To appear in S. Arbeiter and J. Kennedy, eds., *The Philosophy of PM.*

On multiversism

Faced with an intractable problem, some philosophers employ a singular strategy: their idea is to dismiss or dissolve the problem in some way, as opposed to meeting it head on with a proposed solution. Multiversism in many of its varieties has recently emerged as a popular application of this approach to the continuum problem: CH is true in some worlds, false in others; the effort to settle it one way or the other is misguided, a pseudo-problem. My goal here is to examine a few actual and possible implementations of this strategy, but first, in the interest of transparency, I should acknowledge a tendency toward the opposing view of CH. At least for now, I believe that one of the most pressing questions in the contemporary foundations of set theory is how to extend ZFC (or ZFC+LCs) in mathematically defensible ways so as to settle CH (and other independent questions) and to produce a more fruitful theory. It seems best to begin by sketching in my own peculiar take on this opposing view. Then, with this as backdrop, I'll turn to multiversism.

I. Background

Many defenders of the project of settling CH and other independent questions – I'm going to stop saying that, but please take the broader project as understood – many defenders of the project do so in familiar philosophical terms. Perhaps most familiar is a straightforward embrace of an objectively existing realm of abstracta, of sets, in which CH is determinately true or false (though we don't currently know which). This approach is reasonably attributed to Gödel in his famous paper 'What is Cantor's continuum problem?' (Gödel [1947/64]), but others have proposed alternative versions (including me in my youth (Maddy [1990])). Years ago, Paul Benacerraf presented a particularly compelling formulation of an epistemological challenge to Gödel – how can we ordinary physical humans come to know anything at all about this non-spatiotemporal, acausal abstract realm? (Benacerraf [1973]) – and since then, that challenge has evolved through various changes in thinking on general epistemology without losing any of its force (see, e.g., Field [1989], pp. 25-30).

While I think this challenge is serious – my youthful self once tried to meet it – I now see a more fundamental worry about views like Gödel's as logically prior to Benacerraf's. To see this, consider the various defenses that set theorists, past or present, have offered for new assumptions or methods: Cantor was out to extend a theorem on representing functions by trigonometric series; Dedekind wanted characterizations of algebraic ideals or real numbers that don't depend on particular algorithms, series, sequences, or representations, that yield more general theories in abstract algebra and topology; Zermelo hoped to codify the power and reach of the nascent theory of sets, including its key foundational role,[1] while avoiding the threat of contradictions; defenders of large cardinal axioms from inaccessibles to extendables appeal, among other things, to the iterative conception of set, citing their axioms as attempts to maximize the height of V, and to the role of large cardinals as a tool for comparing consistency strengths; contemporary supporters of determinacy hypotheses mount a compelling case so varied and complex that I won't try to summarize it here, but one striking fact is that the determinacy axiom $AD^{L(R)}$ is implied by many, perhaps all, 'natural' theories of sufficiently high consistency strength.[2] These mathematical gains are seen as good reason to adopt and to axiomatize the theory of sets.

At this point, Benacerraf challenges Gödel to explain how these methods manage to track his seemingly inaccessible realm of abstracta, given what we take to be our limited human abilities. This is where I think something has gone wrong. In addition to the mathematical considerations that support a given hypothesis, the Gödelean metaphysics is imposing an additional extra-mathematical condition, the need to check that the hypothesis squares, somehow, with a viable epistemological account that tracks the demanding metaphysics. Starkly put, it seems to me that the mathematical considerations,

---

1 See [2017], [2019], for what's intended by here by 'foundational role'.

2 This is what Koellner calls 'overlapping consensus'. See Koellner [2011], §4.5.

strong as they are, should be enough by themselves. Benacerraf wants Gödel to supply a non-trivial epistemology; I think he shouldn't be positing a metaphysics that requires one.[3]

Of course, Gödel takes on (his version of) the Benacerrafian challenge, not my worry. He begins with 'intuitive' or 'intrinsically necessary' claims – for example, axioms that 'force themselves upon us as being true' (Gödel [1964], p. 268). These are so-called 'intrinsic' considerations. In addition, as is well-known, he recognizes a second mode of justification in terms of an axiom candidate's 'success' – 'extrinsic' considerations. The nature of this success depends on the analogy with natural science. First, there are 'verifiable consequences', that is:

> consequences demonstrable without the new axiom, whose proofs with the help of the new axiom, however, are considerably simpler and easier to discover, and make it possible to contract into one proof many different proofs. (Gödel [1947/64], p. 182/261)

Presumably, the analogy here is with observations and experimental results. More ambitiously,

> there might exist axioms so abundant in their verifiable consequences, shedding so much light upon a whole field, and yielding such powerful methods for solving problems … that, no matter whether or not they are intrinsically necessary, they would have to be accepted at least in the same sense as any well-established physical theory. (Ibid.)

Here we have the counterpart to theory formation and testing.

This approach introduces a range of well-honed tools from the philosophy of science: confirmed predictions, theoretical

[3] A quick note on terminology. I use 'metaphysics' for the study of what there is, what the world includes, what's real and what it's like, as opposed to 'epistemology', theory of knowledge, the study of how we come to know these facts. 'Ontology' is the branch of metaphysics focused on the question of what entities there are.

unification, explanatory power and scope, and so on. In his paper 'Mathematical evidence' (Martin [1998]), Tony Martin gives an impressive example from his defense of the consistency of ZF+AD:[4] he proves a general theorem about sets of recursive degrees from AD (the Cone Lemma), then checks that various implications of this theorem for particular sets of degrees were all provable from ZFC. Martin summarizes: 'I take it to be intuitively clear that we have here an example of prediction and confirmation' (ibid., p. 224).

By itself, this science/mathematics analogy solves neither the neo-Benacerafian problem nor my prior methodological problem: the mechanism of intuition still needs to be spelled out and an external requirement is still being imposed. But even if these could be set aside, there'd be other points of tension in its treatment of extrinsic considerations. Some concern the analogy itself: for example, in what sense is proving a theorem from ZF comparable to observing an experimental outcome? Others concern the corresponding scientific principles that purportedly validate their mathematical cousins: for example, is inference to the best explanation really a reliable rule? Are simple theories more likely to be true? But, in addition to these quite legitimate concerns, I think there's another that's particularly salient from a set-theoretic point of view.

Consider again the history of set theory. In the aforementioned algebraic work, Dedekind introduced sets simply because he thought representation-free definitions and a more abstract approach would

[4] That is, ZF plus the Axiom of Determinacy.

generate good mathematics. (He was right about that.) Zermelo, who pioneered something like the intrinsic/extrinsic distinction well before Gödel (in Zermelo [1908]), was happy to lump all his extrinsic considerations together, appealing simply to their promotion of 'productive' mathematics, with no thought to fitting them into an analogy with natural science. Similarly for contemporary defenses of large cardinals as a natural measure of consistency strength or $AD^{L(R)}$ as an apparent consequence of any natural increase along that hierarchy of those strengths.

Now perhaps there's a way to fit all these into a natural-scientific model, but the more one explores the development of set theory, the more efforts to retroactively impose broad categorizations from the philosophy of science begin to recall Wittgenstein's remark about efforts to translate mathematics into the notation of *Principia Mathematica*:

> … if tables, chairs, cupboards, etc. are swathed in enough paper, certainly they will all look spherical in the end.
> (*RFM* III §53, p. 185).

Once again, as a general point, it seems to me that an axiom or concept or method with clear mathematical benefits that blocks no significant mathematical avenues shouldn't be held to any additional philosophical standard. But the specific challenge for those reliant on a science/mathematics analogy is a choice between two unappealing options: reject some seemingly legitimate mathematical justifications because they resist inclusion under the 'science-like' umbrella or take 'science-like' to be so amorphous as to have no actual bite.

Of course, few people these days would explicitly endorse a Gödelian position on the subject matter of set theory. Still, I would argue that many contemporary positions face analogous challenges. This is easy to see for accounts that identify an alternative abstract subject matter for set theory: de-re structures, second-order logical truths, modal facts. Each such account owes an explanation of how our actual set-theoretic methods – including the extrinsic ones – manage to track the relevant metaphysics. They may not face the immediate roadblock of the Benacerraf objection to human knowledge of abstract mathematical *objects*, but this doesn't absolve them of their epistemic duties,[5] nor does it justify the imposition of inappropriate burdens.

Views that appeal to concepts or meanings may sound less loaded at first blush, but when it becomes clear that these aren't the sorts of concepts or meanings studied by cognitive science or linguistics, questions of precisely what they are and how they're known tend to arise anew.[6] Exploring Gödel's conceptualism, Martin writes:

> this concept of set seems perfectly objective … We did not create it, though we have singled it out as something to study. (Martin [2005], p. 362)

Recognizing the epistemological question immediately raised by this objectivity, he continues:

---

[5] Sometimes this obligation is well masked, as for example, Leng's apparently straightforward fictionalism ends up turning on contentful modal facts akin to those of Hellman. See Leng's essay in this volume and my response.

[6] Steel presents a special case: he does appeal to 'meaning' – as in 'the meaning we assign to the word "set"', and to 'synonymy', as in 'translations' from LMV to L_ϵ – and not in any ordinary or scientific senses of these terms, but Meadows and I have argued that these appeals are actually inessential (Maddy and Meadows [2022], §5).

> the objectivity of the concept does not, however, imply that we lack epistemic access to it … We *understand* the concept and we can *explain* it … When we work in the mathematical subject of set theory, we can think about what we are doing as finding out what is implied by the concept. (Ibid.)

No one versed in the Benacerraf challenge is likely to be satisfied by this appeal to 'understanding'. It seems more a label for the problem than a solution: how does our human understanding manage to track the truth about this objectively existing concept? Peter Koellner appeals to Gödel's endorsement of 'the belief that for clear questions posed by reason, reason can also find clear answers'.[7] Koellner intends to place the epistemological onus on the human faculty of Reason and rational intuition, as understood by Charles Parsons (Parsons [2008], chapter 9), but I doubt that Parsons himself would claim to have given a fully satisfactory account of how this faculty functions.

By now, I think it's clear that each of these proposed subject matters – abstract ontology, truth value realism, second-order validity, modal facts, conceptualism, meanings – requires a non-trivial epistemology of one sort or another. Robust Realism is the label I've used for any such reading of set-theoretic practice. Robust Realisms are subject both to a neo-Benacerrafian epistemological challenge and to my own challenge that the additional burdens imposed by any such epistemology would be inappropriate, that considerations of mathematical advantage alone should be enough. Still, despite the fatal shortcomings of Robust Realism in all its

---

[7] The quotation comes from Gödel [1961], p. 381, cited in Koellner [2006], p. 198.

forms, I've suggested that a form of realism is still available ([2011]).

What I call Thin Realism avoids the need for an extra-mathematical epistemology by positing a metaphysically minimal realm of sets that simply *are* the sort of thing accessible by ordinary set-theoretic methods. Though its sets are still objective, nonspatiotemporal, and acausal, this de-natured style of realism is both controversial and unlikely to satisfy anyone with Robust Realistic inclinations – not least of all because it is, I argue, a mere linguistic variant of an alternative that bypasses all matters of truth or reference or ontology or metaphysics more generally.

I call this equivalent position Arealism rather than anti-realism or nominalism because it isn't based on a principled rejection of abstracta for one philosophical reason or another. Instead, I imagine an idealized inquirer beginning from ordinary observations, then generalizing, proposing and testing theories, etc., and soon, in the course of these empirical investigations of the world, coming to develop mathematics, eventually even pure mathematics. How should this inquirer understand this new practice? Its methods often differ from her previous empirical approaches – experiments versus proofs, for example – still, the two practices do share logic and means-ends reasoning. Is mathematics just more of what she's been doing all long, seeking truths about an objective subject matter like the rest of science, or is it something different, immensely useful, but of a different kind? The Thin Realist takes the former stance, the Arealist takes the latter – not out of a prior distaste for abstracta, like an anti-realist or a nominalist, but as a sober-minded assessment of what her methods do and do not support. For her, set theory isn't in the business of describing a subject matter or discovering truths; it's a practice of devising and developing a list of axioms to meet a range of purely mathematical goals. My claim is that neither Thin Realism nor Arealism is either required or forbidden, that either way of speaking is acceptable. To keep things simple, I speak as an Arealist from now on.[8]

The most compelling objection to views like Arealism is based on the role of mathematics in natural science: crudely put, if you believe what science says, and science says there's a function that does so-and-so, then you must believe in functions.[9] I've argued that

[8] Candidly, of the two ways of speaking, I've always found Arealism more congenial; Thin Realism was my attempt to spell out what seemed to me equally legitimate (if to me foreign) ideas of John Burgess and John Steel (see [2011], pp 60-62). For a more principled reason to prefer the Arealist idiom in the present context, Schatz [ta] argues persuasively that Thin Realism is a convenient way of speaking in ordinary mathematics, but that Arealism is best for foundational discussions. Finally, I've indicated elsewhere ([2015], pp. 247-248) that Thin Realism (as I intend it) stands or falls with the objectivity of mathematical depth, and while I tend to believe that depth is objective, I have no strong argument to that effect (other than assuming we all agree that 'group' is an objectively better concept than various logically consistent others in the same vicinity and pointing to this as an example). Arealism has the advantage that it would survive the demise of objective depth.

[9] This argument rests on the assumption that 'science' takes the same attitude toward all the 'objects' it employs when even the most cursory glance reveals that mathematics often appears in contexts of explicit idealization, not intended to be understood as literal truth. (Think, e.g., of fluid dynamics.) With an eye toward the determinateness of CH, our best hope for a literal use is in applications of continuum mathematics to the theory of space-time, but, alas, that hope is soon dashed by physicists' clear acknowledgment that no one yet understands the structure of space-time in the small. In fact, I think this picture of mathematized science as a homogeneous expanse of physical entities and mathematical entities (and perhaps entities hovering somewhere in between) is misleading. A picture closer to the truth sees pure mathematics as an abundant store of abstracta, some of which – when we're lucky, often with a course of non-trivial

this depends on an over-simplified understanding of how mathematized science works, but getting into this here would distract from our main quarry. For now, let me just suggest that the Arealist has the means to turn this objection away.

Finally, then, returning to our central theme, though both the Robust Realist and the Arealist take CH seriously, they do so on starkly different terms. For the Robust Realist, there's a determinate truth value waiting to be discovered; for the Arealist, in contrast, set theory isn't a practice that concerns itself with truth values. Given that CH is independent of the current axioms, the remaining question for the Arealist, what's yet to be discovered, is simply this: is there or is there not an additional axiom candidate with both sufficient mathematical advantages to merit adoption and the specific ability to prove or disprove CH? This presents at least as real a question as the Robust Realist's determinate truth value, but it's a question about 'productive' mathematics, to use Zermelo's term, not about the features of an independent subject matter.

II. A partial taxonomy of multiversisms

Given this particular way of legitimizing the pursuit of a decision on CH, let's now consider the opposing idea, that this pursuit is somehow or other based on an illusion. One traditional

---

tinkering (see the exchange with Wilson in this volume) – can serve to model a worldly situation well enough for a particular purpose. To function in this capacity, the mathematical structures needn't exist in some abstract sense, they need only be described in a useful way by our mathematical theory – the Arealist's stock-in-trade. (See [1997], §§II.6-7, [2008], [2011], chapter 1.)

family of such views includes if-thenism: the only real question is which conclusions follow from which assumptions; as far as ZFC and CH are concerned, that question has been answered; analogous questions can be asked and perhaps answered about the relations of other assumptions to CH. To this, the traditional version adds – these implication relations are all on a par, mathematics is just a matter of what follows from what – which is often intended to deflect tough questions from intrusive philosophers. Of course, everyone knows that this isn't strictly correct, that mathematicians prefer, for example, Peano arithmetic, the group axioms, and ZFC over hosts of random collections of assumptions. Any viable form of if-thenism would need to be enhanced with an account of how and why some assumptions are preferred to others. I bring this up only to note: if this 'how and why' is spelled out in terms of the kinds of mathematical advantages we've touched on here, the resulting Enhanced If-thenism more or less coincides with Arealism.[10] But now, multiversism.

To focus discussion, let's limit consideration to multiverses in which every world thinks ZFC+LCs, but worlds differ on CH. An advocate of such a view might consider CH to have been settled, though not by a yae or nay:

> On the multiverse view … the continuum hypothesis is a settled question; it is incorrect to describe the CH as an open problem. The answer to CH consists of the expansive, detailed knowledge set theorists have gained about the extent to which it holds and fails in the multiverse, about how to achieve it or its negation in combination with other diverse set-theoretic properties. Of course, there are and will always remain questions about whether one can achieve CH or its negation with this or that hypothesis,

[10] See [2022]. Also the 'proofism' of Maddy and Väänänen [2023], pp. 48-49.

> but the point is that the most important and essential facts about CH are deeply understood, and these facts constitute the answer to the CH question. (Hamkins [2012], p. 429)

To the Arealist and her compatriots on CH, this may sound like giving up and declaring victory, but for the multiversist, perhaps discretion is the greater part of valor. Either way, it seems to me there are (at least) three overlapping strains of multiversism, the first of which is ontological, or metaphysical more generally.

1. Metaphysical multiversism

The most straightforward Metaphysical Multiversism is a simple ontological variety. Hamkins puts it this way:

> The multiverse view is one of higher-order realism – Platonism about universes … a realist position asserting actual existence of the alternative set-theoretic universes into which our mathematical tools have allowed us to glimpse. (Hamkins [2012], p. 417)

Hugh Woodin describes his target, generic multiversism, in less dramatic, but still ontological-sounding terms:[11]

> Let the *multiverse* (of sets) refer to the collection of possible universes of sets. … The multiverse is the *generic-multiverse* if it is generated from each universe of the collection by closing under generic extensions … and generic refinements. (Woodin [2011], p. 14)

Such views are often paired with an analogous universist position to set up the contrast, for example, in Neil Barton's taxomony:

> Universism: There is (up to isomorphism) just one maximal unique universe of set theory, and it contains all the sets. Every set-theoretic sentence has a definite truth value in this universe. (Barton [2021], p. 22)

> Multiversism: There are multiple equally legitimate universes of set theory, and no-one universe is especially privileged. (Ibid., p. 49)

[11] This isn't to say that Woodin and Hamkins are in agreement. See §IV, below.

This is Ontological Multiversism, with Ontological Universism posited as its foil, perhaps the most common approach in the literature.

Koellner complicates the discussion by introducing a second pair of contrasting terms: 'pluralism' and 'non-pluralism'. Joan Bagaria and Claudio Ternullo apparently take these as more or less equivalent to 'multiversism' and 'universism' ([Bagaria and Ternullo], p. 3), but Koellner [2019] distinguishes them.[12]

Recounting the history of CH, he first characterizes pluralists as those who

> … maintained that the independence results effectively settle the question by showing it *had no answer*. On this view, one could adopt a system in which, say CH was an axiom and one could adopt a system in which ~CH was an axiom and that was the end of the matter – there was no question as to which of two incompatible extensions was the 'correct' one. (Koellner [2019], p. 2)

'Correct' here apparently means 'true':

> The *non-pluralists* (like Gödel) held that the independence results merely indicated the paucity of our means for circumscribing mathematical truth. (Ibid.)

For the pluralist, while one choice or another for extending ZFC 'has its advantages with respect to certain aims there is no "fact of the matter" as to which one is correct' (Koellner [2011], p. 49). The pluralist/non-pluralist disagreement over whether or not CH has a truth value then links up with ontology and multiversism like this:

> One way of providing a foundational framework for [pluralism] is in terms of the multiverse. On this view there is not a single *universe* of set theory but rather a *multiverse* of legitimate

---

[12] Interestingly, the earlier Koellner [2011], p. 1, seems to collapse this distinction. Non-pluralism 'either implies or is a consequence of the fact that there is an objective mathematical realm', thus linking non-pluralism directly to ontology. Similarly, pluralism 'either implies or is a consequence of the fact … that there is no objective mathematical realm'. (Koellner distinguishes pluralism and non-pluralism at various levels.)

> candidates … none of which can be said to be the 'true' universe. (Koellner [2019], p. 25)

So Koellner's pluralism might be called Truth-value Multiversism with the option of Ontological Multiversism as an add-on.

(Let me pause for a moment to note that Barton employs a surprisingly austere version of pluralism: 'we should tolerate multiple competing *theories* of sets, and not necessarily identify one as privileged' (Barton [2021], p. 75). Presumably, the corresponding non-pluralism would hold that there is one privileged theory of sets, and this theoretical distinction would be expected to map directly to the truth-value or ontological distinction. As it happens, though, Barton draws no such easy correspondence. I come back to this in §III.)

Parallel to the varieties of Robust Realism in §I, there's also the possibility of a multiverse, not of worlds or truth values, but of concepts of set. Hamkins combines this sort of view with his Ontological Multiversism – 'the *multiverse view* … holds that there are diverse distinct concepts of set, each instantiated in a corresponding set-theoretic universe' (Hamkins [2012], p. 416) – but this could stand alone as Conceptual Multiversism. My point is that there are metaphysical multiversisms of ontological, truth-value, and conceptual varieties, and probably more, as well as many varieties of each – but all of them qualify as versions of Robust Realism because they posit an objective, abstract subject matter that demands a non-trivial epistemological account of how our methods manage to track it. So these views suffer from the same shortcomings that afflict universe versions of Robust Realism: both Benacerraf's challenge and my own methodological concern, as well as the Extrinsicness challenge for views that call on the science/mathematics analogy (as they often do).

All this metaphysical talk is heady enough, but it's natural to wonder how these philosophical maneuvers affect the practice of set theory. Given that these positions ascribe a distinctive subject matter to the discipline, it's perhaps also natural to expect its fundamental theory to consist of assertions about that subject matter. Just as the universist takes ZFC (ZFC+LCs) to describe her unique universe (or truths or concept or …), one might expect the multiversist to propose an alternative collection of axioms that describe, instead, the multiverse. Which brings us to the second of the three routes I hope to trace into multiverse thought.

2. Axiomatic multiversism

Fitting as it might seem for a Metaphysical Multiversist to opt for a foundational theory of universes and sets to replace a theory, like ZFC+LCs, focused exclusively on sets, the only available list of first-order axioms in a multiverse language is due to John Steel, whose intent is not at all metaphysical. Steel's aim is to collect together all 'natural' candidates for extensions to ZFC+LCs, where 'natural theories' are those 'considered by set theorists, because they [have] some set-theoretic idea behind them' (Steel [2014], p. 157). In this way, he hopes to compare these theories on 'a neutral common ground', and ultimately perhaps 'to decide whether some such theory is preferable to the others' (ibid., p. 165).[13]

---

[13] As indicated, this language of 'comparing' and 'deciding' if one or another theory is 'preferable' comes from Steel [2014]. More recently, Steel

Steel's idea is to locate this common ground by devising a theory of universes strong enough to prove the existence of a realization for each such candidate but weak enough not to include realizations for theories that would 'lose information' (ibid., p. 167). ('Natural' theories like ZFC+V=L or ZF+AD are represented by inner models of worlds, not by worlds.) Observing in practice that each known 'natural' theory is equiconsistent with a theory of the form ZFC+LCA, for some large cardinal axiom LCA, and that the proofs of these equiconsistencies work via inner models and forcing extensions, Steel's axioms end up codifying something very like Woodin's multiverse – closed under forcing extensions and refinements – except that he adds the simplifying assumption of Amalgamation: any two worlds share a forcing extension. This guarantees that Steel's system MV is sound and complete for its intended set of countable models. (There can be no such axiomatization of Woodin's multiverse.)[14]

has called his earlier talk of comparison 'misleading': 'L_{MV} is not a device for comparing the merits of one of those theories with another, it is a way to make more visible their underlying unity' (p. 17 of Steel's essay in this volume). See footnote 16 for more.

[14] See Maddy and Meadows [2022] for discussion and proofs. Here let me make one correction. Our paper calls Amalgamation into question on the grounds that it limits the array of worlds and hence of theories – how do we know we haven't ruled out some good candidates for extending ZFC+LCs? Though I wasn't able to see this during our exchanges while the paper was being drafted, Steel subsequently got through to me that this is only a problem if you focus on the worlds and wonder whether important ones might be missing. What matters for Steel isn't the worlds but the theories: as long as the axioms of Extension and Refinement are satisfied, all the live candidates will be represented. So, as long as Amalgamation is consistent with Extension and Refinement, which it is, there's no danger of significant omissions. The axiom is then amply justified by its mathematical benefits (e.g., the completeness theorem and the translation function from L_{MV} to L_ϵ). (Steel would say that Amalgamation is justified because it's provably true on the interpretation of L_{MV} determined by the translation function t. See footnote 53 of Maddy and Meadows [2022] for discussion of this approach.)

The system MV offers a second route to multiversism – it could be offered as an alternative to ZFC – but Steel himself is not an Axiomatic Multiversist. His central concern is the status of CH: his axioms are designed to represent the range of viable theoretical options; his leading question is whether one (or some) of those options can be seen to be preferable to others. In other words, he isn't tempted by the typical multiversist move of declaring the question of CH to have no answer, true in some worlds, false in others; he wonders whether one (or some) of those worlds – for him, theories represented by those worlds – stands out from the rest.[15] So, he reasons: what if there were 'a distinguished reference world … an individual world that is definable in the multiverse language' (Steel [2014], p. 168)? Woodin observed that if there were such a definable world, it would be unique, and it would be contained in all the other worlds; it would be what's now called the 'core' of the multiverse, defined as the intersection of all worlds, assuming that intersection is itself a world. Steel argues that this core, if it exists, would represent the sought-after preferred theory,[16] that CH would emerge as the legitimate question whether it is true or false in the core.

---

[15] It's worth noting Hamkins's remark that 'there is no reason to consider all universes in the multiverse equally, and we may simply be more interested in the parts of the multiverse consisting of universes satisfying very strong theories' (Hamkins [2014], p. 436). It's possible in principle that the search for more interesting universes in his multiverse could end up answering to the same mathematical considerations as the analogous search in Steel's multiverse, but this seems unlikely in practice.

[16] This is a non-trivial argument, sketched in the paragraph overlapping pp. 21-22 below. On Steel's more recent understanding of his project (see footnote 13), we might say that the unification of the candidate theories by L_{MV} allows us to see the theory of the core as preferable because it

Of course, since Steel's 2014 paper, Toshimichi Usuba [2017] proved that the multiverse does have a core, ultimately assuming an extendible (Usuba [2019]). For Steel, this means that his foray into multiverse thinking, his willingness to entertain his multiverse of possible theories, has returned the answer that CH is a legitimate question, after all.[17] The focus, then, settles on determining the theory of the core, a question on which MV+LCs is largely uninformative.[18] At that point, the problem devolves into one familiar to the Arealist: how to find good mathematical reasons to adopt axioms beyond ZFC+LCs (whether for the theory of the core or for the theory of V makes no difference). Steel regards V=Ult-L as a promising candidate, but however that may be, multiverse thinking has done its job and drops out of the discussion.

Still, a Metaphysical Multiversist could help themselves to MV+LCs or some variation as a mathematically explicit expression of their multiversism, though for Hamkins, in particular, MV+LCs wouldn't suit, because his multiverse is much more generous than Woodin's or Steel's generic versions.[19] In fact, though Hamkins does offer axioms of his own, he doesn't appear to be an Axiomatic multiversist: his axioms don't take the form of a live first-order alternative to

---

'includes' all the other theories (in the sense of that same passage on pp. 21-22).

[17] That is, if there's good reason to include an extendable among the large cardinals posited in our fundamental theory. I leave this important question aside.

[18] MV+LCs is MV plus the assumption that each world has large cardinals.

[19] See §IV.

ZFC+LCs or MV+LCs; he doesn't propose a multiverse replacement for our universist foundational theory. He sees himself, rather, as offering a fruitful way of thinking about the set-theoretic project:

> The mathematician's measure of a philosophical position may be the value of the mathematics to which it leads. Thus, the philosophical debate here [over the subject matter of set theory] may be a proxy for: where should set theory go? Which mathematical questions should it consider? (Hamkins [2014], p. 440)

He goes on to describe two lines of mathematical thought inspired by the multiverse perspective, namely, the modal logic of forcing and set-theoretic geology. Neither of these is stated in a multiverse language, but in ordinary L_ϵ, and proved in ZFC or an extension thereof. This practice-oriented approach brings us to the third and final route to multiversism.

3. Heuristic multiversism

Consider the role of the iterative hierarchy. For the Metaphysical Universist, it's the abstract subject matter of set theory, but the intuitive picture it provides – of sets arranged in a succession of ranks, taking all possible subsets at each step, extending as far as possible into the transfinite – this picture has also served, for roughly a century now, as a prolific inspiration for all manner of set-theoretic progress: reflection principles for large cardinals, for example, or closure principles for forcing axioms. The intuitive picture provides an effective heuristic for set-theoretic practice, quite independently of any metaphysical status. In an analogous way, a Heuristic Multiversist might hold that the intuitive picture of an array of such hierarchies plays the same sort of role and does a better job of suggesting new mathematical developments – like those Hamkins cites – and promising more. Instead of replacing ZFC, this version of multiversism simply recommends a different attitude toward it, as a theory shared by many distinct universes. This substitution, the Heuristic Multiversist claims, is a more effective inspiration, leading to new developments of ZFC, new theorems of ZFC, and perhaps even new axioms candidates!

For an early version of this approach, consider Kenneth Kunen's highly influential textbook treatment of forcing (Kunen [1980]). In an appendix to the chapter introducing the method, he lays out several ways of understanding what's going on in applications of forcing. His own preference, the method of countable transitive models, is familiar and effective for most purposes, but there are alternatives like the Boolean-valued models deployed in Jech [2003]. Another is what Kunen calls 'forcing over V', the purely syntactic approach with the definable forcing relation. Consider Kunen's intuitive description of how it works. With 'M' understood as the countable ground model in the c.t.m. approach, he writes of 'forcing over V':

> We may think of this approach as putting ourselves (in V) in the place of the M-people of the c.t.m. approach; so we are making up names for, and talking about, objects in some generic extension of V which does not exist at all (to us). (Kunen [1980], p. 234)

In a later rewriting of the book, he puts this even more colorfully:

> We, living in V, are in the position of the M-people … We can dream about some ideal universe $V[G]\supset V$ and make use of these dreams to motivate our discussion of [the forcing relation], but these dreams cannot be mentioned in rigorous proofs of the theorem about [the forcing relation]. (Kunen [2013], p. 281)

With his ‘dreams’, Kunen perfectly and delightfully illustrates how a multiverse intuitive picture can serve as a highly effective heuristic in a context that remains squarely within L_ϵ and ZFC, in other words, without sacrificing Axiomatic Universism. So, Heuristic Multiversism isn’t entirely new!

Now recall Steel’s multiverse project. We’ve seen that his foray into multiverse thinking leads him to focus on the core, and in particular, to propose, at least tentatively, the axiom V=C.[20,21] We now recognize this development as a particularly dramatic application of Heuristic Multiversism, but to probe a bit deeper, consider a question left hanging in our earlier discussion: what argument is there for identifying V with the core? Answer: C is contained in every world, every world is a generic extension of C.[22] Here Kunen’s dreamers have reappeared: the resident of C has imaginative access to every world of the multiverse, and that’s enough. Steel’s Heuristic Multiversism leads to the core and returns us to Axiomatic Universism with a new axiom in L_ϵ, a candidate for addition to ZFC+LCs. (Would that it were more informative!)

---

[20] As mentioned earlier, Steel goes on to suggest that V=Ult-L is true in C, but I leave that further step aside here.

[21] As it happens, this axiom candidate also emerged in the context of set-theoretic geology under the name ‘Ground Axiom’. See Reitz [2007].

[22] See Maddy and Meadows [2022], pp. 146-147. [Added later: In his interview in this volume, Woodin appears to echo this general idea in the context of his own version of the generic multiverse: pressed on why the core is ‘preferable’ to other worlds, he replies, ‘All models of the generic universe are just generic extensions of the root … so all truth is reducible to the root’ (Woodin [ta], p. 7).]

III. Multiversism and the Arealist

With this taxonomy in place, imagine the Arealist presented with the possibility of multiversism. Keeping the focus on CH, given the Arealist's belief – tentatively, of course – that it remains a real question, she would be expected to resist the characteristic multiversist dissolution of the problem, and for this reason, expected to side with the universist. Surprisingly, though, the various characterizations of the universist/multiversist dichotomy we've reviewed don't deliver this outcome so straightforwardly. Consider, then, her attempts to locate herself on one side or the other of this dichotomy.

Starting from the ontological Hamkins/Barton version from §II, she sees herself nowhere; she's no more inclined to assert the existence of 'just one maximal unique universe' than of 'multiple equally legitimate universes'. Koellner's choice, again from §II, between pluralism and non-pluralism is even more befuddling. On the one hand, she agrees with his pluralist that there is 'no fact of the matter' about CH; on the other, she holds out hope that it isn't a matter of indifference which of CH and ~CH we adopt, siding this time with the non-pluralist. Another of Koellner's characterizations focuses the point more precisely: for the pluralist,

> although there are *practical* reasons that one might give in favour of one set of axioms over another – say, that it is more useful for a given task –, there are no *theoretical* reasons that can be given. (Koellner [2011], p. 1)

In contrast, for the non-pluralist, 'theoretical reasons *can* be given for new axioms' (ibid.). For the Arealist, the distinction between practical and theoretical reasons is obscure; being 'more useful for a given task' is just the sort of thing that counts as a good reason, full stop. So, as perhaps would be expected, she sees herself in neither Truth-value Multiversism nor Truth-value Universism. To fully inhabit the Arealist's perspective, we must eschew all metaphysics – ontology, truth values, concepts, meanings – and attend exclusively to mathematical needs and benefits.

In this respect, the prospects are better for Axiomatic Multiversism. One version of this idea would be the austere pluralism of Barton, mentioned in passing above, to repeat: 'we should tolerate multiple competing *theories*, and not necessarily identify one as privileged' (Barton [2021], p. 75). The corresponding Axiomatic Universist would presumably hold that one theory is properly preferred, for now, presumably ZFC+LCs. Here the Arealist would seem a clear fit for team universist, but Barton imposes unexpected obstacles. One is predictable, if onerous: the universist …

> … thinks that every statement of first-order set theory has a determinate answer, [so] there is a unique privileged set theory; the true one. (Barton [2021], p. 75)

Obviously, the Arealist won't endorse the idea of a 'true' set theory; only slightly less obviously, she sees the question – can the first-order statement CH be satisfactorily settled one way or the other? – and thus the question of determinateness, as open. Further sealing the case, Barton imposes a second, purely methodological condition on his universist: she cannot 'tolerate the use of different theories' (ibid.). As Barton himself points out, no reasonable set-theoretic practitioner could accept this restriction:

> In virtue of the kinds of ways in which she might be ignorant, [she] should tolerate many different theories of sets (at least for now) (ibid.).

If this Axiomatic Universist can't endorse, for example, the development of both ZFC+LCs+V=Ult-L and ZFC+LCs+MM, then the Arealist once again falls between stools.

Barton aside, there's a more straightforward version of the Axiomatic Universism/Multiversism distinction, namely, the choice between ZFC+LCs and a multiverse theory like MV+LCs. A less specific contrast would be between those who hold that our fundamental theory should be written in the language of sets, L_ϵ, and those who hold that it should be written in the language of worlds and sets, something like L_{MV}. Here, at last, the Arealist's position is straightforward, siding with the Axiomatic Universist in both cases (tentatively, as always). The trouble this time is that no multiversist known to me takes the other side, the Axiomatic Multiversist side, in either of these forms. So the Arealist once again finds herself outside any live universist/multiversist debate.

This leaves the final option, Heuristic Universism versus Heuristic Multiversism. Here, perhaps, the distinction will gain some real traction, recommending that the practice be guided by contrasting intuitive pictures. There can be little doubt of the effectiveness of the iterative picture, and Kunen's, Steel's, and Hamkins' examples demonstrate the initial promise of a multiverse picture. Still, any hope of pinning the Arealist down to one argumentative position over the other is once again dashed; from her point of view, the question is, why choose? Metaphysics aside, nothing precludes exploiting now one, now the other of these intuitive guides to maximal effect. One needn't forgo large cardinals to pursue set-theoretic geology or vice versa! So, the opportunistic Arealist remains stubbornly unclassified.

IV. Digression: two contrasting multiversisms

That's pretty much the story I want to tell here, but my tight focus on the supposed conflict between universism and multiversism has obscured the contrast within multiversism between the broad Hamkins multiversism and the narrower Woodin-Steel generic multiversism. It seems irresponsible to leave the subject without at least a nod in this direction, besides which I think we're now in a position to make a few potentially fresh observations from the Arealist point of view. The story begins with their respective motivations.

For Hamkins,

> The most prominent phenomena in set theory has been the discovery of a shocking diversity of set-theoretic possibilities. Our most powerful set-theoretic tools, such as forcing, ultrapowers, and canonical inner models, are most naturally and directly understood as methods of constructing alternative set-theoretic universes.
>
> This abundance of set-theoretic possibilities poses a serious difficulty for the universe view … We have a robust experience in these worlds, and they appear fully set theoretic to us. The multiverse view … explains this experience by embracing them as real, filling out the vision hinted at in our mathematical experience, that there is an abundance of set-theoretic worlds into which our mathematical tools have allowed us to glimpse. (Hamkins [2014], p. 418)

Hamkins motivation is clear: he's out to explain mathematical experience, the phenomenology of set-theoretic practice. He does so by positing a metaphysics that takes that experience at face value, 'placing no undue limitations on what universes might exist' (Hamkins [2014], p. 437). Indeed, he suggests that 'there seems no reason to restrict attention only to ZFC models' and goes so far as to include second-order arithmetic as 'set-theoretic in a sense' (ibid., p. 436). So my focus here on multiverses where the worlds agree on ZFC+LCs, though differing on CH, is already a distortion, a taming, of Hamkins' position.

Compare now what Woodin describes as the source of his generic multiverse:

> The refinements of Cohen's method of *forcing* in the decades since his initial discovery of the method and the resulting plethora of problems shown to be unsolvable, have in a practical sense almost compelled one to adopt the generic-multiverse position … The argument that Cohen's method of forcing establishes that the Continuum Hypothesis has no answer, is implicitly assuming the generic-multiverse conception of truth. (Woodin [2011], pp. 16-17)

Here Woodin sees forcing in particular as 'almost compelling' us to generic multiversism and thus as a threat to the determinacy of independent statements like CH. If forcing extensions were all obviously unintended, not viable, they wouldn't present such a problem, but they aren't. As Steel points out, they often represent 'natural' theories, legitimate contenders for extensions of ZFC+LCs; this line of thought is what inspires Steel's sharper generic multiverse with Amalgamation. For Woodin and Steel, the goal isn't to explain mathematical experience but to explore the possibility that forcing has undermined the universist's picture of the world of sets.

Enter the Arealist. How does the contrast between Hamkins generous multiverse and the Steel/Woodin generic multiverse look from this perspective? On the motivational contrast just noted, the

Arealist's sympathies lie with the generic version: her potential interest in multiversism, like theirs, arises from a concern with CH, not from an analysis of set-theoretic experience. Motivations aside, the core disagreement here is perhaps most often seen as metaphysical – how many set-theoretic universes are there? – but these intramural debates between forms of Metaphysical Multiversism strike the Arealist as ill-posed.[23] She does care about axiomatics, but Hamkins hasn't offered a clear alternative to ZFC and its extensions, and as far as heuristics are concerned, those cited by Hamkins seem equally available to Woodin and Steel,[24] with Steel's case for V=C as an added attraction. In sum, then, once the metaphysical questions have been set aside, the generic multiverse appears to provide a more promising heuristic, at least for now. Still, just as the Arealist remains open to the heuristic benefits of both Universism and Multiversism, she's perfectly happy to explore both versions of Multiverism. Once again, without the illusory metaphysical contrast, there's no need to rule out any helpful heuristic.

IV. Conclusions

I hope to have shown that multiversism is no one doctrine, indeed no one type of doctrine, that many versions are uncomfortably

---

[23] In particular, the Arealist sees no basis for Hamkins's inference from what happens in model theory, as a branch of mathematics conducted within ZFC and its extensions, to a conclusion about an ambient, pre-theoretic metaphysics.

[24] E.g., the modal logic of forcing and set-theoretic geology are both studies of forcing models.

metaphysical, and that the contrast between universism and multiversism is considerably muddier than the either-or presentations suggest. Looking past the metaphysics, different questions take center stage: should our fundamental assumptions involve just sets or sets and worlds; should our axioms be formulated in L_{ϵ} or something like L_{MV}; should our fundamental theory be ZFC and its extensions or something like MV and its extensions; which ways of thinking, which heuristics can deliver the best mathematics? Some of these are questions of Axiomatic Universism versus Axiomatic Multiversism. Multiverse axioms have indeed been offered, most prominently by Steel and Hamkins, but no one appears to be arguing that replacing ZFC+LCs with such a theory would leave us better off mathematically: Steel's line of multiverse thinking ends up endorsing the extension of ZFC+LCs with V=C (back to Axiomatic Universism), and Hamkins employs his multiverse thinking to suggest new theorems in ZFC+LCs (Heuristic not Axiomatic Multiversism). For now, Axiomatic Multiversism is unmotivated; Axiomatic Universism remains dominant.

On the question of heuristics, Heuristic Universism, in the form of the iterative picture, has proved itself dramatically since Zermelo's time, but we've seen that Heuristic Multiversism has made a recent showing, as in Kunen's 'forcing over V', Steel's case for V=C, and Hamkins' set-theoretic geology. For the Arealist, these new developments are unalloyed good news, the rise of a new guide to good mathematics. In the end, it's only the warring metaphysical camps that make us think we need to choose between two effective heuristics; this impediment removed, the Arealist can reap the rewards of both.

Shouldn't anyone be pleased with a shiny new tool added to the toolbox?

So, what morals should we draw? Obviously, I think we can and should approach matters of set-theoretic method as Arealists, unencumbered by metaphysical delicacies, helping ourselves to a full range of heuristics. From this perspective, metaphysics-free multiversism has its heuristic successes: (1) developments like set-theoretic geology, (2) the possibility of an Axiomatic Multiversism promoting MV+LCs or something like it as our fundamental theory, and especially, (3) Steel's employment of Heuristic Multiversism to mount a case for V=C. Returning at last to CH, as long as our language remains L_ϵ and our theory ZFC and its extensions, the Arealist's understanding of CH survives: is there or is there not an axiom candidate with both sufficient mathematical advantages to merit adoption and the specific ability to prove or disprove it? In this form, CH remains a legitimate question. And finally – until someone seriously proposes a multiverse alternative to ZFC+LCs as our fundamental theory – until that happens – my considered opinion on universism versus multiversism is that we'd be better off retiring the whole debate.[25]

[25] Thanks to Joan Bagaria, Juliette Kennedy, Claudio Ternullo, and Georgio Venturi for their kind invitation to address their conference in Barcelona and to the audience there for a stimulating discussion. I'm also grateful to John Steel for his thoughts on §II.2, to Toby Meadows and Jeffrey Schatz for illuminating discussions related to the addendum to footnote 22, and to Adam Chin, Chris Misch, and Evan Sommers for helpful comments on an earlier draft.

References

Bagaria, Joan, and Claudio Ternullo

[2023] 'Steel's programme: evidential framework, the core and ultimate-L', *Review of Symbolic Logic* 16, pp. 788-812.

Barton, Neil

[2021] 'Indeterminateness and 'the' universe of sets: multiversism, potentialism, and pluralism', in M. Fitting, ed., *Selected Topics from Contemporary Logics*, (College Publications), pp. 105-182.

Benacerraf, Paul

[1973] 'Mathematical truth', reprinted in Benacerraf and Putnam [1983], pp. 403-420.

Benacerraf, Paul, and Hilary Putnam, eds.

[1983] *Philosophy of Mathematics*, second edition, (Cambridge: Cambridge University Press).

Feferman, Sol, Charles Parsons, and Stephen Simpson, eds.

[2010] *Kurt Gödel: Essays for his Centennial*, (Cambridge: Cambridge University Press).

Field, Hartry

[1989] 'Introduction', in *Realism, Mathematics and Modality*, (Oxford: Basil Blackwell), pp. 1-52.

Gödel, Kurt

[1947/64] 'What is Cantor's continuum problem?', reprinted in Benacerraf and Putnam [1983], pp. 470-485, and Gödel [1990], pp. 176-187, 254-270.

[1961] 'The modern development of the foundations of mathematics in the light of philosophy', in Feferman et al [1995], pp. 375-387

[1990] *Collected Papers*, volume II, S. Feferman et al, eds., (Oxford: Oxford University Press).

[1995] *Collected Works*, volume III, S. Feferman et al, eds, (Oxford: Oxford University Press).

Hamkins, Joel

[2012] 'The set-theoretic multiverse', *Review of Symbolic Logic* 5, pp. 416-449.

Jech, Thomas

[2003] *Set Theory*, (Berlin: Springer).

Koellner Peter

[2006] 'On the question of absolute decidabilty', *Philosophia Mathematica*. Reprinted in Feferman et al [2010], pp. 189-225.

[2011] 'Independence and large cardinals', in The Stanford Encyclopedia of Philosophy (Summer 2011 Edition), Edward N. Zalta (ed.), URL = https://plato.stanford.edu/archives/sum2011/entries/independence-large-cardinals/.

[2019] 'The continuum hypothesis', in The Stanford Encyclopedia of Philosophy (Spring 2019 Edition), Edward N. Zalta (ed.), URL = https://plato.stanford.edu/archives/spr2019/entries/continuum-hypothesis/.

Kunen, Kenneth

[1980] *Set Theory: an Introduction to Independence Proofs*, (Amsterdam: Elsevier).

[2013] *Set Theory*, (London: College Publications).

Leng, Mary

[ta] 'Enhanced "if-thenism", fictionalism, and realist anti-platonism', this volume.

Maddy, Penelope

[1990] *Realism in Mathematics*, (Oxford: Oxford University Press).

[1997] *Naturalism in Mathematics*, (Oxford: Oxford University Press).

[2008] 'How applied mathematics became pure', *Review of Symbolic Logic* 1, pp. 16-41.

[2011] *Defending the Axioms*, (Oxford: Oxford University Press).

[2015] 'Afterward', special issue on Mathematical Depth, *Philosophia Mathematica* 23, pp. 245-248.

[2017] 'Set-theoretic foundations, in A. Caicedo, J. Cummings, P. Koellner, and P. Larson, eds., *Foundations of Mathematics: Essays in Honor of W. Hugh Woodin's 60th Birthday* (Providence, RI: American Mathematical Society, pp. 298-322.

[2019] 'What do we want a foundation to do?', in S. Centrone, D. Kant, and D. Sarikaya, eds., *Reflections on the Foundations of Mathematics: Univalent Foundations, Set Theory and General Thoughts*, (Cham, Switzerland: Springer), pp. 293-311.

[2022] 'Enhanced if-thenism', in *A Plea for Natural Philosophy, (*Oxford: Oxford University Press), pp. 253-293.

[ta] 'Reply to Leng', this volume.

Maddy, Penelope, and Toby Meadows

[2020] 'A reconstruction of Steel's multiverse project', *Bulletin of Symbolic Logic* 26, pp. 118-169.

Maddy, Penelope, and Jouko Väänänen

[2023] *Philosophical Uses of Categoricity Arguments*, (Cambridge: Cambridge University Press).

Martin, D. A.

[1998] 'Mathematical evidence', in H. G. Dales and G. Oliveri, eds., *Truth in Mathematics*, (Oxford: Oxford University Press), pp. 215-231.

[2005] 'Gödel's conceptual realism', *Bulletin of Symbolic Logic* 11, pp. 207-224. Reprinted in Feferman et al [2010], pp. 356-373.

Parsons, Charles

[2008] *Mathematical Thought and its Objects*, (Cambridge: Cambridge University Press).

Reitz, Jonas

[2007] 'The ground axiom', *Journal of Symbolic Logic* 72, pp. 1299-1317.

Schatz, Jeffrey

[ta] 'Arealism, Thin Realism, and the problem of objective depth', in C. Antos-Kuby, N. Barton, and G. Venturi, eds., *Palgrave Companion of Philosophy of Set Theory*, (London: Palgrave).

Steel, John

[2014] 'Gödel's program', in J. Kennedy, ed., *Interpreting Gödel*, (Cambridge: Cambridge University Press), pp. 153-179.

[ta] 'Generically invariant set theory', this volume.

Usuba, Toshimichi

[2017] 'The downward directed grounds hypothesis and very large cardinals', *Journal of Mathematical Logic* 17.

[2019] 'Extendible cardinals and the mantle', *Archive for Mathematical Logic* 58, pp. 71-75.

Van Heijenoort, Jean, ed.

[1967] *From Frege to Gödel*, (Cambridge, MA: Harvard University Press).

Wilson, Mark

[ta] 'The flight from application', this volume.

Wittgenstein, Ludwig

[1939/40] Part III, *Remarks on the Philosophy of Mathematics*, revised edition, G. E. M. Anscombe, trans., G. H. von Wright, R. Rhees, G. E. M. Anscombe, eds., (Cambridge, MA: MIT Press, 1978), pp. 143-221.

Woodin, Hugh

[2011] 'The continuum hypothesis, the generic-multiverse of sets, and the Ω-conjecture', in J. Kennedy and R. Kossak, eds., *Set Theory, Arithmetic, and Foundations of Mathematics*, (Cambridge: Cambridge University Press), pp. 13-42.

[ta] 'A conversation with Hugh Woodin', this volume.

Zermelo, Ernst

[1908] 'A new proof of the possibility of a well-ordering', S. Bauer-Mengelberg, trans., reprinted in van Heijenoort [1967], pp. 183-198, and Zermelo [2010], pp. 121-159.

[2010] *Collected Works*, volume I, H.-D. Ebbinghaus and A. Kanamori, eds., (Berlin: Springer).