Set theory's foundational role

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Introduction

Nowadays, standard orthodoxy holds that set theory provides a foundation for mathematics. This idea isn't new; Zermelo's initial axiomatization had explicit foundational aspirations:[1]

> Set theory is that branch of mathematics whose task is to investigate mathematically the fundamental notions 'number', 'order', and 'function', taking them in their pristine, simple form, and to develop thereby the logical foundations of all of arithmetic and analysis. (Zermelo [1908a], p. 200/189)[2]

Since then, it's become clear that every classical mathematical object[3] can be matched with a set-theoretic surrogate and that all classical mathematical theorems about those objects can be proved from the axioms of set theory – what we might call the set-theoretic 'reduction'.[4] Over a century later, it's easy to forget what a remarkable achievement this is, but a bare mathematical fact, however impressive, rarely wears its significance on its sleeve. What exactly is the import of this display of set-theoretic reach? Why is it

[1]See Zermelo's retrospective description quoted in Ebbinghaus [2007], p. 28: 'I began, under the influence of D. Hilbert, to whom I owe more than to anybody else with regard to my scientific development, to occupy myself with questions concerning the foundations of mathematics'.

[2]The first page number refers to the reprinting in van Heijenoort [1967], the second to the reprinting in Zermelo [2010].

[3]With the exception of some categories, e.g., the category of all groups or the category of all categories. See §2 for discussion.

[4]I use this term as nothing more than a label for the mathematical fact just described.

regarded as 'foundational'?

One popular approach to these questions begins with an analysis of the concept or meaning of 'foundation' then examines how the set-theoretic case does or doesn't match up. Whether or not there's such a concept or meaning to be examined – I have my doubts – there definitely are specific desiderata that tend to come up in these discussions and specific services that set theory actually tends to provide in practice. Instead of focusing on what properly counts as a 'foundation', it seems to me more productive to skip over the label and attend directly to these individual roles, their perceived benefits, and how the reduction of mathematics to set theory might serve them. This is the approach of §1.

Of course, there are dissents from this orthodoxy, the most vigorous of which traces to the rise of category theory in the 1960s: set theory is criticized as unable to provide a proper foundation for category theory and the new mathematics it facilitates – then category theory itself is proposed as a replacement. These ideas are the subject of §2. More recently, in the final decade or two of the last century, a number of set theorists began to entertain the possibility that ZFC's picture of a single universe of sets could give way to a form of multiverse thinking – eventually this grew into the suggestion that a multiverse theory might provide a better foundation than ZFC. This is the topic of §3.[5]

1 Set theory

To clear the way, let me begin by acknowledging that set theory doesn't serve a notorious pair of purposes that some philosophers tend to emphasize, that it doesn't provide either a metaphysical or an epistemological foundation for mathematics. The metaphysical foundation would take identifying a set-theoretic surrogate for a mathematical object as uncovering or discovering what that object really has been all along: to our surprise, 2 really is $\{\emptyset, \{\emptyset\}\}$; $\sqrt{2}$ really is $(\{r \in \mathbb{Q}^+ | r^2 < 2\}, \{r \in \mathbb{Q}^+ | r^2 > 2\})$. The epistemological foundation would see a proof from the set-theoretic axioms as the route

[5]§§1 and 2 draw on themes from the first two sections of Maddy [2017] and Maddy [2019], but with departures occasioned by a new analysis of multiversism in §3 – quite different from Maddy [2017], §3 – inspired by Steel [2024].

through which we come to know a mathematical theorem. The (unrealized) hope here is that the set-theoretic embedding would reduce all philosophical questions of mathematical ontology and epistemology to the corresponding questions for set theory itself (presumably easier, or at least, more focused).

In fact, I don't think this sort of payoff ever was the goal of set-theoretic foundations. No **Metaphysical Insight** into the nature of mathematical objects is claimed, just that set-theoretic surrogates can play the requisite mathematical roles. So, for example, all that matters for an ordered pair is that $(x, y) = (x', y')$ when and only when $x = x'$ and $y = y'$; after that, the Kuratowski definition is a convenient convention. Similarly, any sequence of sets that satisfies the Peano Axioms – written set-theoretically, so that induction says 'for all $A \subset \mathbb{N}$, if $0 \in A$ and $Sn \in A$ whenever $n \in A$, then $A = \mathbb{N}$' – can serve as the set-theoretic natural numbers; after that, the von Neumann ordinals are generally chosen for practical reasons, including the ease of generalizing them to the transfinite ordinals. As Yiannis Moschovakis describes the plan in his introductory textbook:

> We will discover within the universe of sets *faithful representations* of all the mathematical objects we need, and we will study set theory on the basis of the lean axiom system of Zermelo **as if all mathematical objects were sets**. (Moschovakis [1994], pp. 33–34, emphasis in the original)

Whatever metaphysical puzzles mathematics may pose remain untouched.

Similarly, the observation that set-theoretic foundations don't identify a basic **Epistemic Source** for mathematical knowledge would seem obvious: surely Euclid, Euler, and Gauss knew plenty of mathematics without being familiar with set theory. Russell remarked early on that the logical order needn't track the epistemic order, that fundamental logical axioms are justified by their well-known mathematical consequences, not vice versa;[6] likewise, Zermelo famously defended that Axiom of Choice on the basis of its consequences.[7] The question of how we come to know the mathematics we do isn't solved by the set-theoretic reduction.

[6]See Russell [1907].
[7]See Zermelo [1908b], Zermelo [1908a].

To isolate the plausibly 'foundational' roles that set theory *does* play, it helps to return to the pre-history of the subject in the dramatic mathematical developments of the 19th century. One salient example of the conceptual challenges these expansions of the subject posed comes early in the century, when 'imaginary points' (those with complex coordinates) and 'points at infinity' (where parallel lines intersect) were introduced into geometry. Their mathematical utility was unquestioned, but for a field so wedded to geometric intuition, an appeal to 'invisible points' was understandably troubling. A relatively stable reorientation was achieved by Karl von Staudt in mid-century: for example, the point of intersection of the parallel lines in one direction is identified as (what we would recognize as) the equivalence class of any one of those lines under parallelism.[8] At the time, this move was regarded as purely logical, so von Staudt was demonstrating that these new entities were logical extensions of intuitive geometry, in other words, trustworthy, after all. Later in the century, the use of proto-set-theoretic methods became more recognizable, for example, in Dedekind's replacements of Kummer's 'peculiar sort of imaginary divisors'[9] with sets of divisors (ideals) and the geometric, intuitive real numbers with sets of rationals (cuts).[10] The set-theoretic reduction gradually emerged from developments like these.

But this isn't the whole story; a second 19th century theme also contributed to the early growth of set theory, namely, the rise of pure mathematics. In the 17th century, Galileo famously held that the Great Book of Nature is written in the language of mathematics, and Sir Isaac Newton largely agreed.[11] In the 18th, Euler carried on in the same vein, sometimes allowing physical intuition to substitute for mathematical rigor. Over the course of the 19th century – as, for example, differential equations were revealed as smoothed-out approximations to a more complex, indeed random, atomic structure – this reading of mathematics as uncovering literal truths

[8]See Wilson [1995], pp. 125-140, for discussion and references. Wilson traces the influence of this geometric work of von Staudt on Frege's definition of number in Frege [1884]. (Recall that the illustrative example with which Frege prepares the reader is the 'direction of a line'.)

[9]This characterization from Kummer is quoted on p. 164 of Avigad [2006]'s penetrating methodological analysis of Dedekind's theory of ideals.

[10]Dedekind [1872].

[11]See Maddy [2008] or chapter I of Maddy [2011] for a more complete version of the story in this paragraph.

about the world began to rupture. On parallel track, mathematicians were branching out into the study of a variety of structures that interested them mathematically, with no immediate concern for physical applications at all – so that, by the 1880s, Cantor could declare that 'the essence of mathematics lies in its freedom' (Cantor [1883], p. 896). But with this freedom came a question: without physical intuition to guide us, how can we be sure a given new structure is actually coherent? Enter Hilbert with his program of axiomatization and relative consistency proofs, eventually backed, via the reduction, by Zermelo's axiomatization of set theory.

So, finally, the central question: what roles are sets playing here, especially roles that tend to be thought of as 'foundational'? One obvious candidate is Dedekind's use of set-theoretic tools to pin down the notions of continuity and real number so that the basic theorems of analysis could be rigorously proved (that is, without fudges from geometric intuition). The set-theoretic reduction alone simply identifies surrogates for mathematical items already in good working order, so something more is happening here: the then-current understandings of the real numbers and of continuity weren't sharp enough to support proofs of the basic theorems; set-theoretic tools allowed Dedekind to sharpen them sufficiently to do that job, providing an improvement, not just a surrogate.[12] Dedekind, for one, saw himself as providing 'a truly scientific foundation' for the subject, so perhaps it's reasonable to count this **Elucidation** among the foundational roles that set theory can and has played.

It might seem that von Staudt's methods run parallel to Dedekind's, but this isn't quite right. In von Staudt's case, the practice of complexified analytic geometry was functioning – the problem was how to understand the points with imaginary coordinates in geometric terms. What von Staudt did with his proto-set-theoretic tools was provide a better answer to that question. Stretching the family resemblance one bit further, another of Dedekind's projects aimed to uncover the 'absolutely indispensable foundation' for the natural numbers – not on 'notions or intuitions of space and time' but on 'the pure laws of thought' (Dedekind [1888], pp. 790-791).[13] Here the

[12]The replacement of the imprecise notion of function with the set-theoretic version is a similar example. See Maddy [1997], 118-126, for an overview.

[13]See Maddy and Väänänen [2023], §2, for discussion.

immediate need wasn't as pressing as for the real numbers or points with complex coordinates, but in all three cases, a kind of mathematically-significant foundational clarity is at issue. And though von Staudt and Dedekind both imagined that they were appealing to logic alone – 'the pure laws of thought' – their methods were subsequently revealed to be set-theoretic.[14] I propose to include all these under the general heading of **Elucidation**.[15]

A second answer to our question – what plausibly 'foundational' goal is set theory serving? – arises from the second historical theme just outlined, from the rise of pure mathematics to Hilbert's program of axiomatization to Zermelo [1908a]. By this route, the uncertainty that a particular exercise of Cantorian freedom is coherent, not chimerical, was to be removed by embedding it in a more familiar mathematical structure and from there, ultimately, in set theory. Naturally, the hope then was to prove the consistency of the set theoretic axioms. Zermelo tried:

> In the present paper I intend to show how the entire theory created by Cantor and Dedekind can be reduced to a few definitions and ... axioms. ... The further, more philosophical, question about the origin of these principles and the extent to which they are valid will not be discussed here. I have not yet even been able to prove rigorously that my axioms are consistent, though this is certainly very essential; instead I have had to confine myself to pointing out now and then that the antinomies discovered so far vanish one and all if the principles here proposed are adopted as a basis. (Zermelo [1908a], pp. 200-201/191)[16]

Still, showing that the known paradoxes are blocked is some comfort, and the upshot of a reduction is that the new undertaking is at least no riskier

[14]See Maddy and Väänänen [2023], pp. 4-5, for Dedekind's reaction to this realization.

[15]Perhaps it's worth emphasizing that neither von Staudt nor Dedekind uncovered what points at infinity or real numbers really are (**Metaphysical Insight**) and that we don't come to know the facts of arithmetic by deriving them from Dedekind's axioms (**Epistemic Source**). Van Staudt showed that a workable surrogate for points at infinity could be reached by (what he regarded as) pure logic. Dedekind found a surrogate for real numbers that better served analysis and revealed that arithmetic could be characterized using only tools from (what he regarded as) pure logic.

[16]Apparently we have Hilbert to thank for encouraging Zermelo to publish Zermelo [1908a] without the desired consistency proof. See Ebbinghaus [2007], p. 78.

than Zermelo's simple axioms.

Of course, ordinary empirical evidence for the consistency of ZFC has accumulated over many years of far-reaching developments in set-theoretic mathematics. There's also the evolving intuitive picture of the iterative hierarchy that continues to guide the pursuit of new directions in the subject (at least so far) without a mishap. Today, being no more risky than ZFC is widely considered a fairly strong assurance that it's safe to continue.[17] But what about stronger theories? As is well-known, the answer begins with set theory's production of a staggering hierarchy of large cardinal axioms (LCAs), with ever-increasing consistency strength. Of course such increases typically come with corresponding decreases in our confidence in the consistency of the relevant cardinal, but informed opinion ranges from cautious optimism[18] to confidence even for some of the very largest of the known large cardinals.[19] Along the way, so many 'natural'[20] extensions of ZFC have been shown to be equiconsistent with ZFC plus one or another large cardinal axiom that these assumptions are now 'widely accepted as *the* measuring rod of exhaustive principles against which all possible consistency strengths can be gauged' (Kanamori [2009], p. 298). In this way, ZFC and its extensions provide the valuable service of **Risk Assessment**: the place of a theory's consistency strength in the hierarchy of large cardinals gives a reasonable measure of how safe it can be expected to be.

Useful as they are, **Elucidation** and **Risk Assessment** aren't the only mathematical benefits arising from the set-theoretic reduction. As we've seen, Hilbert's idea was to defend the coherence of a branch of pure mathe-

[17]E.g., even Vladimir Voevodsky, father of Univalent Foundations, grants that 'ZFC retains its key role as the theory which is used [to] ensure that the more and more complex languages of the univalent approach are consistent' (Voevodsky [2014]).

[18]E.g., Steel [2014], p. 164: 'we have good evidence that the consistency hierarchy is not a mirage, that the theories in it we have identified are consistent. This is especially true at the lower levels, where we already have canonical inner models', i.e., at least up through Woodin cardinals.

[19]E.g., Kanamori [2009], p. 328: 'attitudes about and expectations concerning I1-I3 have evolved since their formulation, from skepticism toward confidence and acceptance'.

[20]Steel [2014], p. 157, gives a rough sense of how this informal term is to be understood: 'by "natural" we mean considered by set theorists, because they had some set-theoretic idea behind them. Here the standards are very liberal, as the many thousands of pages published by set theorists will testify'.

matics by axiomatizing it and proving its relative consistency by constructing a model, embedding it, in a more trustworthy theory, ultimately in ZFC. But this approach only guarantees that the theory of that branch holds within its corresponding model; the theorems proved there don't necessarily apply in a model of the theory of another branch. The trouble is that the various branches of pure mathematics are in fact very much intertwined, from coordinate geometry, to analytic number theory, to algebraic geometry, to topology, to modern descriptive set theory (a confluence of point-set topology and recursion theory), to the kind of far-flung interconnections recently revealed in the proof of Fermat's Last Theorem. John Burgess lays out the problem:

> Interconnectedness implies that *it will no longer be sufficient to put each individual branch of mathematics separately on a rigorous basis* ... To guarantee that rigor is not compromised in the process of transferring material from one branch of mathematics to another, it is essential that the starting points of the branches being connected should ... be compatible. ... The only obvious way to ensure compatibility of the starting points ... is ultimately to derive all branches from a common, unified starting point. ([Burgess, 2015], pp. 60–62)

The set-theoretic reduction provides just such a 'unified starting point': the various mathematical items are all represented side-by-side, by their set-theoretic surrogates in the set-theoretic universe, the same **Generous Arena**; the various mathematical theorems are all proved from the same spare list of set-theoretic axioms, a **Shared Standard** of proof. It seems natural to classify these accomplishments 'foundational'.

The final, plausibly 'foundational' job I'd like to highlight also has its roots in Hilbert-style thinking. In the course of 19th century developments like those sketched here, the completed infinite had permeated throughout the fields of pure mathematics, and doubts lingered about the security of the entire endeavor. Hilbert's thought was to calm these fears by proving the consistency of the new infinitary mathematics using only indubitable finitary methods – but this task that could only be undertaken if the whole subject could be brought together in a single, purely syntactic formulation. This was provided by the set-theoretic reduction and Zermelo's simple axiomatization.

Without this **Metamathematical Corral**, neither Hilbert's goal – a finitary proof of the consistency of classical mathematics – nor Gödel's proof – that this can't be done, even with the resources of classical mathematics itself – couldn't have been precisely posed, much less settled.

In sum, then, we've isolated five substantial benefits of the set-theoretic reduction – **Elucidation**, **Risk Assessment**, **Generous Arena/Shared Standard**, and **Metamathematical Corral** – all of which could plausibly be considered 'foundational' as the term is ordinarily used. I think we can agree that these are welcome results, that they should be considered among the successes of the theory as a whole. Before we leave set theory to consider a pair of its rivals, it's worth noting the methodological consequences that follow if we assume that providing these foundational services is among the goals of set-theoretic practice. In the interest of effective **Risk Assessment**, strong large cardinal axioms, indeed the strongest possible, should be serious candidates for addition to ZFC; if there's good evidence for the consistency of such an assumption and no evidence that it blocks any fruitful or potentially fruitful avenues, then it should be embraced. To avoid the possibility that regarding V as the **Generous Arena** will limit the Cantorian freedom of pure mathematics, set theorists should strive to include as many structures as possible. In the common metaphor, V should be as tall and wide as set theorists can make it – the practice should be guided by a methodological admonition to *Maximize*. At the same time, as **Shared Standard** especially highlights, there should be a single, explicit set of axioms from which all the theorems are derived – the practice should also be guided by a methodological admonition to *Unify*.[21] I return to this tidy and appealing picture of set theory and its foundational role in §3,[22] but first there's the forceful protest that set theory should be replaced by a theory of categories.

[21] *Maximize* and *Unify* are identified as methodological maxims in service of the foundational goal in Maddy [1997].

[22] The contrast between the *Maximize* and *Unify* of §1 (from, e.g., Maddy [1997]) and the versions of §3 (from, e.g., Steel [2014]) is explored more fully in work-in-progress, §§4.3 and 8.3, where I side with Steel against my earlier self.

2 Category Theory

Category theory was first developed by Samuel Eilenberg and Saunders Mac Lane in the context of algebraic topology (Eilenberg and MacLane [1945]), but it soon spread to functional analysis, algebraic geometry, and beyond. By the mid-1960s, William Lawvere observed:

> In the mathematical development of recent decades one sees clearly the rise of the conviction that the relevant properties of mathematical objects are those which can be stated in terms of their abstract structure rather than in terms of the elements which the objects were thought to be made of. (Lawvere [1966], p. 1)

A growing number of mathematicians found the category-theoretic setting 'much more natural and readily-useable than the classical one' (ibid.). From there, he continues

> The question thus naturally arises whether one can give a foundation for mathematics which expresses wholeheartedly this conviction concerning what mathematics is about, and in particular in which classes and membership in classes do not play any role. (ibid.)

Lawvere is explicit on his meaning here: for him, a 'foundation' is 'a single system of first-order axioms in which all usual mathematical objects can be defined and all their usual properties proved' (ibid.), in other words, a category-theoretic reduction. He doesn't explain what such a reduction would achieve, but he presumably imagines it would simply recover the benefits of its set-theoretic analog.

At the consequential set theory symposium at UCLA in 1967, Mac Lane picked up the theme of category theory as 'an effective method of organizing parts of mathematics' (Mac Lane [1971], p. 231).[23] More recently, McLarty elaborates and seconds this approach:

[23]McLarty [2013], p. 77, explains that Mac Lane first read the work of Lawvere, a graduate student at the time, 'to see specifically what went wrong'. Finding nothing, he decided 'it was a good idea' and communicated Lawvere [1964] to the National Academy.

> Mac Lane ... took a *foundation of mathematics* to be a body of truths which organize mathematics as do I here. More specifically, that is truths which actually serve in practice to define the concepts of mathematics and prove the theorems. I do not merely mean truths which could in some principled sense possibly organize the practice but truths actually used in textbooks and journal articles, and discussed in seminar rooms and over beer, so their notions do occur in practice. (McLarty [2013], p. 80)

The call is for a more practice-oriented, category-theoretic foundation to replace the overly-remote orthodoxy of ZFC and its extensions.

It seems to me there are two interrelated critiques of set-theoretic foundations at work here. The first is that set-theoretic surrogates introduce arbitrary excess structure that's irrelevant and distracting (for example, Zermelo's ordinal and von Neumann ordinals for the natural numbers or Dedekind cuts vs. Cauchy sequences for the reals). So, far example, Mac Lane [1986] (p. 407) cites Hermann Weyl as complaining that set theory 'contains far too much sand'. The other concerns more abstract areas of mathematics, like Mac Lane and Eilenberg's algebraic topology: the set-theoretic surrogates there are complex sets that no set theorist would ever have noticed but for the work of algebraists, and their set-theoretic complexities (sand?) add no insight or guidance for further development of the actual mathematics. In contrast, category-theoretic formulations provide what might be called **Essential Guidance**: they get to the mathematical core of the idea without irrelevant substructure, and they point the way to new developments. This is a new, plausibly 'foundational' goal, to clarify, organize, and guide the day-to-day work of mathematicians.[24]

There can be no doubt that category theory provides this kind of **Essential Guidance** to large swaths of contemporary mathematics, especially, but not only, in its more algebraic branches. All the same, it would likewise be hard to deny that, for example, the theory of Dedekind cuts reveals something essential to the real numbers and points the way to further study of continuous structures. Here Mac Lane seems to agree:

[24]McLarty [2013], p. 77, makes this novelty explicit: 'Many kinds of foundational study are pursued for many reasons. But the relevant sense here is a foundation as a body of truths deductively organizing the actual practice of mathematics'.

> Categories and functors are everywhere in topology and in parts of algebra, but they do not as yet relate very well to most of analysis. (Mac Lane [1986], p. 407)

(Perhaps this isn't so surprising, given that set theory arose out of analysis and category theory out of algebra.) One close observer, A. R. D. Mathias [2001] (p. 227) concludes that 'there are portions of mathematical reasoning for which the language of categories gives a very smooth presentation [and] portions for which is clumsy'. Of course, the same can be said of set theory.[25] Mac Lane concludes that 'there is as yet no simple and adequate way of conceptually organizing all of Mathematics' (Mac Lane [1986], p. 407), that is, no single foundation provides **Essential Guidance** for all of mathematics.

To these category-theoretic critiques of set-theoretic foundations – excess structure and lack of **Essential Guidance** in areas he cares about – Mac Lane adds a third:

> Categorical algebra has developed in recent years as an effective method of organizing parts of mathematics. Typically, this sort of organization uses notions such as that of the category **G** of all groups. This category consists of two collections: the collection of all groups G and the collection of all homomorphisms ... of one group G into another one; the basic operation in this category is the composition of two such homomorphisms. To realize the intent of this construction it is vital that this collection **G** contain *all* groups; however, if 'collection' is to mean 'set' ... this intent cannot be directly realized. (Mac Lane [1971], p. 231)

There's no set of all groups anymore than there's a set of all sets. Alexander Grothendieck provided a practical workaround for this problem in his theory of 'universes', described here by Burgess:

> For applications ... one doesn't need a category of literally *all* groups ... It is always enough to have a category of 'enough' groups, though how many is enough may vary from application

[25] There are also areas where the two approaches complement each other. See early work of Andreas Blass and co-authors, e.g., Blass and Ščedrov [1983]; more recently, e.g., Bagaria and Brooke-Taylor [2013], Bagaria et al. [2015], Meadows [2023].

> to application. ... Grothendieck's hypothesis is that every set, however large, belongs to some local universe [that is, some V_κ]: The 'global universe' ... is simply the union of increasingly large local universes. (Burgess [2015], p. 174)

If **Risk Assessment** were the issue, this would be enough – category theory is no worse off than ZFC+'there's a proper class of inaccessibles' – but Mac Lane isn't satisfied; he wants 'the category of all categories *überhaupt*' (Mac Lane [1971], p. 234). The hope is that a category-theoretic foundation, unlike a set-theoretic foundation, can deliver this result.

The challenge of developing such a foundation is sharply formulated in Solomon Feferman [1977]: we want a first-order theory of categories that generates all-inclusive categories like the category of all groups or the category of all categories while allowing all the routine operations typically employed in the practice.[26] Feferman [2013] lays out the partial results achieved by himself, McLarty, and others, all of which fall short, either by failing to validate all the ordinary operations or by compromising the unlimited categories that motivated the project in the first place. He suggests some further directions for the quest, but Ernst [2015] soon explained why a solution was so elusive: unlimited category theory, understood as Feferman specifies, is inconsistent.[27] So it's no shortcoming of set theory that it can't found the categories *überhaupt* that Mac Lane sought – they can't be founded in any consistent theory.

Fortunately, Lawvere [1966] proposes a more viable option for category-theoretic foundations that's been corrected and improved by others, culminating in McLarty [1991]. Lawvere announces the project in his title – 'The Category of Categories as a Foundation' (CCAF) – and begins with a familiar gesture toward **Essential Guidance** for the familiar fields:

> A foundation of the sort we have in mind would seemingly be much more natural and readily-useable than the classical one [i.e.,

[26] These are requirements R1-R3 of Feferman [1977], pp. 154, 156, or Feferman [2013], p. 9. Feferman [2013], p. 9, includes R4: establish consistency 'relative to a currently accepted system of set theory', in other words, set-theoretic **Risk Assessment**.

[27] In particular, the non-trivial argument of Ernst [2015] shows that the unlimited category of graphs can be constructed by the ordinary operations but leads to a contradiction.

> set theory] when developing such subjects as algebraic topology, functional analysis, model theory of general algebraic systems, etc. (Lawvere [1966], p. 1)

The axioms for this system would begin from the Eilenberg-Mac Lane definition of a category and from there ...

> The author believes, in fact, that the most reasonable way to arrive at a foundation meeting these requirements is simply to write down axioms descriptive of properties which the intuitively-conceived category of all categories has until an intuitively-adequate list is attained. (Ibid.)

This he does, adding, for example, a few elementary categories and closure under standard operations like those listed by Feferman. The resulting 'elementary axioms produce the major theorems of general category theory' (McLarty [1991], p. 1243), but 'prove the existence of few specific cases':

> I regard these axioms as a background theory to be used with axioms on particular categories categories or functors. ... This approach was suggested by Lawvere [1966]. (McLarty [1991], p. 1259)

So, for example, we might add an axiom for a category of rings or a category of topological spaces or a category satisfying the needs of synthetic differential geometry. Or, more to the point for foundational purposes, we could posit the category introduced in Lawvere [1964], that is, one satisfying 'The Elementary Theory of the Category of Sets' (ETCS).

ETCS is a natural version of set theory formulated in terms of objects (sets) and arrows (functions between them) rather than sets and elements. Though ETCS by itself is fairly weak – equivalent to ZC with bounded Separation – category-theoretic versions of Replacement can be added.[28] In the end, Augmented-ETCS (ETCS+Replacement) functions very much like more familiar Zermelo-style systems:

[28] See, e.g., Mitchell [1972], Osius [1974], Shulman [2019], McLarty [2004].

> The foundational point is: Once you get beyond axiomatic basics, to the level of set theory that mathematicians normally use, ZF and ETCS are not merely inter-translatable. They work just alike. (McLarty [2004], p. 41)

It follows that this category-theoretic approach can replicate familiar set-theoretic reduction that underlies set-theoretic foundations.[29] Does it follow that category theory can step into the foundational roles in which set theory has tended to be cast?

Assuming Dedekind's **Elucidation** of the reals is typical, Augmented-ETCS is up to the job:

> The [category-theoretic] construction of the Dedekind reals does not mimic Dedekind's construction. It *is* Dedekind's construction. Admittedly Dedekind did not work in topos theory. Nor did he work in ZF, Zermelo's theory, or even in Cantor's set theory, none of which existed in 1858 when he produced his definition of the reals ... He worked with sets and subsets and orderings, all of which can be formulated in Cantor's set theory, or in ZF, or in [Augmented-ETCS] – the latter two achieving a level of formal rigor that Cantor did not. (McLarty [2004], p. 39)

Risk Assessment is less straightforward. LCAs can be transferred from ZF to the ETCS by standard translations, which are enough for some purposes. So, for example, such a translation of a category-theoretic presentation of a branch of algebraic topolopy would suffice to place it in set theory's **Generous Arena/Shared Standard**, though no one would want to carry out the actual mathematics in set theory – category theory would remain, undeniably, the best context for that. In contrast, for category theory to take over the job of **Risk Assessment** from set theory, it wouldn't be enough simply to shoehorn notions and methods from ZF into ETCS by this kind of brute force; here the category-theoretic version is out to replace set theory as the context where the assessment is carried out, where the actual mathematics is

[29]Philosophers skeptical of the foundational suitability of ETCS have raised a number of objections, mostly challenging the autonomy of category-theoretic set theory in one way or another, but it seems to me that McLarty and others have cogent replies. See Ernst [2017] for an overview with references.

done. For this to work, there would have to be practically workable category-theoretic versions of LCAs[30] and of the proof techniques used in set-theoretic equiconsistency proofs.[31] For now, the extent to which an extension of ETCS can do this remains an intriguing open question.[32]

But now consider **Generous Arena/Shared Standard**. Many structures of classical mathematics are brought together in the category satisfying Augmented-ETCS, but this is only one in the range of categories that would be posited by adding axioms to the background theory CCAF in the full-flourishing of category-theoretic mathematics. The objects of the category SDG (smooth differential geometry), for example, aren't brought side-by-side with the inaccessible cardinals of Augmented-ETCS; the theorems in the former theory aren't proved from the same assumptions as those in the latter. Nor does CCAF itself bring the objects of these categories together: it just has an axiom for each, asserting that there is such-and-such a model. This is to reintroduce the separation between the branches of mathematics that Burgess warned so eloquently against.[33] Whatever its merits, this approach cannot serve the quite real mathematical needs met by set-theory's ability to bring surrogates for all branches of mathematics into one **Generous Arena** with a **Shared Standard** of proof: 'we want one framework theory, to be used by all, so that we can use each other's work' (Steel [2014], p. 164). The reduction alone doesn't deliver these benefits.[34]

In sum, then, defenders of category-theoretic foundations have introduced

[30]McLarty [2004] gives such a treatment of measurable cardinals, generalizable to supercompacts, but analogous questions for many other LCAs remain largely unexplored.

[31]Typically, inner models and forcing (see §3). There has been work on category-theoretic versions of forcing – beginning from the work of Lawvere and Tierney (see Tierney [1972]) – but inner model theory may be more problematic.

[32]See Hanson [2024] for a recent overview.

[33]Indeed, at one point, Mac Lane councils: 'do not construct the reals, but describe them axiomatically as an ordered field, complete in the sense that every bounded set has a least upper bound' (Mac Lane [1981], p. 468).

[34]I suspect that those who take set-theoretic foundations and category-theoretic foundations to be in some sense equivalent have in mind a comparison of Augmented-ETCS and ZFC+LCAs. But its hard to see that the category theorists' desire for **Essential Guidance** in the parts of mathematics where its methods are superior would be satisfied if all their work were required to take place within a theory of sets, even if those sets were characterized as a categorically. This would be just another form of set-theoretic dominance.

a new foundational goal, **Essential Guidance**, and argued persuasively that an approach like that of CCAF with Augmented-ETCS meets that goal for a wide swath of contemporary mathematics – while also acknowledging that set theory serves this purpose in other aspects of the subject. Beyond that, both Augmented-ETCS and ZFC+LCAs deliver **Elucidation**, but category-theoretic **Risk Assessment** remains open. Finally, **Generous Arena/Shared Standard** – and for similar reasons, **Metamathematical Corral** – remain the province of set theory. We now turn to a challenge to set-theoretic foundations from within.

3 The multiverse

The internal challenge to the set-theoretic foundation laid out in §1 arises from a certain tension in that simple picture. Set theory is to serve as the **Generous Arena/Shared Standard** for classical mathematics, gathering together surrogates for all its objects and premises for all its proofs. So, for example, if an analyst or for that matter a mathematical physicist wants to know if they can assume the real numbers are well-ordered, the answer is provided by set theory: yes, they are. In order to play this role effectively, set theory adopts the methodological principles *Maximize* – so as not to curtail either pure or applied mathematics – and *Unify* – so as to provide unequivocal answers.

But now suppose our analyst or physicist has a particular argument or construction in mind and asks a further question: is this well-ordering simply definable? On the picture in §1, set theory is now less decisive. There's a temptation to answer with a story something like this: 'Well, if you work in ZFC, this isn't settled, but if you add LCAs, as set theorists tend to do, then the answer is no. Still, if you're keen on that argument or construction, you could decide to work in ZFC+$V=L$. L is well-understood and you'd have all the powers of ZFC, so this is a fairly generous arena with a fairly rich shared theory. When collaborating with others across the branches of mathematics, you'd have to be careful to note the extra assumption of $V=L$ – there'd be limits on import and export – but as long as you're careful about this, sure, feel free to use this definable well-ordering to prove your theorem (as a theorem about L) or to model your physical system (within L).'

On the one hand, this seems an entirely reasonable and appropriate response. On the other hand, it represents a departure from the strict von Staudt-style understanding of the reduction: no surrogate has been located in V; the object doesn't appear, the model isn't constructed, in the standard **Generous Arena**; the relevant theorems aren't proved on the customary **Shared Standard**. If we're willing to go this far, why not invite the mathematician or physicist to use whatever consistent theory they like, Hilbert style, to view themselves as working within any old model of that theory, since we know V must include them? If ZFC+V=L is acceptable, why not ZFC+∼Con(ZFC)? At that point, there's nothing much left of **Generous Arena/Shared Standard**.

What's gone wrong here is related to a more general sense in set-theoretic circles that the advent of forcing in the 1960s and the subsequent explosion of independence results has undermined the notion that set theory is the study of a single universe or the pursuit of a single axiom system. Multiversism comprises various rivals to this 'universism' – from Joel Hamkins' 'entire cosmos of set-theoretic universes' (Hamkins [2012], p. 418) to the generic versions of Hugh Woodin [2011] and John Steel [2014] (also Steel [2024]).[35] Only Steel's version offers a first-order axiom system that stands as a potential alternative to ZFC and its extensions, and consequently, as a potential challenge to the set-theoretic foundations of §1. Indeed, it seems to me that Steel's thinking promises to illuminate the quandary of the previous paragraph: identifying a range of 'natural' models or theories that goes beyond a von-Staudt-like adherence to V alone but stops short of a Hilbert-like indiscriminate profusion. Let me explain.

Steel's approach can be understood as guided by the principles of *Maximize* and *Unify* from §1, but interpreted in his own, more subtle forms.[36] So, for example, where *Maximize* in §1 counseled the set theorist to include as many and various structures as possible in V, Steel's version is the admonition to maximize what he calls 'interpretive strength', the ability to 'interpret' as many theories as possible. For Steel, this isn't a simple call for higher and higher consistency strengths; not every interpretation in the usual sense, not every model, counts as an 'interpretation' – let's call it a

[35]See Maddy [2024] for an overview of leading multiversisms.

[36]Recall footnote 22.

suitable interpretation – in Steel's sense.[37] We've already implicitly assumed that interpreting ZFC+V=L in L is suitable, and the same could be said of inner models in general.[38] Also, though forcing extensions first appeared merely as technical devices, subsequent simplifications and experience have transformed them into mathematically workable outer models on a par with the familiar inner models. Steel includes these two constructions as suitable interpretations – should there be more?

Here the line of thought turns to some plain empirical facts. The first, noted in passing in §1, is that the large cardinal axioms proposed to date tend to line up in a simple hierarchy of consistency strengths.[39] The second, also noted in §1, is that the consistency strengths of 'natural'[40] extensions of ZFC tend to match up with this large cardinal hierarchy, each equiconsistent with ZFC+LCA for some LCA or another.[41] The third is that the proofs of these equiconsistencies tend to take a particular form: an upper bound on the consistency strength of a natural theory T is established by showing it's true in a forcing extension of the relevant ZFC+LCA; a lower bound is established by constructing an inner model of the relevant ZFC+LCA in a model of T. As long as these three patterns hold, it's reasonable to think that all natural extensions of ZFC will have suitable interpretations among the inner models and forcing extensions of theories of the form ZFC+LCA for some LCA.

This is the germ of Steel's multiverse. We saw in §1 how the set-theoretic imperative to *Maximize* arises from the foundational goal of set theory: set-theoretic foundations shouldn't curtail the freedom of pure mathematics or,

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[37]Steel speaks of 'meaning-preserving' interpretations, but Maddy and Meadows [2020] argue that appeal to 'meaning' is inessential to the project. 'Meaning' in a sense recognizable to a linguist, say, wouldn't play the role Steel envisions, and I admit to some skepticism that a philosophical theory could do better.

[38]An inner model is a formula $\varphi(x)$ such that ZFC proves $\forall x, y, z(x \in y \wedge y \in z \rightarrow x \in z)$, $\varphi(\alpha)$ for every ordinal α, and $\sigma^{\varphi(x)}$ for every axiom σ of ZF.

[39]It's worth noting that as natural theories increase in consistency strength, they agree on increasing ranges of 'concrete' mathematical statements, facilitating import/export rules. See Steel [2014], §3, or Maddy and Meadows [2020], §3.

[40]Recall footnote 20.

[41]Steel [2024], p. 22, footnote 25, adds a caveat: 'There are some qualifiers ... The most important is that this picture has as of now been only partly verified, and there are many fundamental open problems'.

for that matter, the structures available to natural science (as for our imagined mathematical physicist). In §1, this was read as recommending that V be, in the common metaphorical terms, as tall and wide as possible. In contrast, if *Maximize* read in Steel's way, as recommending there be as many suitable interpretations as possible, to accommodate as many natural extensions of ZFC as possible, this isn't enough – the forcing extensions and inner models of the large cardinal ranks must be included, as well. But now, with all these theories, all these models, what happens to *Unify*?

In fact, this echos yet another of Mac Lane's critiques of set-theoretic foundations:

> The Zermelo-Frankel axioms do not settle the famous continuum hypothesis – there are models of ZF for which the hypothesis holds and others for which it fails. The demonstration of the latter fact by Cohen's method of forcing suggests other 'independence' results and leads to the construction of many alternative models of set theory. Another result is the introduction of a considerable variety of axioms meant to supplement ZF. For these reasons 'set' turns out to have many meanings, so that the purported foundation of all Mathematics upon set theory totters. (Mac Lane [1986], pp. 358-359)

Steel [2024] (p. 23) cites Mac Lane's worry and takes up his challenge. From his perspective, all the natural extensions of ZF are valuable – they offer the mathematician or the physicist a variety of suitable interpretations in which to work – what Mac Lane is missing is how they interrelate. To remedy this,

> We seek a language in which all these theories can be unified ... in a way that exhibits their logical relationships, and shows clearly how they can be used together. That is, we want one neat package they all fit into. (Steel [2014], p. 165)

This is Steel's version of *Unify*: not a single extension of ZFC, but an overarching system in which all the natural extensions can be represented and their interrelations displayed – that is, represented and displayed in such a way that import and export conditions are clear, and 'we can use each other's work' (Steel [2014], p. 164).

Steel implements this idea in a first-order language $\mathscr{L}_{MV}$ with two sorts, for sets and worlds, with a $\mathscr{L}_{MV}$-theory MV that's intended to include a world, or inner model of a world, for each natural extension T. MV asserts, for example, that ZFC is true in each world and that a forcing extension or refinement of a world is a world.[42] MV is equiconsistent with ZFC, and likewise MV+LCAs (MV plus the assumption that ZFC+LCAs is true in each world) and ZFC+LCAs. Given the three empirical facts sketched above, every natural extension of ZFC should be represented by a world or an inner model of a world in the theory of MV+LCAs.

In this way, Steel's MV+LCAs has made the 'underlying unity' of 'the many different extensions of ZFC that we have investigated ... more visible' (Steel [2024], p. 23). As to their 'logical relationships', he then asks if there's a 'distinguished reference world ... an individual world that is definable in the multiverse language' (Steel [2014], p.168). If there is such a definable world, Woodin observed that there's only one; it's contained in every other world (which is to say that the intersection of all worlds is itself a world); and every world is of the form $C[G]$ for some C-generic G. Since Steel [2014] appeared, Usuba [2017], [2019] established that, assuming the existence of an extendible cardinal,[43] the multiverse does have a core C. The forcing relation is definable in C, so a suitable interpretation for any natural extension of ZFC can be understood with tools present in C. (Treating the definable forcing relation as encoding a description of the forcing extension is familiar from, for example, Kunen's influential textbook, in his discussion of 'forcing over V': 'we are making up names for, and talking about, objects in some generic extension of V which does not exist at all' (Kunen [1980], p. 234). Similarly, Steel [2024] (p. 30) uses the notation V^G for 'the (imaginary) ... generic multiverse generated by V' to be 'unpacked more fully using the definability of the forcing relation'.[44])

[42]MV also says that every world is a transitive proper class, that sets occur only in worlds, and, crucially, that any two worlds have a common forcing extension. This last, the Axiom of Amalgamation, is responsible for the fact that MV is axiomatizable. (See Maddy and Meadows [2020] for discussion and proofs. On p. 150, Meadows and I raise a worry about Amalgamation that now seems to me misguided. See Maddy [2024], footnote 16.)

[43]Goldberg [2024b] has shown that the extendible is essential. The existence of an extendible is a non-trivial assumption, well worth examining, but I leave it aside here.

[44]These are examples of what I've called 'Heuristic Multiversism': employing an intuitive picture of a multiverse to facilitate mathematical thinking. See Maddy [2024].

Assuming, then, that the multiverse has a core and that the three empirical facts hold true, C is a remarkable world where all natural extensions of ZFC and their interrelations are in effect accessible. The thought naturally arises that $V{=}C$ might be an attractive axiom. To express this idea in a multiverse language, let $\mathscr{L}^+_{MV}$ be $\mathscr{L}_{MV}$ with a new constant $\dot{V}$ and let MV$^+$ be MV extended to $\mathscr{L}^+_{MV}$ with the added axiom $\dot{V}$=C (that is, $\forall U(\dot{V} \subseteq U)$).[45] With this extended language, any φ in $\mathscr{L}_\in$ can be translated into $\mathscr{L}^+_{MV}$ as $\dot{V} \models \varphi$. In the other direction, ψ in $\mathscr{L}^+_{MV}$ can be sent to the $\mathscr{L}_\in$ formula $(V^G, V) \models \psi$, where V interprets $\dot{V}$ and V^G is the 'imagined' multiverse generated by V (actually described via the definable forcing relation, as we've seen). If these translation functions are called u and t, respectively, then $t(\dot{V}{=}C)$ says, roughly, 'V is the core of its own imagined multiverse'.[46] Abbreviating this $\mathscr{L}_\in$ formula to $V{=}C$, we can formulate two attractive theories: one in $\mathscr{L}_\in$ – ZFC+LCAs+$V{=}C$ – the other in $\mathscr{L}^+_{MV}$ – MV$^+$+LCAs+$\dot{V}{=}C$.[47]

Building on Meadows [2021], Steel observes that the two theories are equivalent in the sense that ZFC+LCAs+$V{=}C \vdash \varphi \leftrightarrow t(u(\varphi))$ and MV$^+$+LCAs+$\dot{V}{=}C \vdash \psi \leftrightarrow u(t(\psi))$. He concludes that they 'are equivalent as foundations' (Steel [2024], p. 31).[48] This isn't to say that he thinks they're equally good for day-to-day use – ZFC+LCAs+$V{=}C$ is simpler, with 'a more familiar syntax, so in the end, it would be better to work with it' (ibid., p. 32) – but it also isn't to deny the significance of MV$^+$+LCAs+$\dot{V}{=}C$.[49] By providing suitable interpretations for all natural

[45]By the Downward Directed Grounds Theorem of Usuba [2017], $\forall U(\dot{V} \subseteq U)$ is equivalent to the claim '$\dot{V}$ has no generic refinements' – the Ground Axiom of Hamkins and Reitz (see Reitz [2007])

[46]Parallel to footnote 45, this is equivalent to 'V has no generic refinements', which is expressible in $\mathscr{L}_\in$ by an important theorem of Laver and Woodin.

[47]For those hoping for a strong new axiom candidate, it's perhaps disappointing to note that the $\mathscr{L}_\in$ assumption $V{=}C$ tells us very little – e.g, if α is an ordinal, any model of ZFC has a forcing extension with $V{=}C$ that leaves the first α levels unchanged (Reitz [2007]) – though see Goldberg [2024a].

[48]The equivalence of the two theories also has a practical effect: 'anyone who is able to understand and work with the hypothesis $V{=}C$ will understand the simple equivalence ... and be able to shift between $\mathcal{L}_\in$ and $\mathcal{L}_{MV}$ easily' (Steel [2024], p. 32). Here Steel describes precisely the set theorists in a position to take advantage of the *Heuristic Multiversism* described in footnote 44.

[49]This nuance of Steel's thinking is missed in Maddy [2024]. Though he doesn't propose this multiverse theory as a rival or alternative to universe theories – the challenge posed

extensions of ZFC, Steel's multiverse delivers a satisfying answer for our puzzled mathematician or physicist: here is a range of natural theories, with mathematically workable models, whose interrelations are well-understood; if you locate your structures of interest in any one of them, you can feel confident that powerful mathematical tools will be available and any import/export limitations will be clear. This is the advertised path between the von-Staudtian and Hilbertian extremes.

With these two theories, two expressions of the same underlying idea, Steel has succeeded in implementing both *Maximize* and *Unify* on his understanding of those principles: he maximizes the natural theories suitably interpreted by including both inner models and forcing extensions; he then unifies all these either in a multiverse theory or in a universe theory that effectively encodes the multiverse. The result is a **Generous Arena** that includes not only the bounty of V as described by ZFC+LCAs, but a wide range of worlds or imagined worlds embodying all the natural extensions of ZFC+LCAs. The new **Shared Standard** allows for a principled range of natural axiomatic starting points whose interrelations are mathematically tractable. And finally, for purposes of **Metamathematical Corral**, either of Steel's theories brings together a new mathematical multiplicity under one first-order umbrella. Though proving the consistency of either unifying theory would be a grand achievement, implying the consistency of all natural theories at once, of course the familiar Gödelian forces are in play.

In sum, then, Steel's theories play new, improved forms of the familiar set-theoretic foundational roles of **Generous Arena/Shared Standard** and **Metamathematical Corral**. **Elucidation** and **Risk Assessment** can presumably be carried out in ZFC+LCAs or in ZFC+LCAs+V=C or in any world of MV$^+$+LCAs+$\dot{V}$=C; the former is also well-served by Augmented-ETCS, while the latter remains an open question. Category-theoretic foundations are less equipped for the other three set-theoretic jobs but they introduce and meet the new foundational goal of **Essential Guidance** for a wide swath of contemporary mathematics. Set theory, too, could be said to provide this for other branches of the subject. Finally, nothing at all precludes the identification of other goals that could reasonably be classified as 'founda-

at the end of my essay – he does see it as a significant complement, to be preserved, not simply kicked away once we reach ZFC+LCAs+V=C.

tional' or of other theories that might facilitate those goals. The foundational study of mathematics remains an appropriately open-ended undertaking.[50]

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