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ORIGINAL ARTICLE

Wittgenstein on mathematics

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Abstract

The mature Wittgenstein's groundbreaking analyses of sense and the logical must—and the powerful new method that made them possible—were the result of a multi-year process of writing, re-arranging, re-writing and one large-scale revision that eventually produced the *Philosophical Investigations* and *RFM* I. In contrast, his struggles during the same period with questions of arithmetic and higher mathematics remained largely in first-draft form, and he drops the topic entirely after 1945. In this paper, I argue that Wittgenstein's new method can be applied to the cases of arithmetic and set theory and that the result is innovative, recognizably Wittgensteinian, and independently appealing. I conclude by acknowledging the reasons Wittgenstein himself might have had to resist applying his own proven method to the case of mathematics—particularly to set theory—and by indicating why I think those reasons are ultimately unsound.

I. | INTRODUCTION

Understanding Wittgenstein on any topic is a notoriously tricky undertaking, but his post-*Tractarian* thinking on mathematics is particularly difficult to pin down, much less to assess, because it never reached the stable equilibrium of the *Tractatus* or the *Philosophical Investigations* (*PI*). This alone doesn't distinguish it from, say, the philosophy of psychology or colour or external world scepticism, but unlike these others, the philosophy of mathematics was a central preoccupation of his post-*Tractarian* thinking, lecturing and writing—that is, until he dropped the topic cold in the mid-1940s and never returned. The circumstances of this turn are suggestive, but for now, my point is that any effort to attribute a philosophy of mathematics to the mature Wittgenstein is necessarily engaged in speculation about how the raw material of his copious unpublished writings on the subject might be reasonably pieced together.

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So much for disclaimer. What I propose to do is to begin from a particular reading of Wittgenstein's method in the mature writings on rule-following and logic (Section II), then use that method as an interpretive guide to his contemporaneous writings on mathematics.[1] This is fairly straightforward for arithmetic and number theory (Section IV) but becomes more speculative for the case of set theory (Section V). Still, the hope is that this approach yields an original and illuminating, even attractively Wittgensteinian, view of mathematics. I conclude by acknowledging the reasons Wittgenstein himself might have had to resist applying his own proven method to the case of mathematics—particularly to set theory—and by indicating why I think those reasons are ultimately unsound. But first, some background.

II. | BACKGROUND

Wittgenstein's post-*Tractarian* writings can be divided into stages in various ways, but only two of these matter here: the transitional[2] period of *Philosophical Remarks* (*PR*) and the *Big Typescript* (*BT*) (1929–1936)[3] and the mature period of the *PI* and *Remarks on the Foundations of Mathematics* (*RFM*) (1936–1945). The composition of the *PI* was a multi-year process of writing, re-arranging, re-writing and one large-scale revision.[4] Through all the variations of the early drafts, beginning in 1937 and well into the 1940s, there were always two parts, corresponding roughly to what would become *PI* §§1–189 and *RFM I*. Then, in the draft of 1944, there is an abrupt and dramatic rethinking: The much-revised second part, on mathematics, is almost entirely replaced; the new draft consists of a version of *PI* §§1–189, a few pages on rule-following from *RFM I*, more-or-less *PI* §§189–197, plus new material corresponding roughly to *PI* §§198–421. In other words, all the material on mathematics beyond rule-following is gone, replaced by the private language argument and the philosophy of psychology. By 1945, more material on the philosophy of psychology had been added to make up the now-familiar *PI*.

Two points here are worth carrying forward. One is that the status of Part I of *RFM* is quite different from the rest of the book. Like *PI* §§1–189, *RFM I*

[1]This approach is also applied in Maddy (2022), to the question of external world scepticism in *On Certainty* (*OC*). (Note: Wittgenstein's writings are cited with the customary abbreviations as indicated in the text and the bibliography (*OC*, *PI*, *RFM*, etc.).)

[2]I prefer this term to the currently more common 'middle' period (e.g. Stern 2018) because Wittgenstein does seem to be in transition between the poles of the *Tractatus* and the *PI*. Still, I completely agree with Stern (2018: 2) that there was no single decisive moment that 'the path … was a long and complicated one, with many turning points and branching paths along the way'.

[3]Although they date to the same period, the *Blue and Brown Books* (1933–5) contain nothing like the sustained discussions of mathematics to be found in *PR* and *BT*, so I leave them aside here.

[4]See Baker and Hacker (2005: 1–6, 36–39), for a detailed account.

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went through multiple re-writings, re-arrangings and revisions between 1937 and 1942, while the rest of *RFM* is first draft material, despite its ultimate excision from the *PI*, *RFM I* is more authoritative than Wittgenstein's other unpublished writings. The other salient point is that Wittgenstein gave up on mathematics after 1945.

Another important source, the *Lectures on the Foundations of Mathematics* (*LFM)*—Cora Diamond's reconstruction of Wittgenstein's 1939 lectures from four sets of contemporaneous notes—is in its own special category: On the one hand, these obviously aren't Wittgenstein's own words; on the other, they report things Wittgenstein was apparently willing to say in his own voice, which eliminates the problem of locating his actual views in the welter of cross-talk in *RFM*. Under the circumstances, as a third takeaway, it seems reasonable to treat *LFM* as more-or-less on a par with the parts of *RFM* other than *RFM I*.

III. | THE MATURE APPROACH TO SENSE

A responsible reading of the discussion of sense in the mature period—including the central material on rule-following in the *PI* and on logic in *RFM I*—would take more space than is available here, but we need a brief sketch to illustrate the method I propose to take as settled and to apply later to the case of mathematics.[5] The *Tractarian* picture theory having collapsed, Wittgenstein ran through various alternatives during the transitional period, including a fairly straightforward verificationism at various points, especially in *PR*,[6] and a more relaxed understanding of language as a system of arbitrary grammatical rules, especially in *BT*.[7] All these new alternatives appear to involve rules of one sort or another, which troubles Wittgenstein throughout the transitional period, for example:

> If I have been given a general … rule, I must recognize each time anew that this rule can be applied *here*, too (that it holds for *this* case as well). No act of foresight can spare me this act of *insight*. For the form to which the rule is applied is in fact a new one at each new step. – But here it's not a matter of an act of *insight*, but of an act of *decision*.[8]

[5]See Maddy (2014) for more.

[6]For example, *PR* §43, p. 77: 'to understand the sense of a proposition means to know how the issue of its truth or falsity is to be decided'.

[7]For example, *BT* §18, 61e/77: 'All one can do … is to define the grammatical game, state its rules and leave it at that'. See also *BT* §60, 207e/264–265: 'The description of how a proposition is verified is a contribution to its grammar … The question about verification is only a particular form of the question "what does one do with this proposition?"'.

[8]*BT* §110, 379e. See also *BT* §149, 171.

By the *PI*, Wittgenstein is able to write instead:

> It would almost be more correct to say, not that an intuition was needed at every point, but that a new decision was needed at every point.[9]

'Almost … correct', but not correct, as the Wittgenstein of the *Investigations* has found his way around the puzzles of rule-following, sense and logic.

The reading of this resolution that I find most illuminating brings three distinct voices into direct engagement.[10] Two of these are already suggested in the transitional period remarks just cited. The first, the Voice of Temptation, insists that something or other—my training or intentions, my insight or intuition—serves to predetermine the correct application of a term to a new case; the second, the Voice of Correctness, appeals to the familiar Wittgensteinian counters to undermine Temptation and concludes that nothing determines correctness in a new case and that 'an act of *decision'* is needed every time. These two voices compete in the transitional period and a goodly portion of the *PI*, as well, but the novelty of the *PI* is the third voice, the Commentator, revealed in that 'almost correct'.

The Commentator's strategy begins with an ordinary observation: We never actually face the sorts of unresolvable conflicts that Correctness imagines, like the one over simple addition. Still, his protests do serve to undermine the sort of 'sublime' rules that Temptation insists upon—unsullied by contingencies like the way human subjects react to training, the role arithmetic plays in our lives, or the ubiquity of medium-sized objects stable enough to count. What Correctness misses is that those protests have no force against our actual rule-following practices, which depend, at the most general level, on exactly those three fundamental contingencies. In sum, then, Temptation's mistake is insisting on idealized rules; Correctness's mistake is to think that turning away Temptation is enough to undermine actual rule-following; in other words, they both implicitly presuppose Temptation's picture what effective rules must be like. In the face of this conflict, Wittgenstein's Commentator simply points out that their presupposition is false, that we don't need rules like that, and that we get by just fine with the ordinary rules that we have.

In this way, the mature Wittgenstein resolves, or dissolves, many concerns of the transitional writings: Words of natural language have a wide range of uses—no unified account of 'sense' is to be had. This much is common to the *PI* and *RFM I*, but *RFM I* goes on to ask about logical rules. While a philosopher might be brought to admit that the proper use of words depends on various contingencies, logical inference may be another matter. Predictably, Temptation protests:

[9] *PI* §186.

[10] This breakdown into three voices comes from Stern (2004) (who credits Cavell) as elaborated in Maddy (2014, 2022, Forthcoming).

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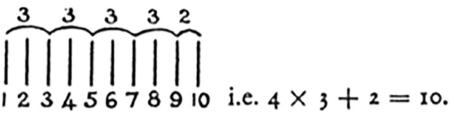
> But doesn't e.g. 'f(a)' have to follow from '(x)f(x)' if '(x)f(x)' is meant in the way that we mean it?[11]

Correctness replies that meaning depends on rules and, as above, that rules don't actually predetermine correct and incorrect responses, but this time, it's more of a challenge for the Commentator to convince them both that logical compulsion, like other rule-following, also depends on the trio of natural reactions, interests and practices, and the good behaviour of the world. This is where we hear, for example, of the odd shopkeeper, who uses rubber rulers so as to charge different customers different amounts (*RFM* I.5); both his natural reactions and his interests differ from ours. The *PI* also includes the example of cheese-weighing, where the world is uncooperative: What if 'lumps suddenly grew or shrank with no obvious cause[12]?' But both these cases involve measurement, not logic.

For Wittgenstein, logic and elementary arithmetic—calculations such as 2 + 2=4—are fundamentally intertwined, the same sort of thing, so here our example is the poor nut-sharer, doing arithmetic in a world where objects regularly just vanish:

Imagine someone bewitched so that he calculated:



> Now he is to apply this calculation. He takes 3 nuts four times over, and then 2 more, and he divides them among 10 people and each one gets *one* nut; for he shares them out in a way corresponding to the loops of the calculation, and as often as he gives someone a second nut it disappears.[13]

RFM I gives an eloquent description of the very general facts about the world on which our logic and elementary arithmetic depend:

> Put two apples on a bare table, see that no one comes near them and nothing shakes the table; now put another two apples on the table; now count the apples that are there … The result of the counting is probably four (We should present the result like this: when, under such-and-such circumstances, one puts first 2 apples

[11] *RFM* I.10, 41.
[12] *PI* §142.
[13] *RFM* I.137.

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> and then another 2 on a table, mostly none disappear and none get added.) And analogous experiments can be carried out, with the same result, with all kinds of solid bodies. – This is how our children learn sums; for one makes them put down three beans and then another three beans and then count what is there. If the result at one time were 5, at another 7 … then the first thing we said would be that beans are no good for teaching sums. But if the same thing happened with sticks, fingers, lines and most other things, that would be the end of all sums.[14]

The moral is that, despite the philosopher's attachment to "the crystalline purity of logic,"[15] once we open our eyes to the actual use of logic and elementary arithmetic in our lives, we see that they're contingent, too, that they only appear necessary because we feel compelled to infer and add as we do: "it is *we* who are inexorable in applying these laws."[16]

In sum, then, the mature writings on rule-following and logic offer a methodological template for approaching philosophical questions about a given practice. It begins with (at least) two voices—Temptation and Correctness—that personify often familiar opposing views, views that many philosophers, including Wittgenstein himself in certain moods, find attractive. A third voice—the Commentator—calls attention to the ordinary facts that actually make the practice work: our natural reactions, our customs and goals, and very general facts about the world. Of course, neither Temptation nor Correctness is satisfied with these mundane contingencies, so the Commentator goes on to isolate the idealizing preconception, the fixed idea about how the practice *must* work, that blinds them to the force of these simple facts. Human dealings aren't sublime, but they function effectively, nonetheless.

This is the approach I hope to apply to the case of mathematics in Sections IV and V, but first a lookback at the transitional period.

IV. | TRANSITIONAL THINKING ABOUT MATHEMATICS

The story of Wittgenstein's post-*Tractarian* engagement with mathematics beyond arithmetic equations begins with the various puzzles that bedevil him throughout the transitional writings, as he struggles to settle on a new theory of sense. Whatever the shortcomings of the *PR*'s verificationism as an account of ordinary sentences about the world, for mathematical claims, in Wittgenstein's

[14] *RFM* I.37, 51–52. I would say, though Wittgenstein wouldn't, that the long experience of our species in a world that behaves in this way explains why our natural inclinations are as they are and that evolutionary pressures have moulded our basic cognitive machinery so that it feels natural to add and infer as we do.

[15] *PI* §108.

[16] *RRM* I.118, 82.

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hands, it's disastrous. We have: 'how a proposition is verified is what it says'. For an ordinary sentence, we might think of its sense as complete specification of the various experiences that would confirm it; analogously, since a mathematical proposition is verified by its proof, its sense would be a complete specification of its proof. But if you have a complete specification of a proof, then you have the proof—a mathematical proposition without a proof has no sense, is senseless.[17]

The unwelcome consequences cascade from there. How can we look for a proof of a mathematical proposition when its sense is its proof? We have no idea what to look for in advance of having found it.[18] Wittgenstein sees the danger—'my explanation mustn't wipe out the existence of mathematical problems'[19]—but his various efforts at best cover only elementary cases.[20] Another aspect of the underlying problem is another disanalogy with ordinary claims:

> 'From two things, I infer that he's at home: first his jacket's in the hall, and also I can hear him whistling'. Here we have two independent ways of knowing. … whereas a mathematical proof is an analysis of the mathematical proposition.[21]

Two proofs imply two senses, two distinct propositions: 'there can't be two independent proofs of one mathematical proposition'.[22] Often enough, the purported second proof works by situating the proposition in a new context:

> If I know the rules of elementary trigonometry, I can check the proposition $\sin(2x) = 2\sin(x) \cdot \cos(x)$, but not the proposition $\sin(x) = x - (x^3/3!) + \ldots$ the sine function of elementary trigonometry and that of higher trigonometry are *different* concepts … as far as the system of elementary trigonometry is concerned, the second proposition make *no sense*.[23]

[17]See, for example, *PR* §162, 192: 'a mathematical proposition is only the immediately visible surface of a whole body of proof … A mathematical proposition – unlike a genuine proposition – is *essentially* the last link in a demonstration'. For a more familiar constructivist, for example the sense of p is 'there exists a certain type of proof'; for Wittgenstein, the sense of p *is* the proof.

[18]See, for example, *PR* §148, 170: 'What is it that goes on when, while we've as yet no idea how a certain proposition is to be proved, we still … proceed to look for a proof?'—*PR* §155, 183: 'If I hear a proposition of, say, number theory, but don't know how to prove it, then I don't understand the proposition, either'.

[19]*PR* §148, 170. See also *PR* §§151, 176.

[20]For example, Wittgenstein suggests 'a proof of relevance' (*PR* §149, 171, *BT* §110) or 'method of checking' (*BT* §120, e.g. 423e/625: 'there is a check for propositions of the form "there is a *k* between *n* and *m* such that …," which refer to intervals') or a 'method of search' (*BT* §122).

[21]*PR* §153, 179.

[22]*PR* §155, 184.

[23]*PR* §151, 177.

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This raises a new cry of dismay:

> Wouldn't this imply that we can't learn anything new about an object in mathematics, since, if we do, it is a new object?[24]

Obviously, Wittgenstein isn't comfortable with any of this.

Despite the trigonometric example, Wittgenstein is thinking mostly of arithmetic, where mathematical induction presents a somewhat different challenge. This isn't typically a case of more than one proof of the same claim; Often enough, the inductive proof is the only proof. Suppose, we want to show that all natural numbers have a certain feature: for all n, $\varphi(n)$. Perhaps simple computation can establish $\varphi(1)$, $\varphi(2)$, $\varphi(3)$, … , but we're looking for a proof that pulls all these special cases together and proves them all at once. Suppose, then, we're given a proof by induction, showing first '$\varphi(1)$' and second 'for all n, $\varphi(n)$ implies $\varphi(n+1)$'. But this second claim again concerns all natural numbers—how are its individual cases pulled together and proved all at once? What the inductive proof actually gives us is a mechanism for generating a proof in any particular case: 'Go in *this* direction … and you will arrive home'.[25] This is more systematic than the brute calculations we had before, but it's still just a way of generating the individual proofs of $\varphi(1)$, $\varphi(2)$, $\varphi(3)$, … , not a way of pulling them all together and proving them all at once.

Wittgenstein draws a stark moral from this analysis:

> The most striking thing about a recursive proof is that what it alleges to prove is not what comes out of it.[26]

We imagine that 'for all n, $\varphi(n)$' makes a claim about each and every individual natural number that's analogous to the way 'all the cats in this house are black' makes a claim about each and every individual cat in the house, but for Wittgenstein, the sense of these propositions is given by their verification conditions, and on examination, those are far from analogous:

> Compare the generality of genuine propositions with generality in arithmetic. It is differently verified and so is of a different sort.[27]

[24] *PR* §155, 183.
[25] *PR* §164, 196.
[26] *PR* §163, 193.
[27] *PR* §§166, 200.

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The inductive proof purports to provide a proof of $\varphi(n)$ for each and every individual number n, on analogy with a direct inspection of each individual cat, but it doesn't.

Turning to *BT*, although the analysis of sense is more lenient than the verificationism of *PR*, the position on mathematics is essentially the same: 'If you want to know *what* was proved, look at the proof'.[28] The familiar unpalatable consequences follow. For example, he asks again, can I search for a proof of a mathematical proposition?

> If I'm searching for something—I mean, the North Pole, or a house in London—I can *completely* describe what I am looking for before I've found it (or have found that it isn't there) … Whereas if I'm 'searching' for something in mathematics … I cannot describe what I am looking for … for if I could describe it in every particular, then I would already *have* it.[29]

Searching for a proof isn't like a Polar Expedition (a recurring image[30]): If I could mount such a search, I'd already be there; if I had a mathematical proposition with sense, I'd already have the proof. As before, an unproved claim is senseless, no proposition has two different proofs, and so on.

The treatment of mathematical induction in *BT* is also similar to that of *PR*, although more extended (*BT* §§126–135). Once again, there's the theme that an inductive proof provides, not proof of each $\varphi(n)$, but a form for generating a series of proofs: 'A recursive proof is … a law for the construction of proofs'.[31] Once again, it follows that when we take the proof by induction to establish 'for all n, $\varphi(n)$', this is generality of a different kind:

> This form of expression allowed the case of *all* numbers to be confused with the case of 'all the people in this room'[32]

To accept this new form of generality is to generate a new system: 'We fail to see that in doing so, we are starting a totally new calculus'.[33]

In the context of *BT*, where the sense of a proposition is given by its role in a system of grammatical rules, this idea of a 'new calculus' has more structure than the 'new concept' of sine in *PR*'s trigonometric example. In the new

[28] *BT* §120, 425e.

[29] *BT* §119, 421e/621.

[30] From, for example, *PR* §161.

[31] *BT* §133, 468e. See also *BT* §127, 446e, §128, 448e, or §133, 471e.

[32] *BT* §130, 457e/683. See also *BT* §130, 460e, or §133, 470e.

[33] *BT* §134, 474e.

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system, the inductive proof is simply an application of the grammatical rules—as Wittgenstein now says, it's 'criterial'.

> We have to keep in mind that '(*n*)f(*n*)' isn't a proposition until I have a criterion of truth and that then it only has the sense this criterion gives it.[34]

In other words, coming at the end of a proof by induction isn't evidence for the truth of a general claim about numbers, a *symptom*; coming at the end of a proof by induction is *what it is to be* a general claim about numbers. The concept of such a generality is *introduced* by the adoption of a grammatical rule, and furthermore, these rules are entirely arbitrary—'grammar is not answerable to any reality'[35]—so there's no danger of conflict with any pre-existing fact of the matter.

But now Wittgenstein considers a disturbing possibility. Suppose we've taken an inductive proof to establish that 'for all *n* and *m*, n + m = m + n'.[36] This implies, in particular, that 45 +88 = 88 + 45. It's tempting to think that, in the elementary system, we can also establish this same identity by direct calculation, but of course, '45+88=88+45' in the elementary system doesn't mean the same as the similarly shaped identity in the extended system. Still, the extended system must have its own rules for calculations, so the sentence '45 + 88 = 88 + 45' there would seem to have two proofs: one via the inductively proven generality, the other by direct calculation. After the inductive proof, he remarks:

> But how do I know that [45+88=88+45] without having proved it? How can a general proof give me a particular proof? For after all I could carry out the particular proof, and then how do the two proofs meet?[37]

Now the familiar discomfort of the two-proof problem becomes even more poignant, as Wittgenstein wonders: 'And what happens if they don't agree'?[38] The official response would be that they can't disagree because they have different proofs and therefore different meanings, yet Wittgenstein wavers. He's

[34] *BT* §128, 499e.
[35] *BT* §56, 184e.
[36] His actual case in *BT* §§126–135 and elsewhere is associativity, in particular, the inductive proof of associativity in Skolem's 1923 paper, 'The foundations of elementary arithmetic', reprinted in Skolem (1967: 305–306). Skolem develops what's now known as primitive recursive arithmetic, where free variables provide a limited form of implicit universal quantification.
[37] *BT* §129, 452e/674. Here he's imagining, not performing the simple additions, but running through each of the individual proofs of commutativity stepping up to 88, but either way, it's a simple matter of direct calculation.
[38] *BT* §129, 452e/674.

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tempted to think that the sentence '45 + 88 = 88 + 45' in the extended system does have two proofs, but that apostasy leads to another: If so, how can the rule of induction be arbitrary; don't its implications then have to agree with the results of straightforward computation?

PR and *BT* also contain some discussion of infinitary mathematics beyond arithmetic, especially the theory of sets. Although set theory would seem to be a thriving system with clear grammatical rules, which ought to be enough at least for *BT*, set theory is highly disfavoured. As *RFM* treats the topic in more detail, I leave this to Sections V and VI, but it would be malpractice to leave the transitional writings without citing this famous passage:

> The philosopher notices changes … which today's mathematician passes over calmly, with a blank expression on his face. – What will distinguish the mathematicians of the future from those of today will really be a greater sensitivity; and *that* will – as it were – prune mathematics; for then everyone will be more concerned with absolute clarity than with the invention of new games.[39]

This culminates in one of Wittgenstein's most beloved aphorisms:

> Philosophical clarity will have the same effect on the growth of mathematics as sunlight has on the growth of potato shoots. (In a dark cellar they grow several meters long.)[40]

We eventually follow Wittgenstein into that cellar, but for now, our concerns surround the senselessness of an unproved mathematical claim, the impossibility of proving a mathematical claim in more than one way, the particular puzzle of mathematical induction, and suchlike.

V. | MATURE THINKING ON PROOF AND NUMBER THEORY

Given that the problematic conclusions of the transitional period all spring from the rule-based accounts of sense, we should expect Wittgenstein's new mature thinking about sense to change the landscape, hopefully for the better. He now understands that words of natural language function not on a single unified recipe but in myriad different ways, that those functions are fixed by

[39] *BT* §122, 432e/643.

[40] *BT* §122, 433e/643.

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the role of those words in our linguistic practices, and that those practices rest on the three pillars of our natural reactions, our interests and goals, and the way the world is. In this light, it seems we should turn our attention to the inclinations of mathematicians, the customs and goals of their practice and the worldly features that make that practice possible.

This Wittgenstein does, to bracing effect. For example, there's this passage, which the editors date to 1939–1940:

> Of course it would be nonsense to say the *one* proposition cannot have two proofs – for we do say just that.[41]

Or these remarks from the early 1940s:

> Ought I to say that the same sense can have only *one* proof? …. It all depends *what* settles the sense of a proposition, what we choose to say settles its sense. The use of the signs must settle it.[42]

> A proof of a proposition shows me what I am prepared to stake on its truth. And different proofs can perfectly well cause me to stake the same thing.[43]

Of course, I've selected these remarks from the welter of voices that compete in these writings, but I do so guided by the model of the *PI* and *RFM I*: Temptation thinks the sense of a mathematical proposition is fixed independently of the proof, independently of the practice; Correctness, on full display in the transitional writings, thinks the sense of a mathematical proposition is fixed by its proof; and the Commentator, as in the passages just quoted, thinks both are assuming that sense must be something perfectly explicit and determinate, when human practice doesn't actually work that way.

Another bit of evidence here comes from *LFM*:

> The word 'proof' changes its meaning, just as the word 'chess' changes its meaning. By the word 'chess' one can mean the game which is defined by the present rules of chess or the game as it has been playing for centuries past with varying rules.[44]

[41] *RFM* III.58, 189.

[42] *RFM* VII.10, 366–367.

[43] *RFM* VII.43, 409.

[44] *LFM* III, 39.

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Then crucially:

> *We* fix whether there is to be only one proof of a certain proposition, or two proofs, or many proofs. For everything depends on what we call a proof.[45]

It seems relatively clear, at least as clear as matters can be in Wittgensteinian interpretation, that he takes his new understanding of sense to solve the problems of unproved claims and claims with multiple proofs that so troubled him during the transitional period.

The special case of mathematical induction can be treated along similar lines, and the absence of the topic from *RFM* suggests that Wittgenstein may have been satisfied with some such resolution.[46] Wittgenstein does address the topic in his final lecture of 1939: 'Let's consider a proof by mathematical induction'.[47] He inserts an inductive argument that 'for all n, $\varphi(n) = \psi(n)$',[48] then raises the sticky question about $\varphi(3000) = \psi(3000)$: It now has one proof in elementary arithmetic—a long derivation from 1 to 3000 of successive induction steps[49]—and another in number theory[50]—a universal instantiation from the inductively established generality.

> We could prove this by going through all 3000 steps, but this would be very long. So we take a short cut. Now what's responsible for this? How is it that we can leave all these steps out? Someone might say, 'This is rash'. But of course it isn't. … You haven't the faintest doubt you can do this.[51]

Our natural reaction to mathematical training is to accept inductive proof as obviously legitimate and the simple equations of the extended practice of number theory as provable in two, indeed a variety of ways.[52] When he raises the

[45]*LFM* III, 39.

[46]Goldfarb (2018: 252, footnote 17) notes discussions of Skolem's proof in manuscripts as late as 1940. For example, this mature-sounding passage comes from a 1940 manuscript: 'the induction scheme represents a technique for forming expressions … why shouldn't that be useful'? (IDP: MS 177, 170). As far as I can tell, no mention of mathematical induction made its way into a typescript after the mid-30s. There is a passing reference pencilled into TS 222, the basis for *RFM I*, between what became §31 and §32, but it's too fragmentary and garbled even to parse, let alone interpret (which likely explains why it was deleted by the editors of *RFM*). Thanks to Sorin Bangu for his help with the *Nachlass*.

[47]*LFM* XXXI, 287.

[48]Strictly speaking, Wittgenstein's formulates a quantifier-free version of the usual inductive proof, where generality is carried by free variables. See Marion and Okada (2018) for discussion.

[49]Calculating $\varphi(3000)$ and $\psi(3000)$ directly and comparing the results might also work, making three proofs.

[50]I use this term for the extended system of elementary arithmetic plus induction.

[51]*LFM* XXXI, 287–288.

[52]See footnote 50.

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challenge from *BT*—what happens 'if we should go through the 3000 steps and *not* get the predicted result'[53]—his immediate answer is, 'we're going to say that we've made a mistake',[54] that we've miscalculated. The practice rests in part on the fact that 'We should in the great majority of cases get the same result'.[55]

The problem Wittgenstein is facing here can be approached from two different angles. First from the perspective of *BT*: If a powerful infinitary superstructure is to be added to the existing computational system of elementary arithmetic, not just any superstructure will do; its implications have to cohere with those direct calculations. This means that the new grammatical rules can't be arbitrary, after all, and this in turn suggests that a substantive justification is required. Second, from the point of view of the mature theory, Wittgenstein's understanding of rule-following is a three-legged stool, and we have, so far, only the first—natural reactions—and the second—a functioning practice—which leaves the third—very general facts about the world. We've seen that our elementary arithmetic rests on a world of commonplace objects suitable for counting (recall the nut-sharer); how can the number-theoretic superstructure be relied on not to clash with the finitary computational system, especially given that the world doesn't appear to contain a simply infinite sequence of the sort number theory seems to posit?

Wittgenstein's response to these worries can be understood again on the model of the three voices. Temptation claims to grasp an abstract omega sequence of some kind, in which finitary elementary arithmetic can be embedded and mathematical induction is true. Correctness insists that this supposed grasp can't bear the weight intended:

> You are so to speak shewing us the picture of an unsurveyable series reaching into the distance. But what if the picture began to flicker in the far distance?[56]

As with sense, the Commentator takes the critique from Correctness to be effective against Temptation's ideal but not against our actual number-theoretic practice. Simple observation reveals that our mathematical training equips each of us with an intuitive picture of an omega sequence and that these individual pictures are similar enough to support our practice:

[53] *LFM* XXXI, 289.

[54] *LFM* XXXI, 289.

[55] *LFM* XXXI, 290. Goldfarb (2018) traces Wittgenstein transitional thinking about mathematical induction, not through *PR* and *BT*, but in *M*. In his concluding paragraph, he cites this passage from the final lecture of *LFM* and remarks: 'What is most striking to me in the 1939 lecture is that many of the same points as he articulated in the period of *M* still persist, but expressed in a different framework … more akin to Wittgenstein's later view … most importantly, … he's not attacking the mathematics … his attention … has a much more "anthropological" feel: how people calculate, what empirical facts are involved in our counting and calculating, and so on' Goldfarb (2018: 252).

[56] *RFM* V.10, 268.

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> Did you ever hear of such a case? Did you ever hear of a man imagining that the number series goes the other way? No. But these are very important facts. If *many* people did such things, this would affect the nature of our calculus.[57]

Both Temptation and Correctness make the mistake of assuming that mathematical induction requires justification in terms of some kind of abstract necessity. In response, the Commentator observes—as a matter of contingent fact—that human cognitive mechanisms facilitate an intuitive picture that's sufficiently shared, sufficiently determinate and sufficiently compatible with elementary arithmetic to guide our productive mathematical theory of numbers.[58] We have no guarantee that this will work, but we succeed, nonetheless. And that's enough.

VI. | MATURE THINKING ON SET THEORY[59]

What we've seen so far is a fairly straightforward reading of how Wittgenstein came to terms with arithmetic and number theory, but higher mathematics, especially set theory, is a different story. Still, it seems to me that an insightful and illuminating three-voice approach can be reconstructed from *RFM* and *LFM* even if Wittgenstein himself was reluctant to draw these conclusions beyond the natural numbers. This reconstruction is the topic of this section; his reluctance is taken up in the next.

Wittgenstein's sustained discussions of set theory focus on Dedekind's foundations for the calculus—specifically Dedekind cuts—and Cantor's transfinite numbers—specifically Cantor's theorem. In the course of *RFM*, one or another voice charges these with being ‘fanciful’ or ‘imaginary’[60] or ‘fantastic’.[61] In *LFM*, Wittgenstein himself lodges a similar criticism of the claim that a line intersects a circle at two points, even when those points are imaginary:

> Some mathematicians get an aesthetic pleasure from their work. [Inserts a drawing of a circle with a vertical line a small distance to its right.] ‘The line cuts the circle but in imaginary points.’ This has a certain charm … if you like that kind of thing.[62]

[57] *LFM* XXXI, 291.

[58] I think Wittgenstein is right about this (see, e.g. Maddy 2018), although I think we can (and do) go on to explore the sources of this psychological fact.

[59] This section elaborates §5 of Maddy (Forthcoming).

[60] *RFM* V.29, 285.

[61] *RFM* V.5, 260.

[62] *LFM* I, 16.

Considering the same example in *RFM* some years later, he writes, ‘this is essentially a perspective, and a far-fetched one’,[63] but he immediately adds, ‘which does not express any reproach’.[64] So these passing shots aren't necessarily decisive. My goal here is to find a credibly Wittgensteinian story that turns away a corresponding reproach of set theory.

In Dedekind's case, Temptation thinks that ‘the real numbers are there spread out in the number line’,[65] or more dramatically that

> God knows *all* irrational numbers … They are already all there, even though we only know certain of them.[66]

From this point of view, what Dedekind does is twofold: He uses the idea of cuts in the line to characterize what it is to be continuous and uses cuts in the rationals to provide the corresponding real numbers. On the other side, Correctness, ever more loquacious, insists that:

> The picture of the number line is an absolutely natural one up to a certain point; that is to say so long as it is not used for a general theory of real numbers.[67]

He complains of a sort of mission-creep with the notion of a ‘cut’, which starts out innocently enough:

> That every rational number can be called a principle of division of the rational numbers is perfectly clear. Now we discover something else that we can call a principle of division, e.g. what corresponds to $\sqrt{2}$. Then other similar ones – and now we are already quite familiar with the possibility of such divisions.[68]

This works because these principles of division make it ‘possible to say of any arbitrary rational number that it is on one side or the other of the cut’.[69] Where Dedekind goes wrong, Correctness continues, is that he slips from this so-called ‘intensional’ understanding to the more general ‘extensional’ understanding of a cut: the set-theoretic idea of any division of the line into upper and lower, whether or not it's effected by such a principle. The

[63] *RFM* V.21, 280.
[64] *RFM* V.21, 280.
[65] *RFM* V.37, 290.
[66] *RFM* VII.41, 408.
[67] *RFM* V.32, 286.
[68] *RFM* V.34, 288.
[69] *RFM* V.34, 288.

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extensional notion is objectionable because it rests on Temptation's idea that the points on the line and the corresponding numbers exist independently of our definitions and constructions. Here, Temptation and Correctness agree that the viability of Dedekind's move hinges on the abstract existence of the reals.

Once again, the Commentator might endorse Correctness's initial critique of Temptation's fantasy …

> The mathematician is an inventor, not a discoverer.[70]

… but deny that this entails rejection of Dedekind's mathematics. The move from intensional to extensional is justified, not by God's irrationals, as the false presupposition demands, but by several ordinary contingent facts: Dedekind and other mathematicians then and now find the move entirely natural, and it not only coheres with existing mathematical practice but also solves one of its most pressing problems (how to prove the basic theorems of the calculus). As an added bonus, Dedekind's clarifications turn out to be immensely fruitful—for example, a clear understanding of the basics of the calculus made it possible to extend the theory into higher analysis.[71] This is justification enough.

The famous discoverer/inventor remark comes, not from the explicitly set-theoretic contexts of Dedekind and Cantor, but from a notable passage in the more authoritative *RFM* I. It begins:

> What, then – does [mathematics] just twist and turn about within these rules? – It forms ever new rules: is always building new roads for traffic; by extending the network of old ones.[72]

This can be read as a description of what Dedekind has done: not simply following where the existing roads lead but forging a new one of his own devising. Temptation raises the alarm:

> But then doesn't it need a sanction for this? Can it extend the network *arbitrarily*?[73]

Presumably, Correctness would see no constraint, echoing the insistence in *BT* that rules answer to no reality. But then we hear:

[70] *RFM* I.168, 99.

[71] For example, the Lebesgue integral or the differential manifolds of general relativity.

[72] *RFM* I.166, 99.

[73] *RFM* I.167, 99.

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> A mathematician is always inventing new forms of description. Some, stimulated by practical needs, others, from aesthetic needs, – and yet others in a variety of ways.[74]

Here, the Commentator might be insisting that moves like Dedekind's are constrained—contra Correctness—but not by the abstract world of irrationals—contra Temptation. This is what we've just seen: Dedekind's invention was stimulated by mathematical needs and goals and was accepted because it successfully met them.

Wittgenstein's much-discussed treatment of Cantor's theorem in *RFM II* follows a similar trajectory. Temptation sees Cantor in general and this proof in particular as introducing the mathematical community to a previously unknown 'paradise' of the infinite: 'It introduces us to the mysteries of the mathematical world'.[75] Correctness protests that the proof doesn't establish what's been claimed for it, offering a particularly compelling thought experiment:

> The diagonal procedure for the production of a real number [might have been] well-known before the invention of set theory, and familiar even to school children. … The question set would perhaps be: write down a decimal number which is different from the numbers … (Imagine a long series.)[76]

The child hits on the diagonal procedure—a clever move, A+—but she wouldn't draw any conclusion about infinite numbers.

> This changes the aspect of Cantor's discovery … [It] might very well have consisted *merely* in the interpretation of this long familiar elementary calculation.[77]

Correctness is right that the diagonal procedure alone doesn't force us to interpret it in Cantor's bold new framework—the set of reals has a number defined in terms of one-to-one correspondence—just as nothing forces Dedekind to move to the extensional notion of a cut. So interpreted, what Cantor gives us is a 'puffed up proof'.[78]

Here, as with Dedekind, the Commentator agrees with Correctness that Cantor's interpretation of the diagonal argument isn't forced upon us, doesn't represent a discovery within Temptation's paradise. But again, Correctness

[74] *RFM* I.167, 99.
[75] *RFM* II.40, 137.
[76] *RFM* II.17–18, 131.
[77] *RFM* II.17, 131.
[78] *RFM* II.21, 132.

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takes this rebuttal of Temptation to amount to a case against Cantor's conclusion, implicitly endorsing Temptation's assumption that it must be a discovery to be acceptable. In contrast, the Commentator takes Cantor to have 'built a new road', one that comes naturally to mathematicians (once it's pointed out) and that enables tremendous mathematical riches, the entire theory of sets; this is how mathematics works. Both voices are led astray by their shared presupposition:

> If you say that mathematical propositions are about a mathematical reality … it has very definite consequences. And if you deny it, there are also queer consequences … Both would be quite wrong.[79]

They both think that mathematical practice requires a metaphysical justification from outside itself. It doesn't.

VII. | CONCLUSION

Needless to say, this is not the tone Wittgenstein takes in his evaluation of set theory.

> I believe, and hope, that future generations will laugh at this hocus-pocus.[80]

Interestingly, as far as I can tell, the condemnations of set theory are considerably more plentiful in *PR* and *BT* than in *RFM* or *LFM*. For example:

> The theory of aggregates buys a pig in a poke.[81]

> Mathematics is ridden through and through with the pernicious idioms of set theory.[82]

> Set theory is wrong because it apparently presupposes a symbolism, which doesn't exist instead of one that does exist (is alone possible). It builds on a fictitious symbolism, therefore on nonsense.[83]

> The general discussions of set theory … always strike us as blather … We feel that something is going on here that isn't quite right.[84]

[79] *LFM* XIV, 141.
[80] *RFM* II.22, 132.
[81] *PR* §170, 206.
[82] *PR* §173, 211.
[83] *PR* §174, 211.
[84] *BT* §137, 493e/747.

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And, of course, *BT* is also the cite of the infamous ‘pruning’ passage quoted at the end of Section III, above. The relative rhetorical cooling between the transitional and mature writings is perhaps some indication that the new approach to sense has opened up the deeper understanding of set theory suggested in the previous section, even if Wittgenstein can't bring himself to fully embrace it.

The remaining negativity in Wittgenstein's treatment of set theory no doubt arises, in part, from his annoyance at widespread glorification of what Dedekind and Cantor achieved, to the celebration of their exciting discovery of new mathematical realms—of extensional reals, of transfinite numbers—when what they really did was form new productive concepts:

> The dangerous, deceptive thing about the idea: … ‘The set [of reals] is not denumerable’ is that it makes the determination of a concept – concept formation – look like a fact of nature.[85]

Still, I don't think that all of Wittgenstein's lingering antipathy can be accounted for in this way.[86] Instead, I'd like to suggest that a pair of closely related empirical errors block him from accepting set theory as a viable mathematical practice, though (as was hopefully established in the previous section) this benign take is entirely open to him.

One of these errors is contained in another famous passage, this time from *LFM*:

> Hilbert: ‘No one is going to turn us out of the paradise which Cantor has created’. I would say, ‘I wouldn't dream of trying to drive anyone out of this paradise’. I would try to do something quite different: I would try to show you that it is not a paradise – so that you'll leave of your own accord. I would say, ‘You're welcome to this; just look around you’.[87]

The suggestion is that once Wittgenstein establishes that Cantor hasn't in fact opened up a wondrous new realm, mathematicians will simply lose interest in the transfinite:

> If you can show that there are numbers bigger than the infinite, your head whirls. This may be the chief reason this was invented. The

[85] *RFM* II.19, 131.

[86] Maddy (2014: 106–107) includes a second psychological influence, Wittgenstein's distaste for scientific progress. I suspect that for the case of modern mathematics, Wittgenstein simply can't bring himself to move beyond Correctness. Here, I reach for empirical explanations instead.

[87] *LFM* XI, 103.

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> misunderstandings we are going to deal with are misunderstandings without which the calculus would never have been invented, being of no other use, where the interest is centered entirely on the words which accompany the piece of mathematics you make.[88]

> I am *not* saying that transfinite propositions are *false*, but that the wrong pictures go with them. And when you see this the result may be that you lose interest.[89]

This seems to me factually wrong. Set theorists have always differed widely in their beliefs about the nature of their discipline, and though some may approximate the Voice of Temptation, a great many others don't—yet they all continue to pursue the theory with equal zeal. For many, if not all, the mathematics itself is their primary motivation, not some metaphysical whiz-bang. If convinced by Wittgenstein that Temptation is wrong, they might well shrug this off as mere philosophy and continue their work as before.

This leads to what I take to be Wittgenstein's second error, namely, that he didn't see the use of set theory, what's actually interesting about it.

> These considerations say lead us to say that $2_0\aleph < {}^{0}\aleph$. … That is to say: … we can say *this* and give *this* as our reason. But if we do say it … It is for the time being a piece of mathematical architecture which hangs in the air, and it looks as if it were, let us say, an architrave, but not supported by anything and supporting nothing.[90]

This claim ignores many uses of set theory that were obvious even in Wittgenstein's day. We've seen that Dedekind's account of the reals clarified the underpinnings of the calculus and enabled higher analysis and that Cantor opened up the vast and flourishing field of higher set theory. Meanwhile, Zermelo's axiomatization and subsequent reductions established set theory an arena generous enough to contain versions of all mathematical structures and a theory strong enough to prove all mathematical theorems: even those who disapprove of the higher flights of the subject aren't above calling on the clarity and power of its language and various of its foundational functions.[91] And there is so much more.

These blind spots may have blocked Wittgenstein from applying his Commentator's method to the case of set theory, but we've also seen that, in his maturity, he's more restrained in his opposition than he had been during the

[88] *LFM* I, 16–17.

[89] *LFM* XIV, 141.

[90] *RFM* II.35, 135.

[91] See Maddy (2017, 2019).

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transitional period and often uncomfortable siding unequivocally with the Voice of Correctness. This ambivalence may well explain why he eventually reconfigured the *PI* to avoid the philosophy of mathematics and dropped the subject entirely after 1945. Still, I submit that his writings on mathematics contain much of value, starting with his case that our contingent practice has the wherewithal to stand alone, without abstract metaphysical backing, and his insight that groundbreaking contributions like those of Dedekind and Cantor hinge on forming powerful new concepts, not discovering surprising new realms. For these gifts, we can certainly be grateful![92]

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