *a* holds for all objects that are contained under <circle>, even if they differ greatly from *a* in magnitude or position. For Kant, this proof procedure has implications for what mathematical concepts must be like: they are not 'given' but 'made arbitrarily'.[33] A concept that is given, such as an empirical concept like <gold>, is possessed by the subject *before* she comes to arrive at its definition, which can then only be reached, if at all, by slow and uncertain analysis. Mathematical concepts, on the other hand, are 'made' concepts: 'in mathematics, we do not have any concept at all prior to the definitions, as that through which the concept is first given'.[34] I cannot have the definition of the concept <circle> only *after* I begin reasoning about circles, because I need to be able to keep track of precisely which features of this circle can be generalized to all circles. Since mathematical concepts are made arbitrarily, there is nothing contained in the concept that I am not aware of, and so I can be completely certain of what holds for the individual *a* solely in virtue of the fact that it is an F.[35]

Another distinctive feature of mathematical concepts also follows from these considerations: the possession of the concept <circle>, for example, is both sufficient and necessary for the representation of particular circles.

[29]See A713/B741; *Logic* Ak 9: 23, translated by Michael Young in *Lectures on Logic*, 536. I refer to passages in Kant's *Logic* using page numbers from the standard Akademie ('Ak') volumes, or simply by paragraph ('§') numbers, where applicable.

[30]A714/B742: 'Thus philosophical cognition considers the particular only in the universal, mathematical cognition considers the universal in the particular, indeed even in the individual, yet nonetheless a priori and by means of reason, so that just as this individual is determined under certain general conditions of construction, the object of the concept, to which this individual corresponds only as its schema, must likewise be thought as universally determined.'

[31]I refer to concepts in brackets.

[32]A713/B741.

[33]See *Logic*, §102. At A729/B757, Kant says that only mathematical concepts contain 'an arbitrary synthesis that can be constructed *a priori*'.

[34]A731/B759.

[35]A729–30/B757–8: '[O]nly mathematics has definitions. For the object that it thinks it also exhibits *a priori* in intuition, and this can surely contain neither more nor less than the concept, since through the definition of the concept the object is originally given, i.e., without the definition being derived from anywhere else.'

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Mathematical concepts differ from other made concepts: if I introduce the concept <golden mountain> by stipulation, I am not thereby able to produce an object that is a golden mountain; but according to (1), I am able to do so with mathematical concepts. Further, the representation of a particular circle is only possible if it is constructed according to the rules internal to the concept <circle>.[36] With empirical concepts, it is possible for me to intuit a house without possessing the concept <house> and so without having the capacity to intuit it *as a house*.[37] Not so with mathematical concepts. I cannot even intuit a circle unless I already possess the mathematical concept <circle> and have used it in constructing the circle.[38]

In sum, then, what is distinctive about mathematics is a certain relation between concepts and objects, universals and particulars. In mathematics, possessing a concept is prior to representing the objects that are contained under it. Once I possess the concept, I can exhibit or construct the objects falling under the concept; and since I construct the objects falling under the concept, as soon as I possess the concept, I can be assured of the existence of objects falling under the concept. Moreover, all of the (mathematical) properties that belong to the objects contained under the concept belong to it in virtue of the fact that it is contained under the concept.

Cassirer saw substantial overlap between Kant's theory of the mathematical method and Dedekind's foundations of arithmetic. Since this connection seems implausible at first, it will help to get some material out on the table. In perhaps the most famous passage in Dedekind's *Was sind und was sollen die Zahlen?*, Dedekind writes:

> If in the consideration of a simply infinite system N set in order by a transformation ϕ we entirely neglect the special character of the elements; simply retaining their distinguishability and taking into account only the relations to one another in which they are placed by the order-setting transformation ϕ, then are these elements called *natural numbers* or *ordinal numbers* or simply *numbers*, and the base-element 1 is called the *base-number* of the *number-series* N. With reference to this freeing the elements from every other content (abstraction) we are justified in calling numbers a free creation of the human mind.[39]

The picture Dedekind lays out in this passage has three elements.[40] First, the abstraction Dedekind describes takes us from a given simply infinite

[36]B154–5: 'We cannot think of a line without *drawing* it in thought, we cannot think of a circle without *describing* it, we cannot represent the three dimensions of space at all without *placing* three lines perpendicular to each other at the same point.'

[37]*Logic*, Ak 9: 33.

[38]See here A730/B758.

[39]Dedekind, 'The Nature and Meaning of Numbers', §73 (page 68), in *Essays on the Theory of Numbers*, 29–115, translated by Beman.

[40]I cannot defend this reading here. A defence can be found in Erich Reck's 'Dedekind's Structuralism: An Interpretation and Partial Defense'.

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system[41] (it does not matter which) to a *new* system of objects, which differ from all other simply infinite systems in that their elements – the numbers – have no more properties or no more of a *nature*, than can be expressed in terms of the basic relations described by the axioms for simply infinite systems. Each of a natural number *n*'s *essential* properties is a *relational* property of *n* to *some other natural number m*.[42] Second, once we have described what a simply infinite system is – effectively, once we have isolated Dedekind's axioms for simply infinite systems – and have assured ourselves of the consistency of this notion, we can be assured of the *existence* of the numbers, those objects which constitute that simply infinite system all of whose elements have no more of a nature than their positions within the system. Third, since we introduce the natural numbers all at once by abstraction from a simply infinite system, the nature and identity of all of the numbers is determined together: *reference* to each individual number is only possible in the context of the whole system of numbers and the axioms describing them. (I couldn't come to refer to the natural numbers 'one at a time'.)

Cassirer endorses each of the three elements in Dedekind's foundations for arithmetic – both for arithmetic *and* for geometry.[43] Moreover, he thinks that Dedekind's approach is a version of the Kantian idea that mathematicians begin with defined concepts and then straightaway 'construct' the objects falling under them.[44] For Kant, as soon as I possess

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[41]Informally, a 'simply infinite system' is a model of the Peano axioms. Less informally, a simply infinite system is a set of objects N, along with a 1–1 mapping ϕ, such that the image of ϕ is N – {1}, for some element 1 in N, and such that N is the smallest such set. The natural numbers are a simply infinite system, because the function that takes every number n to its successor (n + 1) is a 1–1 mapping from the natural numbers to itself, and the set of successors is the set that includes every natural number except 1. There are other models of the Peano axioms besides the natural numbers, but Dedekind's point is that the natural numbers are unique among models of the Peano axioms because they have no other nature besides what is characterized in the Peano axioms.

[42]A natural number can of course have *non-essential* relational properties with objects that are not numbers: for instance 1 is related to the moon such that 1 is the number of moons of the earth. But this is not an essential property of 1.

[43]See *SF*, 36: '[The numbers] are not assumed as independent existences [*selbstständige Existenzen*] present prior to any relation, but they gain their whole being [*Bestand*], so far as it comes within the scope of the arithmetician, first in and with the relations, which are predicated of them. They are *terms of relations* [*Relationsterme*] that can never be "given" in isolation but only in community with one another.'

[44]Cassirer uses unmistakable Kantian language to describe Dedekind's approach. At *EP II*, 714, Cassirer says that Dedekind's foundations of arithmetic are expressions of Kant's idea that mathematical concepts 'owe their existence not to abstraction but to construction [*Konstruktion*]'.

See also Cassirer, *Kants Leben und Lehre*, translated by Haden as *Kant's Life and Thought*, 159: 'Here [in *a priori* synthesis] we begin with a specific constructive connection, in and through which simultaneously a profusion of particular elements, which are conditioned by the universal form of the connection, arises for us. [. . .] We think the construction of the system of natural numbers, according to one basic principle, and we have included in it from the first, under the mathematical concept <triangle>, I can introduce objects, triangles, falling under it. On Cassirer's Dedekindian picture, the *existence* of mathematical objects, like natural numbers, follows from the fact that we can speak consistently about the system of mathematical concepts.[45] For Kant, mathematical concepts, like <triangle>, make possible the representation of the objects, the triangles, falling under them. For Cassirer, *reference* to mathematical objects, like natural numbers, is always in the context of some background system of concepts. For Kant, all of the mathematically relevant properties that a triangle has, it has in virtue of its falling under the concept <triangle>. Similarly, for Cassirer, mathematical objects, like a natural number, have no more properties, or no more of a *nature*, than can be expressed in terms of the basic relations of the system of concepts.

There is an important difference between Kant's account and the Dedekindian picture. The idea that a mathematical object has no non-structural properties does not fit with Kant's view that the objects of geometry are constructed in the one, infinitely given *locus in quo* of *empirical* objects. On this view, since points in space are necessarily locations that *physical objects* can occupy, geometrical objects would have essential relations to non-geometrical objects. Here Cassirer departs from Kant: for him, Kant was wrong to think that geometry is possible only if, as the 'Transcendental Aesthetic' claims, we can represent in an *a priori* way physical or empirical space and time. Like Russell in his logicist period, Cassirer argues that the development of non-Euclidean geometry shows that the propositions of pure mathematics – pure mathematics, as a non-empirical science – are not *about* physical space and time.[46] There are many logically possible spaces, and no mathematically possible geometry is any more true or real than another.[47] This concession, however, does not touch the fundamentally 'constructive' character of geometry. The claim that geometry is about empirical space is, for Cassirer, independent of the idea that mathematicians proceed constructively from arbitrarily defined concepts.[48] And so we can reject the former while retaining the latter.

definite conditions, all the possible relations between the members of this set.' Without the reference to *a priori* synthesis, this passage might just as easily be describing Dedekind's approach or Kant's. This is Cassirer's point.

[45]As I'll show in the next section, Cassirer's condition is actually stronger than consistency here.

[46]*SF*, 106: 'The role, which we can still ascribe to experience, does not lie in grounding the particular systems, but in the selection that we have to make among them.' When Russell moved to logicism around 1900, he rejected *both* the necessity of pure intuition in geometry *and* the Kantian philosophy of geometry of his early *Essay* – indeed, he rejected the latter partially *because* he rejected the former. Cassirer's Kantianism, on the other hand, does not allow for such an inference.

[47]*ETR*, 432.

[48]Of course *Kant* thought the two were linked: he thought that our pure intuitions of space and time make possible mathematical constructions. The point, though, is that there needs to be an

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Cassirer's position is thus completely different from the better-known attempts by Kantian philosophers to 'plug the leaks'[49] by identifying weaker and weaker fragments of Euclidean geometry that could function as the *a priori* theory of empirical space. For Cassirer, there can be no mathematical propositions – not even the axioms of geometry – grounded in the pure or empirical intuition of space.[50] (Nevertheless, the further progress of mathematics – represented by Dedekind's foundations of arithmetic – has just further confirmed Kant's idea that mathematics is fundamentally constructive.)

Cassirer's appropriation of Dedekind's model removes the largest obstacle that could prevent us from ascribing to the ideal elements in geometry the same ontological status as that ascribed to the real elements.

> If even the elementary forms of mathematics, the simple arithmetical numbers, the points and straight lines of geometry, are understood not as individual things, but only as links in a system of relations, then the ideal elements may be said to constitute 'systems of systems.' They are composed of no different logical stuff than these elementary objects, but differ from them only in the mode of their interconnection, in the increased refinement of their conceptual complexion.[51]

If neither elementary Euclidean geometry nor complex projective geometry is about empirical space, then we do not need an interpretation in empirical space of the so-called 'imaginary' points to talk meaningfully about them. There is now nothing metaphysically suspect to explain away.

4. KANT'S MATHEMATICAL METHOD AND A WIDER CONCEPTUAL SETTING FOR AN OBJECT

We saw in section 2 that Cassirer's appreciation of the historical evolution of mathematics forced him to confront the philosophical problem of the 'unity' of mathematics over time, and we saw that his solution was to posit a 'general logical structure' – a distinctive 'method' – for mathematics. We have now seen the first element in Cassirer's story: a method distinctive of mathematics that is modelled on Dedekind's foundations of arithmetic.

argument connecting the two ideas, and there is thus logical space for Cassirer to reject that argument.

[49]This phrase is from Alberto Coffa, *The Semantic Tradition*, 57. See KMM, 3.

[50]See *Philosophie der Symbolischen Formen*, translated by Hendel and Manheim as *Philosophy of Symbolic Forms* (hereafter abbreviated '*PSF*') vol. 3, 363. On Cassirer's rejection of pure intuition in mathematics, and the relation between this rejection and the distinctive doctrines of Marburg Neo-Kantianism, see Heis, 'Cassirer's Response to Frege and Russell'.

[51]*PSF* 3, 395.

But this is not a complete characterization of the method of mathematics, and it does not suffice for explaining the unity of mathematics over time. Cassirer wants to acknowledge the fact that mathematics is a 'synthetically progressive science',[52] and he wants to be able to respect the conviction that later developments, like the discovery of the wider projective realm, allow mathematicians to better understand the objects and problems their predecessors had worked on already.

Clearly, the progress of mathematics consists in more than discovering new theorems from given axiom systems and introducing new axiom systems for study. As Hilbert put it in the concluding paragraph of his famous lecture on mathematical problems:

> ... [M]athematical science is in my view an indivisible whole, an organism, whose viability is based on the connection of its parts. For in all the distinctions in the matter of mathematical thinking in particular cases, we are still aware very clearly of the identity of the logical tools, the relationship of the ideas in all of mathematics, and the numerous analogies in its various regions. Also: the further a mathematical theory develops, the more harmonious and unitary its structure arranges itself to be, and unknown relations are discovered between hitherto separated branches of knowledge. And thus it comes about that the unitary character of mathematics is not lost in its expansion, but rather becomes ever clearer.[53]

Hilbert thought that the development of mathematics over time – and this is part of what makes the historical development of mathematics progressive – brings with it a new 'unity' or 'harmony' among its different branches.

Cassirer shares this sentiment,[54] and he thinks that it will play a key role in finding a Kantian way to justify the widespread practice of adjoining ideal elements in geometry. In an important passage, which will take us a bit of time to unpack, Cassirer indicates the conditions that new elements must fulfil within mathematics.

> But here the philosophical critique of knowledge must raise still another and sharper demand. For it is not enough that the new elements should prove equally justified with the old, in the sense that the two can enter into a connection that is free from contradiction – it is not enough that the new

[52]Cassirer, *PSF* 3, 398.

[53]Hilbert, 'Mathematische Probleme', 329.

[54]In a posthumously published essay written late in his career, Cassirer writes: 'The presupposition of a homogeneous universal inter-connection constitutes the problem of mathematics and at the same time its fundamental postulate. Mathematics does not possess this postulate from the beginning, but it demands it; and this demand constitutes for it its methodological leading thread that it makes use of on its way' (*Ziele und Wege der Wirklichkeitserkenntnis*, 40).

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> should take their place beside the old and assert themselves in juxtaposition. This merely formal combinability would not in itself provide a guarantee for a true inner conjunction, for a *homogeneous logical structure of mathematics*. Such a structure is secured only if we show that the new elements are not simply adjoined to the old ones as elements of a different kind and origin, but the new are *a systematically necessary unfolding of the old*. And this requires that we demonstrate a primary logical kinship between the two. Then the new elements will bring nothing to the old, other than *what was implicit in their original meaning*. If this is so, we may expect that the new elements, instead of fundamentally changing this meaning and *replacing* it, will first bring it to *its full development and clarification*.[55]

It will be helpful for understanding Cassirer's idea to distinguish three senses in which we can talk about the 'unity' of mathematics. First, mathematics is a unity in the sense that the different activities called 'mathematics' belong together and are not just arbitrarily collected together: there is a single, unified scientific discipline, 'mathematics', and we can make a principled distinction between what belongs in mathematics and what belongs in the other special sciences.

Second, mathematics is a unity in the sense that advances in mathematics give us new ways of talking about old objects, and do not just switch us to a new subject altogether. The same mathematical problems can be treated in different ways; the same mathematical objects can be embedded in different mathematical structures. In the physical case, Cassirer worries that, without insisting on the persistence of a common stock of invariant cognitions, relativizing Kant's *a priori* concepts and principles will make nonsense of the claim that later physicists are able to understand better the very same world that their predecessors knew. Similarly, Cassirer wants to respect the conviction of mathematicians that a new structure being introduced, like complex projective geometry, is '*a systematically necessary unfolding of the old*' that express '*what was implicit in* [*its*] *original meaning*' – that *new* systems of concepts deepen our understanding of the concepts in terms of which our *old* objects, like the circle, get their meaning.

> In all these cases, logical justification of the new elements is to be found not in the fact that the new dimension in which we are beginning to envisage things somehow *displaces* the relations that were valid within the former dimension, but rather in the fact that it sharpens our eye for them as they are. The look backward from the newly opened field to the old one *first opens up the old field* in its entirety to our thinking and gives us an understanding of its finer structural forms.[56]

[55] *PSF* 3, 392.
[56] *PSF* 3, 393 (emphasis added).

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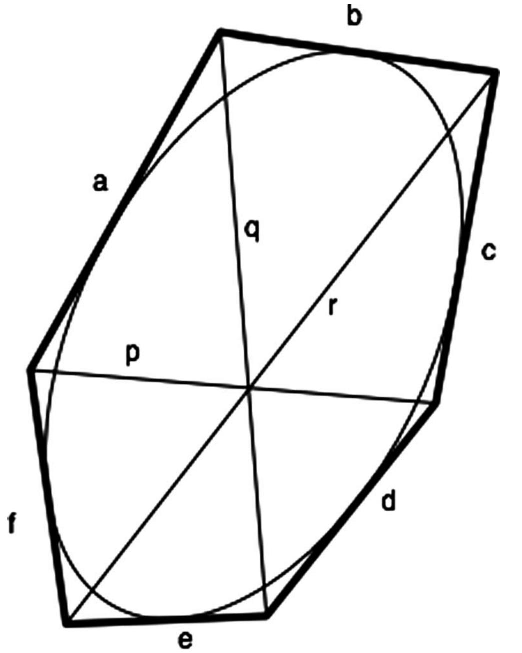

Figure 1. Brianchon's theorem.
The diagonals of a hexagon circumscribed about a real, non-degenerate conic meet in one point, prq.

Thus, Cassirer wants to avoid the view that each newly introduced structure of concepts is a *replacement* of the old structure – that with new structures of concepts, mathematicians just change the subject.

Third, mathematics is a unity in the sense that, as Hilbert puts it, its different parts are connected to one another. Different subfields draw on the same concepts or the same proof techniques. Within a subfield, different phenomena are not simply isolated facts but are united through a new concept or a new proof technique.

For Cassirer, we can understand what it means for one system of concepts 'to make explicit what was implicit in the meaning of an old system' – the unity of mathematics in the second sense – by thinking through what it is for a new system of concepts to unify mathematics in the third sense. To illustrate, let's return to our case study – a case explicitly discussed by Cassirer – and consider in more detail the unification effected by the introduction of new points in projective geometry. Figure 1 gives a diagrammatic representation of Brianchon's theorem, which states that if a hexagon is circumscribed about a conic, its three diagonals meet in one point. Replacing the word 'point' for 'line' (its 'dual') in Brianchon's

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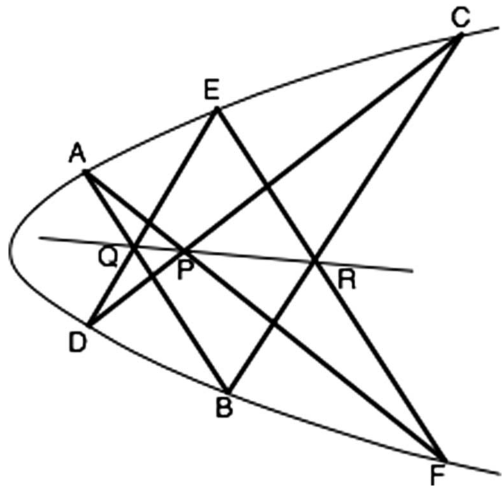

Figure 2. Pascal's theorem.
The intersections of the sides of the hexagon inscribed within a real, non-degenerate conic lie on a line, PRQ.

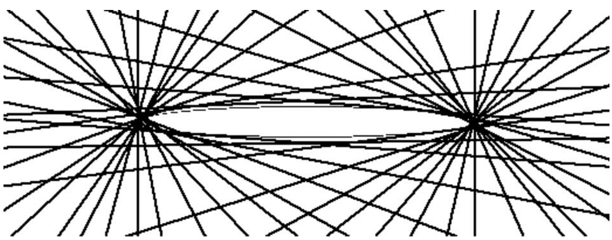

Figure 3. A point pair as a degenerate conic.

theorem, we get Pascal's theorem, that if a hexagon is inscribed within a conic, its three pairs of opposite sides meet in collinear points.[57] In Figure 2, Pascal's theorem is illustrated with a parabola, but this projectively makes no difference (even if it looks different). Now the projective definition of a conic due to Steiner allows for degenerate cases of conics, either as a pair of points or, as in the case we want to consider, a pair of lines. We can make this intuitive by thinking of a hyperbola getting closer and closer to its

[57]In pre-projective geometry, there is an asymmetry between points and lines because any two points determine a line, but only pairs of non-parallel lines determine a point. By introducing points at infinity, any two lines intersect in one point. We can then systematically replace all instances of the word 'line' in plane projective geometry with its *dual*, 'point', without affecting the truth of any of our theorems or the validity of any of their proofs. In fact, Pascal proved his theorem in 1640, and Brianchon proved his in 1806 by dualizing Pascal's.

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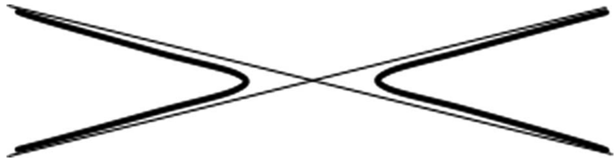

Figure 4. A line pair as a degenerate conic.

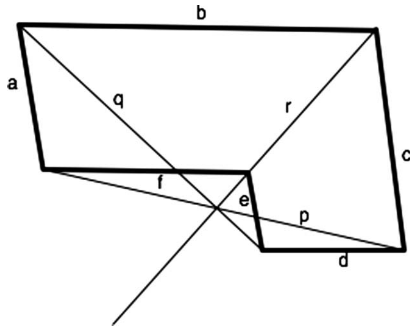

Figure 5. Brianchon's theorem, degenerate case.
The six sides of a hexagonal parallelogram are circumscribed about a pair of points infinitely far away. The diagonals of the hexagon meet in one point.

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asymptotes until it coincides with them, as in Figure 4, or an ellipse getting thinner and thinner until it reduces to two points, as in Figure 3.[58] Thus rephrasing Pascal's theorem for the degenerate case, we see, as in Figure 6: if the six points of a (self-intersecting) hexagon lie on two lines, its three pairs of opposite sides meet in collinear points. (This special case of Pascal's theorem is in fact Pappus' theorem, which had been proved, though by different methods, already in the ancient world.) Dualizing back to Brianchon's theorem, we see: if the six sides of a hexagon intersect in two points, its three diagonals meet in one point. By making free use of the points at infinity, we can let these two points move off the page to infinity, and we get a hexagon composed of parallel lines, as in Figure 5.[59]

[58]For a modern treatment, see Coxeter, *Projective Geometry*, 89–90; projectively, the difference between a degenerate conic and a non-degenerate conic is the difference between a locus of intersections of corresponding lines of two projective *and perspective* pencils, and a locus of intersections of corresponding lines of two projective *but not perspective* pencils. This difference has no effect on the proof of Pascal's theorem.

[59]The use of Brianchon's and Pascal's theorems in the degenerate case to illustrate the power of projective thinking, and, more generally, the power of using ideal elements, is due to David Hilbert's lecture, 'Die Rolle von Idealen Gebilden', delivered in 1919–20 and published in *Natur und mathematisches Erkennen*, edited by David E. Rowe.

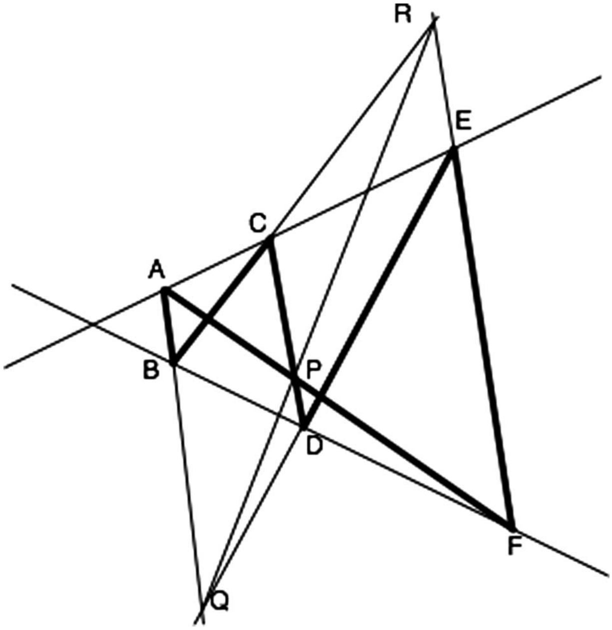

Figure 6. Pascal's theorem, degenerate case.
The six vertices of the overlapping hexagon are inscribed within the line pair, ACE and BDF. The intersections of the sides of the hexagon lie on one line.

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Projectively, these four cases (Pascal's and Brianchon's theorem using non-degenerate conics, and Brianchon's and Pascal's theorem using degenerate conics) are just instances of the same theorem – even if they appear differently at first, and, more importantly, even if they look quite different on the page. By adding infinite points and treating them like any other point, the tedious distinguishing of cases characteristic of pre-projective synthetic geometry is avoided.

With such striking cases of simplification, unification, and generalization in hand, it is easy to understand why the geometer Jakob Steiner, in an enthusiastic passage that Cassirer discusses, thought that modern, projective geometry had uncovered the organism of space, those fundamental relations by which, with great simplicity and parsimony, we can bring order to the chaos of geometrical theorems.[60] Steiner sees the importance of his work in

[60]'The present work has sought to discover the organism by which the most varied phenomena in the world of space are connected to each other. There are a small number of completely simple fundamental relations in which the schematism reveals itself and from which the remaining mass of propositions can be logically and easily developed. By the proper

uncovering the organic unity of geometry or the inner connections of various geometrical phenomena, and thinks that this shows that he has hit on the nature or essence of geometrical objects.

As Cassirer puts it, the new elements are 'an intellectual medium by which to apprehend the true meaning of the old, by which to know it with a universality and depth never before achieved'.[61] One might cash out these metaphors in particular cases by showing of a particular structure of concepts that it allows us to generalize a theorem, or synthesize apparently different mathematical phenomena under one point of view.[62] In our example, introducing points at infinity allows us to unify a series of geometrical theorems together under Pascal's and Brianchon's theorems. Alternately, new elements might reveal a previously hidden relationship between particular mathematical *disciplines* and effect a 'closer and more profound union among them'.[63] As Cassirer notes, the projective use of imaginary elements – like the 'ideal intersection' of two circles that have come apart – provides a distinctly geometrical justification for introducing points whose coordinates are given by complex numbers. This, in turn, allows geometers to make use of the independently discovered algebraic work on complex numbers – thus 'uniting' the two areas of research.[64]

Cassirer thus gets a Kantian picture of what makes mathematics a progressive, unitary discipline by seeing Kant's 'synthesis' as acting *developmentally*, as a deeper or wider system of concepts is synthesized or constructed out of the old.

> What Kant meant is that the distinctive, fundamental character of mathematical synthesis, which he was aiming to explain, comes to light ... in the building up of the mathematical *world of objects*. The formation of the objects of mathematics is 'constructive,' and hence 'synthetic,' because it is not concerned simply with analyzing a given concept into its marks, but because

appropriation of a few fundamental relations, one becomes the master of the whole object; order comes out of chaos, and we see how all the parts naturally fit together, form into series in the most beautiful order, and unite into well-defined groups of related parts. In this way, we come, as it were, into possession of the elements that nature employs with the greatest possible parsimony and simplicity in conferring to figures their infinitely many properties.' Steiner, *Systematische Entwicklung der Abhängigkeit geometrischer Gestalten von einander* (1832), reprinted in *Gesammelte Werke*, vol. 1, edited by K. Weierstrass, 229; my translation. Cassirer quotes and discusses this passage in *The Problem of Knowledge: Philosophy, Science, and History since Hegel*, 50.

[61] *PSF* 3, 393.

[62] See *PSF* 3, 397.

[63] *PSF* 3, 399.

[64] Observations like these would be the starting point for any theory of what it is for a mathematician to have understood or explained a problem or theorem better than he would have otherwise. Cassirer himself discusses various cases, but gives no worked-out theory.
Cassirer's ideas will remind contemporary philosophers of mathematics of the theory of mathematical explanation by 'unification' that was proffered by Philip Kitcher in his *The Nature of Mathematical Knowledge*. (Kitcher was drawing on earlier work by Michael Friedman.)

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> we advance and ascend from determinate fundamental relations, from which we begin, to ever more complex relations, where we let each new totality of relations correspond to a new realm of 'objects'. For every combination, for every new synthesis, there arises a corresponding object which in a strictly methodological sense develops out of preceding ones but in no way coincides with them logically.[65]

The key idea is that new concepts are justified when they provide a deeper understanding of our old objects. In Kant's simpler Euclidean model, the concept <circle> is *prior* to the object, a circle, in the more straightforward sense that we can only represent a circle by constructing it using the concept <circle>. In Cassirer's more sophisticated developmental model, the projective concept <(possibly degenerate) Steiner conic> is not *temporally prior* to the object, a circle, but one might say *logically* or *essentially prior*, in the sense that a circle becomes fully intelligible mathematically only within this wider setting. Using Cassirer's phrase, the *ratio essendi* of the new elements is given by the old elements, but the *ratio cognoscendi* of the old elements is given by the new elements.[66]

I earlier remarked that for Cassirer the fact that a system of concepts is consistent is not sufficient for proving the existence of objects falling under it, and now we see why. A proof (like that given by von Staudt) that complex projective geometry is equiconsistent with Euclidean metric geometry demonstrates, as Cassirer puts it, only the 'can', not the 'must'.[67] But, for the reasons we've just illustrated, Steiner and his successors did not just think that conics *could* be understood projectively; they thought they *had* to be. We thus justify the existence of the new objects in terms of their role in systematizing or explaining the old elements; but once we have the new elements, we see that the old elements become more fully intelligible or understandable in terms of the new.

> The imaginary intermediate members always serve to make possible insight into the connection of real geometrical forms, which without this mediation would stand opposed as heterogeneous and unrelated. It is this ideal force of logical connection that secures them full right to 'being' in a logico-geometrical sense. The imaginary exists, insofar as it fulfils a logically indispensable function in the system of geometrical propositions.[68]

With this, Cassirer's two-stage philosophical justification of the ideal elements in geometry is complete. On the one hand, metaphysical objections to n-dimensional complex projective spaces misfire, because, when we

[65] *Problem of Knowledge*, 75.
[66] *PSF* 3, 393.
[67] *Ziele und Wege*, 37.
[68] *SF*, 83.

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understand the mathematical method as 'constructive' on the model of Dedekind's foundations of arithmetic, these spaces are no different in character from ordinary Euclidean three-dimensional space. On the other hand, as we see in this passage, these ideal elements are acceptable (indeed: necessary) objects of geometrical research because of the work they do unifying and explaining facts about the so-called 'real' points.

Once we have embedded our old objects within a new structure of concepts, the new objects that fall under the widened conceptual structure become independent objects of study, which can themselves be embedded in wider or alternative structures of concepts. We thus get a hierarchical picture of the development of mathematics, as mathematicians discover newer concepts by means of which to construct familiar objects. It is this whole connected process that gives us the mathematical method.

> The actual intellectual miracle of mathematics is that this process [of positing new conceptual structures and the objects that fall under them] ... never finds an end but is repeated over and over, always at a higher level. It is this alone that prevents mathematics from freezing into an aggregate of mere analytical propositions and degenerating into an empty tautology. The basis of the self-contained unity of the mathematical method is that the original creative function to which it owes its beginning never comes to rest but continues to operate in ever new forms, and in this operation proves itself to be one and the same, an indestructible totality.[69]

In keeping with Kant's 'Copernican Revolution', we can see the unity of mathematics (in the first sense specified above) as consisting in this method of connected reconceptualizations of old problems and objects.

Kant himself accepted that mathematics had a distinctive object, but he denied that this fact made mathematics what it is. Philosophers who define mathematics as the science of magnitude, he argued, confuse the effect with the cause: it is *because* mathematics has a distinctive method that it restricts its attention to a particular object.[70] Now some nineteenth-century geometers thought of geometry no longer as the science of three-dimensional Euclidean space, but as the science of n-dimensional complex

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[69] *PSF* 3, 404. Compare here the 1894 remarks by the geometer Felix Klein: 'To the layman the advance of mathematical science may perhaps appear as something purely arbitrary because the concentration on a definite given object is wanting. Still there exists a regulating influence, well recognized in other branches of science, though in a more limited degree; it is the *historical continuity. Pure mathematics grows as old problems are worked out by means of new methods. In proportion as a better understanding is thus gained for the older questions, new problems must naturally arise*' ('Riemann and His Significance for the Development of Modern Mathematics', *Bulletin of the American Mathematical Society*, 1 (April 1895): 170).

[70] See A714–15/B742–3; 'Vienna Logic', *Logic*, Ak 16: 797, *Lectures on Logic*, translated by Young, 258.

projective space. (More recent philosophers might think of all of mathematics as the study of portions of the set-theoretic hierarchy.) Whether or not these are accurate characterizations of the objects of mathematics, they do not express what is fundamental to mathematics. On Cassirer's Kantian view, the ontology always follows after the methodology. What is distinctive of mathematics is the constructive character well illustrated by Dedekind's foundations of arithmetic, and the methodological desiderata well illustrated by the kind of projective unification present in our case study. This two-part story of the method of mathematics is the 'general logical structure' Cassirer was after: it is this story that explains how mathematics can remain a unity over time despite the revolutionary conceptual and ontological changes it has suffered. (Indeed, it is this distinctive mathematical method that explains why mathematics *had* to undergo these revolutionary changes.) Like Kant, then, Cassirer thinks that the unity of mathematics lies not in its objects, but in its method.

5. THE ESSENTIAL APPLICABILITY OF MATHEMATICS AND A NEW PROBLEM FOR A KANTIAN

Our account of Cassirer's philosophical justification of the ideal elements in geometry is now complete. However, there is within Cassirer's thinking another element borrowed from Kant – an element that stands in potential conflict with the position articulated in the previous two sections and thus threatens to undermine the ontological legitimacy of modern geometry. In this section I first introduce the Kantian idea, show how it poses a *prima facie* problem for Marburg Neo-Kantian philosophy of geometry, and then lay out Cassirer's two general strategies for solving this problem.

Kant is emphatic throughout the *Critique* that the synthetic *a priori* cognitions of geometry count as *cognitions* only on account of their necessary link to experience.[71] Kant has no place in his theory for purely mathematical (abstract) objects, if by this we mean genuine objects and not just 'mere figments of the brain'. Rather, a construction in pure intuition provides, not a genuine object, but 'only the form of an object'.[72] Geometrical constructions, then, give rise to genuine knowledge only

[71]Thus, in the 'Principles of Pure Understanding', we read the following. 'Thus although in synthetic judgments we cognize *a priori* so much about space in general or about the shapes that the productive imagination draws in it that we really do not need any experience for this, still this cognition would be nothing at all, but an occupation with a mere figment of the brain, if space were not to be regarded as the condition of the appearances which constitute the matter of outer experience; hence those pure synthetic judgments are related, although only mediately, to possible experience, or rather to its possibility itself, and on that alone is the objective validity of their synthesis grounded' (A157/B196).
[72]A223–4/B271.

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because they establish the (real) possibility of certain empirical objects. But the essential *applicability* of geometry to the objects in space would not be established by the construction if space were not 'the form of all appearances of outer sense' and geometry were not the study of this form. Kant's insistence that geometrical objects are in fact 'forms of objects' or *ways the empirical world could be*, separates his view both from formalism and from Platonism: mathematical statements express *true* judgments that characterize the way the *empirical* world could be. Any Kantian philosophy of mathematics will have to avoid both those positions.[73]

Cassirer feels the pull of this Kantian idea: that the only objects in the strict sense are the empirical objects of natural science, and the only kind of objectivity is the objectivity secured by reference to these objects:

> If it is not possible to prove that the system of pure concepts of the understanding is the necessary condition under which we can speak of a rule and connection of appearances, and under which we can speak of empirical 'Nature' – then this system, with all its consequences and conclusions, would have to still appear as a mere 'figment of the brain' ... The logical and mathematical concepts should no longer constitute tools with which we build up a metaphysical 'thought-world': they have their function and their proper application solely within *empirical science* itself.[74]

The Kantian anti-Platonism comes with the insistence that the meaningfulness of mathematics depends on the fact that propositions about mathematical objects play an essential role in the sciences of physical objects – that in mathematics, 'we are dealing in no sense with some transcendent object, but only with the objective certainty of our empirical knowledge itself.'[75]

Unfortunately, in the context of increasingly abstract mathematical structures like the complex projective plane, this Kantian idea stands in apparent tension with the Marburg Neo-Kantians' desire to understand science, but not condemn or interfere with it.

> Obviously, on this point we are dealing with a general conflict that even today is unsettled. If one considers the development of modern mathematics, one

[73]By 'formalism' I mean the thesis that some or all mathematically acceptable sentences do not express genuine statements that can be true or false, but are uninterpreted strings of figures manipulable according to rules.

Michael Resnik has characterized (ontological) Platonism as the view that ordinary physical objects and numbers are 'on a par' (*Frege and the Philosophy of Mathematics*, 162). Described at this level of generality, then there is a clear sense in which Kant is not a Platonist. The fact that *mathematical* representations can be meaningful and can enter into judgments that purport to be true, has to be explained in terms of the primary relationship between representations and *physical objects*.

[74]KMM, 42–3. Cassirer is of course alluding to A157/B196.

[75]KMM, 48.

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> sees above all that the tendency emerges in it to allow itself to be led merely by the demands of inner logical consistency – without being distracted by the question of its possible applicability ... No one will contest the right of this claim, no one will be permitted to try on philosophical grounds to set limits to the freedom of mathematics, which is the condition of its fruitfulness.[76]

The transcendental method forbids philosophy from declaring any portion of our best mathematical theories as false or meaningless. Some of our best mathematical theories are not applied in natural science. But Cohen and his students wanted to respect Kant's good idea that mathematical judgments get their truth and contentfulness from their use in our experience of empirical objects. How can this tension be defused?

Cassirer has two general strategies for dealing with this problem, the first of which predominates in those earlier writings that are closer to orthodox Marburg Neo-Kantianism, the second of which predominates in Cassirer's later, *Philosophy of Symbolic Forms* writings. Arguing in the first way, Cassirer points out that mathematical concepts that at first seem paradoxical and incapable of application to the physical world – like those of complex analysis – have in fact been employed in physics.[77] Just as a modern Kantian should not view the concepts and laws constitutive of science at a given stage as constitutive of all scientific cognition in general, so too we should not prescribe ahead of time how mathematics, and thus what kind of mathematics, can be applied in physical science.[78]

Though Cohen does not himself take on the challenge posed to Kantian philosophy by the freedom of modern mathematics, this first general approach to answering the challenge is in keeping with the foundational doctrines of the Marburg school. Cohen, in his discussion of mathematics in his systematic treatise *Logik der reinen Erkenntnis* [*Logic of Pure Knowledge*], writes, 'we do not understand mathematics, if we do not recognize it in its purity as the methodological tool of natural science' (485). This characteristic thought, that mathematics is a body of knowledge *because* it is applied in natural science, follows from Cohen's idea that the *fact* whose presuppositions it is the task of philosophy to determine is the fact of

[76]KMM, 47–8. Cassirer is here borrowing the phrase 'freedom of mathematics' from Cantor. On Cantor's views, see especially Howard Stein, 'Logos, Logic, Logistiké: Some Philosophical Remarks on the Nineteenth-Century Transformations in Mathematics'.

[77]See *SF*, 116: 'For it is precisely the complex mathematical concepts, such as possess no possibility of direct sensuous realization, that are continually used in the construction of mechanics and physics.'

[78]Thomas Mormann ('Idealization in Cassirer's Philosophy of Mathematics') also considers Cassirer's views on the relation between mathematical physics and the 'ideal' elements in mathematics. The focus of his paper is what he calls Cassirer's 'sameness thesis': that both mathematical knowledge and physical knowledge are characterized by the introduction of 'ideal elements'. I think Mormann is correct that Cassirer subscribes to this view, and that it plays a central role in his thinking. But my topic here is different: what should Cassirer say about those ideal elements that are not made use of in natural science?

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mathematical *natural science*.[79] When Cohen claims that even pure mathematics needs to be understood as a tool for natural science, he is simply applying his more fundamental idea that mathematical physics is the very paradigm of knowledge – that in terms of which all other sciences are to be understood.

In Cassirer's earlier writings, he shares this fundamental point of view of his teacher.[80] However, starting in the late teens and culminating in the study of the *Geisteswissenschaften* in the three volumes of the *Philosophy of Symbolic Forms* (1923–9), Cassirer came to reject the physicalism of Cohen and Natorp.[81] Each of the special sciences 'frames its own questions', and answers them according to diverse and independent methodologies; none of the methods of the special sciences 'can simply be reduced to, or derived from the others'.[82] When Cohen and Natorp argue that the method of physics acts as the paradigm method for all of the other sciences, which depend on it, they erroneously 'pretend to speak . . . for the whole of knowledge'.[83] The distinctive doctrines of Cassirer's *Philosophy of Symbolic Forms* result from the combination of this new methodological pluralism with the fundamental ideas I described in section 2 of this paper: the various sciences now each present themselves as an independent *fact*, and their independent methods each make possible a distinctive kind of objectivity and a distinctive species of object.[84]

Cassirer's mature methodological pluralism had a clear impact on his philosophy of mathematics. In his earlier period, Cassirer recognizes that

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[79]For instance, in Cohen's 1883 monograph on the calculus, *Das Prinzip der Infinitesimal-Methode und seine Geschichte*, he argues that the concept of the infinitesimal must be grounded in its application – in its transcendental role in making *natural science* possible. He summarizes his approach: 'The grounding of the concept [of the infinitesimal] . . . must fall to the critique of cognition [*Erkenntniskritik*]; that is to say, this grounding must be treated as a *part* and an *example* of the problem that Kant posed in his *new concept of experience or mathematical natural science*. If the concept [of the infinitesimal] is to be grounded, this most powerful tool of experience must be considered as contained in one of the fundamental concepts, in one of the conditions of the possibility of experience' (*Werke*, vol. 5/1, §11; my translation).

[80]See especially KMM, 42–3, 47–8, quoted above. See also *EP* II, 663.

[81]Starting in the 1930s, Cassirer usually associates 'physicalism' with Carnap's claim that the language of physics is a universal language. (See *Ziele und Wege* [written 1937], 6–7; *Zur Logik der Kukturwissenschaften: Fünf Studien*, translated by S. G. Lofts as *The Logic of the Cultural Sciences*, 41.) But his most general criticism of Carnap, that he elevates the method of physics into the method of cognition *überhaupt*, clearly applies to Cohen and Natorp as well.

On Cassirer's evolution away from the Marburg orthodoxy of his earlier career to the methodological pluralism of *Philosophy of Symbolic Forms*, see the excellent discussion in Ferrari, *Ernst Cassirer*, Chapters 2–5.

[82]*PSF* 1, 76, 77, 78.

[83]*Problem of Knowledge*, 11.

[84]Cassirer comes to think that recognizing the objectivity of *cultural sciences* like religious studies, anthropology, and linguistics requires going down a further level and recognizing the objectivity of the various *cultural* forms that these sciences study – an objectivity that is distinct from that provided by the various natural sciences. Hence, 'critique of cognition becomes critique of culture' (*PSF* 1, 80), and myth, art, and language become 'symbolic forms'.

the freedom of modern mathematics poses a problem for Kant's fundamental idea that mathematical judgments get their objective validity from their use in empirical knowledge, and he goes some way towards solving this problem with a general strategy of emphasizing how even the most abstract mathematical structures can come to be applied in unexpected ways in natural science. In his later writings, however, Cassirer realizes that just as biology and the various *Geisteswissenschaften* are autonomous sciences with their own goals, methods, and objects, so too does pure mathematics have its own distinct method and its own distinct desiderata. So, while continuing to emphasize that the applicability of mathematics in natural science is an essential aspect of what makes it a science, Cassirer supplements this more Kantian strategy with a second general strategy for addressing the problem of the freedom of modern mathematics.[85]

This second general strategy is to emphasize the considerations given in the previous section. Since ideal elements are introduced precisely because they unify existing mathematics in such a way that it seems appropriate to say that they reveal the natural home or essential core of more familiar mathematical objects, any use of less abstract mathematics in physics will bring with it all of the more abstract mathematics. Since mathematics is a unity, if mathematical physics forces us to accept some mathematics, it forces us to accept it all.

> We must either decide to brand all of mathematics a fiction, or we must, in principle, endow the whole of it, up to its highest and most abstract postulations, with the same character of truth and validity. The division into authentic and inauthentic, into allegedly real and allegedly fictive elements, always remains a half measure which, if taken seriously, would have to destroy the methodological unity of mathematics.[86]

In fact, from the point of view of pure mathematics, it is not the applicability of complex numbers or n-dimensional complex projective spaces that justify these concepts and demonstrate to the mathematician that there are such objects. Rather, it is the fact that they unify and explain previously known mathematical cases, proofs, and areas of research.[87]

[85]This new strategy is for Cassirer, though, only a supplement to the first. He continues to think that mathematics is not a mere game fundamentally because it is applied in natural science. But – given his increasing appreciation of the methodological autonomy of the various special sciences – he argues that particular mathematical theories can be fully justified if they satisfy the methodological desiderata unique to pure mathematics. And this can be the case even if these particular theories do not get applied in empirical science.

[86]*PSF* 3, 400–1.

[87]See also *Ziele und Wege*, 76.

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6. CONCLUSION

Cassirer's philosophy of mathematics differs in significant ways from that of his contemporaries – a justly celebrated bunch, which includes Frege, Russell, Poincaré, Hilbert, Brouwer, and Carnap. He endorses an ontology of mathematical objects derived from Dedekind: each mathematical object is a position in a relational structure, all of the essential properties of these objects are relations to other objects in the structure, and reference to each object is only possible in the context of the whole system of objects in that structure. He positions his philosophy of mathematics within a comprehensive view – derived from the Marburg Neo-Kantian reading of Kant – of the task of philosophy, and of the nature of objectivity and objecthood. Within this comprehensive view, detailed considerations of the practice of nineteenth-century mathematicians come to play the leading role, and he is led to tell a story about the distinctive methodological desiderata of mathematicians: why they prefer certain kinds of proofs, what gives them reasons to introduce new objects, and what they are looking for in new concepts. This comprehensive view prevents him from endorsing the limitations imposed on the freedom of mathematics by a Kronecker or Brouwer, but his commitment to the essential applicability of mathematics in natural science, which pulls in the opposite direction, compels him to rethink how mathematics is applied in the natural sciences and to reject those philosophies, like formalism or fictionalism, that make this essential applicability inexplicable.

Despite the philosophical subtlety and originality of his philosophy, Cassirer does not, as do his better-known rivals, contribute to technical work in the foundations of mathematics. However, the absence of technical work in Cassirer's writings is tied up with one of his greatest merits, his impressive attention to the history of mathematics. But for readers who have become accustomed to the high level of rigour in, say, Frege's writings, Cassirer's picture will seem incomplete. From his own geometrical research, Frege was well aware of the payoff that introducing new points to the plane provides.[88] Indeed, both Frege and Cassirer think that it is best to understand points at infinity, following von Staudt, as directions of lines in a plane.[89] But while Cassirer was content to speak loosely of points at infinity as 'expressions of relations' between lines, Frege instead thought that we

[88]By introducing points at infinity in projective geometry, Frege wrote, 'we forestall a difficulty which would otherwise arise because of the need to distinguish a frequently unsurveyable set of cases according to whether two or more of the straight lines in the set were parallel or not', and these cases are all 'disposed of at one blow'. Frege, 'On a Geometrical Representation of Imaginary Forms in the Plane' (1873), in *Collected Papers*, edited by Brian McGuinness and translated by Max Black *et al.*, 1.

[89]For Frege, see *Die Grundlagen der Arithmetik*, translated by Austin as *Foundations of Arithmetic*, §64. For Cassirer, see *PSF* 3, 396–7.

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need to identify points at infinity with the extension of the concept <parallel to the line *a*.[90] And Frege of course devoted the bulk of his career to devising a formal language and a rigorous, axiomatic theory in which, *inter alia*, one can give precise definitions of various mathematical objects in terms of concept extensions.

Though there is nothing like the formal content of Frege's *Basic Laws of Arithmetic* in Cassirer's writings, one can, turning the tables, point out that Cassirer's philosophy – with his rich appreciation of the historical development of mathematics, and his desire to understand other mathematical virtues besides consistency and proof strength – brings to the front themes that are either missing from or in the background of Frege's thought. Moreover, close attention to technical questions in the foundations of mathematics seemed to have made some early analytic philosophers simply blind to the kinds of questions that Cassirer worried so much about. Reichenbach, for instance, wrote:

> It has become customary to reduce a controversy about the logical status of mathematics to a controversy about the logical status of the axioms. Nowadays one can hardly speak of a controversy any longer. The problem of the axioms of mathematics was solved by the discovery that they are definitions, that is, arbitrary stipulations which are neither true nor false, and that only the logical properties of a system – its consistency, independence, uniqueness, and completeness – can be subjects of critical investigation.[91]

One could hardly find a point of view further from Cassirer's own.

As I noted in the opening of this paper, historians of early analytic philosophy have come to recognize the importance of Neo-Kantian philosophy during this period, and many have thought that we can make progress on understanding the analytic grandfathers by reading them in conversation with Kant. A conclusion of this paper is that this kind of historical project will be frustrated if we try to think of the emergence of analytic philosophy – as Alberto Coffa does – as an episode in the 'rise and fall of pure intuition'.[92] Understanding the philosophical environs of early analytic philosophy requires knowing which of Kant's doctrines the Kantians of the period thought inspirational and which hopeless. Inasmuch as Cassirer was representative of Kantian philosophy during this period, pure intuition was not a topic that divided Frege, Russell, and Reichenbach from Neo-Kantian philosophy of mathematics.

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[90] *Foundations of Arithmetic*, §§66–8.
[91] Reichenbach, *Axiomatik der relativistischen Raum-Zeit-Lehre*, translated by M. Reichenbach as *Axiomatization of the Theory of Relativity*, 3.
[92] Coffa, *The Semantic Tradition*, 22.

One might argue that, given Cassirer's willingness to discard and re-read large portions of Kant's *Critique*, there is not enough left in Cassirer's Kantianism for it to deserve to be called '*Kantian*'. Those readers of Kant who prefer a greater preponderance of letter over spirit will be sympathetic to this complaint. Nevertheless, a principal finding of our historical research is that, in a period like the end of the nineteenth and beginning of the twentieth century, when Kant interpretation is so entwined with systematic philosophy, there is no real answer to the question what a modern Kantianism should be. This essay has shown, though, that the unorthodox positions that Cassirer was driven to were not unmotivated: he was responding to the full challenge of both of geometry's revolutions, and was trying to stay faithful to the respect for the exact sciences first typified in Kant's *Critique*.

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