Annals of Mathematics, 122 (1985), 335–364

Pseudofree orbifolds

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1. Introduction

The structure of θ_3^H, the integral homology cobordism group of integral homology 3-spheres, is of central importance for understanding smooth 4-manifolds. However until recently the only known nontrivial fact concerning this group was the existence of the Kervaire-Milnor-Rochlin epimorphism $\mu\colon \theta_3^H \to \mathbf{Z}_2$. The first indication that this group might be larger than $\mathbf{Z}_2$ was given by the celebrated work of S. Donaldson [D] which implied that if an integral homology 3-sphere Σ bounds a simply connected 4-manifold whose intersection pairing is definite and not diagonalizable over $\mathbf{Z}$, then Σ cannot bound an acyclic 4-manifold with $\pi_1(\Sigma) \to \pi_1(W)$ an epimorphism. For example, this suggests that perhaps the Poincaré homology 3-sphere has infinite order in θ_3^H. We will show this to be the case.

The Poincaré homology sphere H^3 is an example of a broad class of homology 3-spheres, namely the Seifert fibered homology 3-spheres (equivalently, Brieskorn complete intersections that are homology 3-spheres) $\Sigma(a_1, \ldots, a_n)$, where the notation indicates that $\Sigma(a_1, \ldots, a_n)$ is Seifert fibered over S^2 with exceptional fibers of orders $a_1, \ldots, a_n$ which are necessarily pairwise relatively prime (see § 2). For instance, $H^3 = \Sigma(2,3,5)$.

Starting with a Seifert fibered homology 3-sphere, one is able to construct singular 4-manifolds whose singularities are isolated and which have neighborhoods which are cones on lens spaces with relatively prime orders. We call these singular spaces *pseudofree orbifolds*. They can be desingularized by taking a finite branched cyclic cover. Motivated by recent work of S. Donaldson we shall define a signature defect-type invariant which we reinterpret as the dimension of a moduli space of a set of perturbed deck transformation-invariant Yang-Mills equations on the branched cyclic cover of a pseudofree orbifold related to Σ. We

[1] Partially supported by National Science Foundation Grant MCS8300823
[2] Partially supported by National Science Foundation Grant MCS8002843A01

then use this interpretation to garner new results concerning the 4-manifolds that Σ bounds. This invariant is most easily described for Seifert fibered homology 3-spheres as follows.

Given $(a_1, \dots, a_n)$, let $\alpha = a_1 \dots a_n$. Then the invariant $R(a_1, \dots, a_n)$ is given by

$$R(a_1, \dots, a_n) = \frac{2}{\alpha} - 3 + n + \sum_{i=1}^{n} \frac{2}{a_i} \sum_{k=1}^{a_i - 1} \cot(\pi \alpha k / a_i^2) \cot\left(\frac{\pi k}{a_i}\right) \sin^2\left(\frac{\pi k}{a_i}\right).$$

It turns out that $R(a_1, \dots, a_n)$ is an odd integer (see 6.3). A key theorem is:

THEOREM 1.1. *If $R(a_1, \dots a_n) > 0$ then $\Sigma(a_1, \dots a_n)$ does not bound an oriented smooth 4-manifold V whose intersection pairing is positive definite and whose first homology $H_1(V; \mathbf{Z})$ contains no 2-torsion.*

The orientation which we have assigned to $\Sigma(a_1, \dots a_n)$ is its orientation as the link of an algebraic singularity. This is important since with this orientation $\Sigma(a_1, \dots a_n)$ always bounds an oriented simply-connected smooth 4-manifold with a negative definite intersection pairing, namely the canonical resolution of its associated singularity. It is worth pointing out the following special case of Theorem 1.1.

THEOREM 1.2. *If $R(a_1, \dots a_n) > 0$ then $\Sigma(a_1, \dots a_n)$ does not bound a $\mathbf{Z}_2$-acyclic 4-manifold.*

In this regard one should keep in mind that although $\Sigma(2,3,7)$ has μ-invariant 1, it bounds a rational ball (with $\pi_1(M) = \mathbf{Z}_2$) [FS1] and $R(2,3,7) = -1$. This points out that the invariant $R(a_1, \dots a_n)$ does not carry the μ-invariant.

Further, if $R(a_1, \dots a_n) > 0$ then $\Sigma(a_1, \dots a_n)$ is not oriented cobordant to any (connected) sum of Seifert fibered homology spheres

$$-\Sigma_{j=1}^{r}\left(\Sigma\left(b_{j,1}, \dots b_{j,m(j)}\right)\right)$$

by a positive definite cobordism W where $H_1(W; Z)$ contains no 2-torsion.

Now there are many $\Sigma(a_1, \dots a_n)$ with $R(a_1, \dots a_n) > 0$, for example $R(2,3,6k-1) = +1$ for $k \geq 1$, so that $\Sigma(2,3,6k-1)$ (or any connected sum of such) cannot bound a $\mathbf{Z}_2$-acyclic 4-manifold. In particular $\Sigma(2,3,5)$ has infinite order in θ_3^H.

Here is an outline of the proof of Theorem 1.1 which is carried out in Sections 2–9. Let C denote the mapping cylinder of the Seifert fibration $\Sigma(a_1, \dots a_n) \to S^2$. If $\Sigma(a_1, \dots a_n)$ bounds an oriented positive definite 4-manifold V as in the theorem, then $X = V \cup (-C)$ is a pseudofree orbifold whose singular points have neighborhoods which are cones on the lens spaces $L(a_i, \alpha/a_i)$. We state our main theorem, from which Theorem 1.1 is shown to follow, in Section 2. In order to study the singular geometry of these orbifolds we desingularize everything by taking branched covers in Section 3. In particular, there is a surface F smoothly embedded in C with self intersection number α. The α-fold cover M of $X = V \cup (-C)$ branched over F and the n singular points is a smooth 4-manifold, and there is over M a smooth $\mathbf{Z}_\alpha$-SO(3) vector bundle E whose structure group equivariantly reduces to SO(2). The idea is to consider $\mathbf{Z}_\alpha$-invariant self-dual SO(3)-connections on E. If $R(a_1, \ldots a_n) > 0$ then using ideas of K. Uhlenbeck ([U1], [U2]), Atiyah, Hitchin, and Singer [AHS], and the G-signature theorem, we show in Sections 4–8 that the $\mathbf{Z}_\alpha$-invariant self-duality equations can be perturbed to have a compact moduli space which is a manifold of dimension $R(a_1, \ldots a_n)$ with an odd number of singular points each having a neighborhood which is a cone on a complex projective space $\mathrm{CP}(\frac{1}{2}(R(a_1, \ldots a_n)) - 1)$. If $R(a_1, \ldots a_n) \equiv 1 (\mathrm{mod}\, 4)$ this contradicts the fact that an odd number of such complex projective spaces cannot bound. If $R(a_1, \ldots a_n) \equiv 3 (\mathrm{mod}\, 4)$ a similar contradiction is obtained in Section 9 by considering the reduced gauge group and showing that an odd number of such complex projective spaces cannot bound inside the gauge equivalence classes of connections.

Section 10 is devoted to applications of the main theorem. Therein we give details of the consequences of the theorem to the study of θ_3^H. In particular we prove that θ_3^H has a subgroup of the form $\mathbf{Z} \oplus \mathbf{Z}_{2k}$, $k \geq 0$. Also we discuss applications of the main theorem to constructing examples of Alexander polynomial one knots that are not smoothly slice, to questions regarding representing 2-dimensional homology classes by smoothly embedded spheres, and to the uniqueness up to concordance of simplicial triangulations of topological manifolds.

The authors wish to thank the many mathematicians who have spent time in discussions about this work: in particular, Morris Kalka, Al Vitter, Tom Duchamp, and Howard Sealy for patiently explaining ideas concerning many topics in analysis and differential geometry. Blaine Lawson's beautiful CBMS lectures on Yang-Mills theory were the turning point in the understanding of our program, and his lecture notes [L] were instrumental in our work.

We also recommend that the reader be familiar with [FS2] where we describe the nonequivariant version of the results presented here.

2. Pseudofree orbifolds

A *pseudofree* S^1*-action* is a smooth S^1-action on a smooth $(2n+1)$-manifold such that the action is free except for finitely many exceptional orbits with isotropy $\mathbf{Z}_{a_1}, \ldots, \mathbf{Z}_{a_n}$ where $a_1, \ldots, a_n$ are pairwise relatively prime. The *total isotropy* is the product $\alpha = a_1 \ldots a_n$. A *pseudofree orbifold* $X = Q^5/S^1$ is the quotient of a smooth 5-manifold Q^5 by a pseudofree S^1-action. Then X is a 4-manifold with isolated singularities whose neighborhoods are cones on lens spaces $L(a_i, b_i)$ corresponding to the exceptional orbits in Q^5.

Let $X = Q^5/S^1$ be a pseudofree orbifold, and set $D(X) = X - \bigcup_{i=1}^n \operatorname{int}(cL(a_i, b_i))$. The S^1-action over $D(X)$ is free; hence it is classified by an Euler class $e \in H^2(D(X); \mathbf{Z})$. Since the tubular neighborhood of an exceptional orbit E_i with isotropy $\mathbf{Z}_{a_i}$ in Q^5 is $D^4 \times_{\mathbf{Z}_{a_i}} S^1$ which is diffeomorphic to $D^4 \times S^1$, the part of Q^5 over each $L(a_i, b_i)$ is just $S^3 \times S^1$. From this we see that $i^*(e)$ is a unit in $\mathbf{Z}_\alpha$, where

$$0 \to H^2(D(X), \partial D(X); \mathbf{Z}) \xrightarrow{j^*} H^2(D(X); \mathbf{Z}) \xrightarrow{i^*} H^2(\partial D(X); \mathbf{Z}) = \mathbf{Z}_\alpha.$$

Furthermore the S^1-action on the tube $D^2 \times D^2 \times S^1$ of the exceptional orbit E_i is

$$t \cdot (z, w, s) = (zt^{r_i}, wt^{s_i}, st^{a_i})$$

where r_i and s_i are relatively prime to a_i. The triple $(a_i; r_i, s_i)$ is called the *slice type* of the exceptional orbit E_i with isotropy $\mathbf{Z}_{a_i}$. Note that

$$(D^2 \times D^2 \times S^1)/S^1 = D^2 \times D^2/\mathbf{Z}_{a_i} = cL(a_i; r_i, s_i) = cL(a_i, b_i)$$

where $r_i s_i^{-1} \equiv b_i (\operatorname{mod} a_i)$.

The pair $(D(X), e)$ gives complete classifying information for the S^1-action (Q^5, S^1). Since $i^*(\alpha e) = 0$, there is a (unique) class $f \in H^2(D(X), \partial D(X); \mathbf{Z})$ such that $j^*(f) = \alpha e$. We define e^2 to be the rational number

$$\frac{1}{\alpha}(e \cup f)[D(X), \partial D(X)]$$

where $[D(X), \partial D(X)]$ is the fundamental class of $(D(X), \partial D(X))$. Since X is a rational homology manifold there is a rational intersection form defined on $H^2(X; \mathbf{Q})$. If we identify $H^2(D(X), \partial D(X); \mathbf{Z})$ and $H^2(X; \mathbf{Z})$ and then view $e \in H^2(X; \mathbf{Q})$, the rational self-intersection number of e is just e^2.

In general, given a pseudofree orbifold X, we say that $e \in H^2(D(X); \mathbf{Z})$ is a *pseudofree Euler class* if i^*e is a unit in $H^2(\partial D(X); \mathbf{Z}) = \mathbf{Z}_\alpha$. Such e define pseudofree S^1-actions over X. To see this, let Y be the free S^1-manifold over $D(X)$ with Euler class e. If $p\colon Y \to D(X)$ is the orbit map, then, since $i^*(e)$ is a unit, each $p^{-1}(L(a_i, b_i)) \cong S^3 \times S^1$ and the S^1-action on $S^3 \times S^1 = \partial(D^2 \times D^2 \times S^1)$ is equivalent to $t \cdot (z, w, s) = (zt^{r_i}, wt^{s_i}, st^{a_i})$ where r_i and s_i are relatively prime to a_i. This action then extends in the obvious linear fashion over $D^4 \times S^1$ with one exceptional orbit of isotropy $\mathbf{Z}_{a_i}$.

Before we state our main theorem concerning pseudofree orbifolds we need to introduce two integers. It is convenient to arbitrarily split

$$H^2(D(X); \mathbf{Z}) = \operatorname{Fr} H^2(D(X); \mathbf{Z}) \oplus \operatorname{Tor} H^2(D(X); \mathbf{Z})$$

into free and torsion parts. Define

$$\mu(e) = \begin{cases} \#\{ f \in \operatorname{Fr} H^2(D(X);\mathbf{Z}) \mid i^*f = i^*e,\ f^2 = e^2 \text{ and } \\ \qquad f \equiv e \pmod 2\},\ \alpha \neq 2 \\ (1/2)\ \#\{ f \in \operatorname{Fr} H^2(D(X);\mathbf{Z}) \mid i^*f = i^*e,\ f^2 = e^2 \text{ and } \\ \qquad f \equiv e \pmod 2\},\ \alpha = 2. \end{cases}$$

For a geometrical interpretation of $\mu(e)$ see Propositions 4.2 and 5.4.

Define

$$R(X,e) = 2e^2 - 3 + n + \sum_{i=1}^{n} (2/a_i) \sum_{k=1}^{a_i - 1} \cot(\pi k r_i/a_i)\cot(\pi k s_i/a_i)\sin^2(\pi k/a_i)$$

where $(a_i; r_i, s_i)$ is the slice type at the exceptional orbit with isotropy $\mathbf{Z}_{a_i}$. For a geometrical interpretation of $R(X,e)$ as an index of an elliptic operator and as the dimension of the solution space to certain equivariant Yang-Mills equations, see Theorem 8.2.

We now state a version of our main theorem.

THEOREM 2.1. *Let X be a pseudofree orbifold with pseudofree Euler class e. Suppose that*

(i) *the intersection form on X is positive definite,*
(ii) $H_1(D(X);\mathbf{Z}_2) = 0$,
(iii) $i^*(\operatorname{Tor} H^2(D(X);\mathbf{Z})) = 0$, *and*
(iv) $e^2 \leq 4/\alpha$

(*if $e^2 = 4/\alpha$ assume $e \not\equiv 0 \pmod 2$ and $H^2(\pi_1(D(X));\mathbf{Z}_2) = 0$*). *If $R(X,e) > 0$, then $\mu(e) \equiv 0 \pmod 2$.*

For a statement of the theorem without the technical condition (iii), see Theorem 9.2.

A pseudofree S^1-action on an oriented 5-manifold Q^5 may be viewed as a singular SO(2)-bundle over X. In order to prove Theorem 2.1 we would like to use such singular bundles to imitate the ideas presented in [FS2]. In order to desingularize everything we resort to branched covers and study the equivariant geometry of these covers. Before doing this, we shall explain how Seifert fibered homology 3-spheres fit into the picture.

Let $\Sigma = \Sigma(a_1, \ldots, a_n)$ be a Seifert fibered homology 3-sphere. Note that this means that Σ admits a pseudofree S^1-action with exceptional fibers with isotropy $\mathbf{Z}_{a_1}, \ldots, \mathbf{Z}_{a_n}$; and $\Sigma/S^1 = S^2$. Let C be the mapping cylinder of the orbit map $\Sigma \to S^2$. Then C is a 4-manifold with n singular points which have neighborhoods which are cones on the lens spaces $L(a_i, -\alpha/a_i)$ and the boundary of C is Σ. Now $\Sigma(a_1, \dots, a_n)$ is the link of an algebraic singularity. (See [M], [N]; for example $\Sigma(p, q, r)$ is the link of the isolated singularity at 0 of the equation $x^p + y^q + z^r = 0$ in $\mathbf{C}^3$.) This fact gives Σ a canonical orientation so that oriented this way Σ bounds the canonical resolution, a negative definite simply connected smooth 4-manifold. In fact, blowing up once at the singularity gives C with a negative definite intersection form. Let $W = C - \bigcup_{i=1}^{n} \mathrm{int}(cL(a_i, -\alpha/a_i))$. Then $H_1(W; \mathbf{Z}) = 0$ and $H^2(W; \mathbf{Z}) \cong \mathbf{Z}$. Let β correspond to 1.

As in Theorem 1.1 suppose that Σ bounds an oriented positive definite 4-manifold V whose first homology contains no 2-torsion. By surgering out the free part of $H_1(V; \mathbf{Z})$ we can assume that $H_1(V; \mathbf{Z}_2) = 0$. Let $X = V \cup (-C)$; so X is an oriented positive definite pseudofree orbifold. Its pseudofree Euler class is $e_\Sigma \in H^2(D(X); \mathbf{Z}) \simeq H^2(V; \mathbf{Z}) \oplus H^2(W; \mathbf{Z})$ with $e_\Sigma = (0, -\beta)$ and $e_\Sigma^2 = 1/\alpha$ where $\alpha = a_1 \dots a_n$. Now X satisfies the hypotheses of Theorem 2.1 and $\mu(e_\Sigma) = 1$. Theorem 1.1 follows once we compute the slice types $(a_i; r_i, s_i)$ at the exceptional orbits of the S^1-action classified by e_Σ.

Consider the smooth S^1-action on $\Sigma \times D^2$ given by

$$t \cdot (x, z) = (t \cdot x, tz)$$

where S^1 acts on Σ by the action of the Seifert fibration and on D^2 as multiplication of complex numbers. The orbit space $(\Sigma \times D^2)/S^1 = C$. There is the obvious S^1-action on $V \times S^1$ with orbit space V. These actions agree on the boundaries and glue together to give the pseudofree S^1-action classified by e_Σ.

On the tube of the $\mathbf{Z}_{a_i}$-orbit in Σ the S^1-action is

$$S^1 \times D^2 \times S^1 \to D^2 \times S^1$$
$$t \times (z, s) \to (t^{\alpha/a_i} z, t^{a_i} s)$$

because the Seifert invariant of Σ is $((a_1, \beta_1), \dots (a_n, \beta_n))$ where

$$\sum_{i=1}^{n} (\alpha\beta_i/a_i) = 1,$$

and the slice type of the action at the $\mathbf{Z}_{a_i}$-orbit is (a_i, ν_i) where $\beta_i \nu_i \equiv 1 \pmod{a_i}$. (See [NR].) But the equation $\Sigma\beta_i(\alpha/a_i) = 1$ implies that $\nu_i \equiv (\alpha/a_i) \pmod{a_i}$. In $\Sigma \times D^2$ near a $\mathbf{Z}_{a_i}$-orbit we then have the S^1-action

$$S^1 \times (D^2 \times S^1) \times D^2 \to (D^2 \times S^1) \times D^2$$
$$t \times ((z, s), w) \to ((t^{\alpha/a_i} z, t^{a_i} s), tw).$$

So the rotation numbers are $r_i = \alpha/a_i$ and $s_i = 1$. Thus $R(a_1, \dots, a_n) = R(X, e_\Sigma)$.

3. Branched covers

Let X be a pseudofree orbifold as above with pseudofree Euler class e. Recall that $D(X) = X - \bigcup_{i=1}^{n} \operatorname{int}(cL(a_i, b_i))$. Consider the α-fold cyclic cover of $\bigcup_{i=1}^{n} L(a_i, b_i) = \partial D(X)$ given by

$$H^2(\partial D(X); \mathbf{Z}) \cong H_1(\partial D(X); \mathbf{Z}) \cong H_1\left(\bigcup_{i=1}^{n} L(a_i, b_i); \mathbf{Z}\right) = \oplus \mathbf{Z}_{a_i} \to \mathbf{Z}_\alpha$$

where the unit i^*e goes to a generator of $\mathbf{Z}_\alpha$. Over each lens space this gives α/a_i times the standard cover $S^3 \to L(a_i, b_i)$; and it extends to an α-fold cyclic branched cover of $\bigcup_{i=1}^{n} cL(a_i, b_i)$ branched over the cone points. This branched cover extends to a branched cover over all of X with branch set $F \cup \{\text{cone points}\}$ where F is a surface in the interior of $D(X)$ which represents the Poincaré dual of that unique class $f \in H^2(D(X), \partial D(X); \mathbf{Z})$ such that $j^*f = \alpha e$. (See, for example, Lemma 2.2 of [CG2].) So we have an α-fold cyclic branched cover

$$\lambda\colon M(X) \to X$$

where $M(X)$ is a smooth closed 4-manifold with a smooth $\mathbf{Z}_\alpha$-action and $X = M(X)/\mathbf{Z}_\alpha$.

Let $p\colon Q^5 \to X$ be the pseudofree S^1-action classified by e. Pull this action back over $M(X)$. Each tube $D^4 \times_{\mathbf{Z}_{a_i}} S^1$ of an exceptional orbit in Q^5 pulls back to α/a_i copies of the free S^1-action whose orbit map is p_i in the diagram

$$\begin{array}{ccc} D^4 \times S^1 & \longrightarrow & D^4 \times_{\mathbf{Z}_{a_i}} S^1 \\ {\scriptstyle p_i}\downarrow & & \downarrow \\ D^4 & \longrightarrow & cL(a_i, b_i). \end{array}$$

So when the pseudofree S^1-action is pulled back over $M(X)$ it becomes a principal $S^1 = \mathrm{SO}(2)$-bundle P over $M(X)$. Let $L_e \to M(X)$ denote the associated SO(2) vector bundle. Now L_e carries a smooth $\mathbf{Z}_\alpha$-action by SO(2)-bundle maps covering the $\mathbf{Z}_\alpha$-action on $M(X)$. Note that $L_e/\mathbf{Z}_\alpha$ is just the mapping cylinder of the orbit map $p\colon Q^5 \to X$. Let $E = L_e \oplus \varepsilon$ where ε is a trivial real line bundle over $M(X)$. Then E is a $\mathbf{Z}_\alpha$-equivariant SO(3)-vector bundle over $M(X)$ with a $\mathbf{Z}_\alpha$-invariant SO(2) reduction. In Section 5 we shall study Yang-Mills connections on E which are invariant under the $\mathbf{Z}_\alpha$-action. First we calculate the number of such $\mathbf{Z}_\alpha$-invariant SO(2) reductions of E.

4. $\mathbf{Z}_\alpha$-Invariant reductions of E

Let $X = M(X)/\mathbf{Z}_\alpha$ be as in Section 3 with $E = L_e \oplus \varepsilon$. Given another $\mathbf{Z}_\alpha$-invariant topological reduction $E = \hat{L} \oplus \varepsilon$, let $\hat{P}$ be the principal SO(2)-bundle associated to $\hat{L}$. Then $\hat{P}/\mathbf{Z}_\alpha$ is a pseudofree S^1-manifold over $X = M(X)/\mathbf{Z}_\alpha$ with pseudofree Euler class $\hat{e} \in H^2(D(X); \mathbf{Z})$ so that $\hat{L} = L_{\hat{e}}$.

PROPOSITION 4.1. *$L_e \oplus \varepsilon$ is $\mathbf{Z}_\alpha$-equivalent to $L_{\hat{e}} \oplus \varepsilon$ if and only if*
i) $\hat{e}^2 = e^2$,
ii) $\hat{e} \equiv e \pmod 2$, *and*
iii) $i^*(\hat{e}) = i^*(e)$ *in* $H^2(\partial D(X); \mathbf{Z})$.

Proof. The classification of SO(3)-bundles over a 4-complex due to Dold and Whitney [DW] states that the first Pontrjagin class p_1 and the second Stiefel-Whitney class w_2 completely characterize such bundles. Also recall that the first Pontrjagin class of an SO(2)-bundle is the square of its Euler class.

Let $\lambda: M(X) \to X$ be the branched covering map and let $N = \lambda^{-1}(D(X))$. Consider the cohomology sequences of pairs

$$\begin{array}{ccccccc} 0 \longrightarrow & H^2(D(X), \partial D(X); \mathbf{Z}) & \xrightarrow{j^*} & H^2(D(X); \mathbf{Z}) & \xrightarrow{i^*} & H^2(\partial D(X); \mathbf{Z}) \\ & \downarrow \lambda^* & & \downarrow \lambda^* & & \\ 0 \longrightarrow & H^2(N, \partial N; \mathbf{Z}) & \longrightarrow & H^2(N; \mathbf{Z}) & \longrightarrow & 0 \\ & \parallel & & \parallel & & \\ & H^2(M(X); \mathbf{Z}) & = & H^2(M(X); \mathbf{Z}). & & \end{array}$$

Now $\lambda^* e \in H^2(N; \mathbf{Z}) \cong H^2(M(X); \mathbf{Z})$ is the Euler class of L_e. So under the isomorphism $H^4(M(X); \mathbf{Z}) \cong H^4(N, \partial N; \mathbf{Z})$ we have $p_1(L_e \oplus \varepsilon) = (1/\alpha)(\lambda^*(e \cup f))[N, \partial N] = (e \cup f)[D(X), \partial D(X)] = \alpha e^2$. So if $L_{\hat{e}} \oplus \varepsilon$ is $\mathbf{Z}_\alpha$-equivalent to $L_e \oplus \varepsilon$, by the above classification, $\hat{e}^2 = e^2$; and since $L_{\hat{e}}/\mathbf{Z}_\alpha$ and $L_e/\mathbf{Z}_\alpha$ when restricted to $D(X)$ are stably equivalent SO(2)-bundles, $\hat{e} \equiv e \pmod 2$. So far we have only utilized the information that $L_e \oplus \varepsilon$ is equivalent to $L_{\hat{e}} \oplus \varepsilon$ and that this equivalence respects the action of $\mathbf{Z}_\alpha$ over N.

Now $\lambda^{-1}(cL(a_i, b_i))$ consists of α/a_i disjoint 4-balls. Over each of these balls D^4 we may identify $L_e \oplus \varepsilon|_{D^4}$ with $D^4 \times \mathbf{R}^2 \times \mathbf{R}$ where $\mathbf{Z}_\alpha$ acts via $\zeta \cdot (x, z, s) = (\zeta \cdot x, z\zeta^{k_i}, s)$ where ζ is a fixed generator of $\mathbf{Z}_{a_i}$, k_i is an integer $(\operatorname{mod} a_i)$, and the action on the D^4-factor is that on $D^4 \subset M(X)$. Similarly we may identify $L_{\hat{e}} \oplus \varepsilon|_{D^4}$ with $D^4 \times \mathbf{R}^2 \times \mathbf{R}$ with $\mathbf{Z}_\alpha$-action $\zeta \cdot (x, z, s) = (\zeta \cdot x, z\zeta^{\hat{k}_i}, s)$. Our given $\mathbf{Z}_\alpha$-equivalence of $L_e \oplus \varepsilon$ with $L_{\hat{e}} \oplus \varepsilon$ gives an equivalence of representations of $\mathbf{Z}_{a_i}$ on $0 \times \mathbf{R}^3$; hence $k_i \equiv \hat{k}_i \pmod{a_i}$. However k_i clearly determines the Euler class of the S^1-bundle $S^3 \times_{\mathbf{Z}_{a_i}} S^1 = p_e^{-1}(L(a_i, b_i)) \to L(a_i, b_i)$. It follows then that $i^*(\hat{e}) = i^*(e)$ since the S^1-bundles $p_e^{-1}(\partial D(X))$ and $p_{\hat{e}}^{-1}(\partial D(X))$ are equivalent over $\partial D(X)$.

Conversely, assume i)–iii) hold. Let $L = L_e/\mathbf{Z}_\alpha|_{D(X)}$ and let $\hat{L} = L_{\hat{e}}/\mathbf{Z}_\alpha|_{D(X)}$. Since $i^*(\hat{e}) = i^*(e)$ it follows as above that $L_{\hat{e}} \oplus \varepsilon$ is $\mathbf{Z}_\alpha$-equivalent to $L_e \oplus \varepsilon$ over $M(X) - N$. Over ∂N if we take the quotient by $\mathbf{Z}_\alpha$ we obtain an equivalence of $\hat{L}$ with L, and we need to know that this extends to an equivalence over all of $D(X)$ of $\hat{L} \oplus \varepsilon$ with $L \oplus \varepsilon$. There are two obstructions to extending our equivalence and they lie in $H^2(D(X), \partial D(X); \mathbf{Z}_2)$ and $H^4(D(X), \partial D(X); \mathbf{Z})$. But since $\hat{e} \equiv e \pmod 2$ the first obstruction vanishes, and since $\hat{e}^2 = e^2$ the second vanishes. (See [FU; pp. 223–4].) □

PROPOSITION 4.2. *Suppose $H_1(D(X); \mathbf{Z}_2) = 0$ and $i^*(\mathrm{Tor}\, H^2(D(X))) = 0$. Then, up to orientation, the number of $\mathbf{Z}_\alpha$-invariant reductions of E is just $\mu(e)|H_1(D(X); \mathbf{Z})|$.*

Proof. Given an element $\hat{e} \in \mathrm{Fr}\, H^2(D(X); \mathbf{Z})$ such that $i^*\hat{e} = i^*e$, $\hat{e}^2 = e^2$ and $\hat{e} \equiv e \pmod 2$, there is an SO(2)-vector bundle $L_{\hat{e}}$ such that $L_{\hat{e}} \oplus \varepsilon$ is $\mathbf{Z}_\alpha$-equivalent to $L_e \oplus \varepsilon$. If $t \in \mathrm{Tor}\, H^2(D(X); \mathbf{Z}) \cong \mathrm{Tor}\, H_1(D(X); \mathbf{Z}) \cong H_1(D(X); \mathbf{Z})$, then $(\hat{e} + t)^2 = \hat{e}^2$, $\hat{e} + t \equiv \hat{e} \pmod 2$ since $H_1(D(X); \mathbf{Z})$ is odd torsion, and $i^*t = 0$, so that $i^*(\hat{e} + t) = i^*\hat{e}$. When $\alpha = 2$, $i^*(-e) = i^*(e)$, so that the factor $1/2$ must be included in the definition of $\mu(e)$ to account for a change of orientation. The proposition now follows from (4.1.). □

5. The equivariant geometry of E

In this section we shall describe the setting of our studies, namely the equivariant differential geometry of the $\mathbf{Z}_\alpha$-equivariant vector bundle E over $M = M(X)$.

Choose Riemannian metrics on E and M with respect to which $\mathbf{Z}_\alpha$ acts by isometries. For any vector bundle F over M let $\Omega^k(F) = \Gamma(\Lambda^k T^*M \otimes F)$ be the space of k-forms on M with values in F. In particular, $\Omega^0(F)$ is the space of smooth sections of F. For our purposes we shall view a Riemannian connection as a linear map $\nabla\colon \Omega^0(E) \to Q^1(E)$ which satisfies

$$\nabla(f\sigma) = df \otimes \sigma + f \nabla \sigma$$

and

$$d\langle \sigma_1, \sigma_2 \rangle = \langle \nabla \sigma_1, \sigma_2 \rangle + \langle \sigma_1, \nabla \sigma_2 \rangle$$

where $f\colon M \to \mathbf{R}$, and $\langle\ ,\ \rangle$ is the Riemannian metric on E given above.

Let $\mathscr{C}$ denote the space of all Riemannian connections on E. For $\nabla \in \mathscr{C}$ let $R^\nabla \in \Omega^2(\mathrm{Hom}(E, E))$ denote the curvature of ∇. The gauge transformation group $\mathscr{G}$ of E is the group of all bundle automorphisms of E (fixing M) which restrict to orientation-preserving linear isometries on each fiber of E. The adjoint bundle $\mathfrak{g}_E$ of E is defined by

$$\mathfrak{g}_E = \{L \in \mathrm{Hom}(E, E) | L_x \in so(E_x) \text{ on each fiber } E_x \text{ of } E\}.$$

Of course, for $\nabla \in \mathscr{C}$ we have $R^\nabla \in \Omega^2(\mathfrak{g}_E)$. The map $E \to \mathfrak{g}_E$ given by

$u \to u \times$ —, where "$\times$" denotes fiberwise cross product, induces an isomorphism of bundles $E \cong \mathfrak{g}_E$. Since the difference of two connections on E lies in $\Omega^1(\mathfrak{g}_E)$, $\mathscr{C}$ is an affine space. The gauge transformation group $\mathscr{G}$ acts effectively on $\mathscr{C}$ via $g(\nabla) = g \circ \nabla \circ g^{-1}$; i.e. for $\sigma \in \Omega^0(E)$ and $v \in TM$,

$$g(\nabla)_v(\sigma) = g\big(\nabla_v(g^{-1}\sigma)\big).$$

Let $\mathscr{B} = \mathscr{C}/\mathscr{G}$ denote the moduli space of connections on E and let $\pi\colon \mathscr{C} \to \mathscr{B}$ be the orbit map.

A connection $\nabla \in \mathscr{C}$ is called *self-dual* if $*R^\nabla = R^\nabla$ where "$*$" is the Hodge $*$-operator on M. Let $\mathscr{A}$ denote the subspace of $\mathscr{C}$ consisting of all self-dual connections of E. The action of $\mathscr{G}$ on $\mathscr{C}$ preserves $\mathscr{A}$; we let $\mathscr{M} = \mathscr{A}/\mathscr{G} = \pi(\mathscr{A})$ denote the moduli space of self-dual connections on E.

The compatible actions of $\mathbf{Z}_\alpha$ on E and M induce an action of $\mathbf{Z}_\alpha$ on $\mathscr{C}$. If $\nabla \in \mathscr{C}$, $h \in \mathbf{Z}_\alpha$, $\sigma \in \Omega^0(E)$, and $v \in TM$ are given then first define the action of $\mathbf{Z}_\alpha$ on $\Omega^0(E)$ by

$$h(\sigma) = h \circ \sigma \circ h^{-1}$$

where h^{-1} is a diffeomorphism of M and h is a bundle map. Then set

$$h(\nabla)_v(\sigma) = h\big(\nabla_v(h^{-1}\sigma)\big),$$

i.e. set $h(\nabla) = h \circ \nabla \circ h^{-1}$. This action of $\mathbf{Z}_\alpha$ on $\mathscr{C}$ preserves $\mathscr{A}$. Then define the *invariant connections* of E to be

$$\mathscr{C}^\alpha = \big\{\nabla \in \,|h(\nabla)(\sigma) = \nabla(\sigma) \text{ for all } \sigma \in \Omega^0(E) \text{ and } h \in \mathbf{Z}_\alpha\big\}.$$

Also let $\mathscr{A}^\alpha = \mathscr{C}^\alpha \cap \mathscr{A}$. The subgroup $\mathscr{G}^\alpha \subset \mathscr{G}$ of $\mathbf{Z}_\alpha$-equivariant gauge transformations

$$\mathscr{G}^\alpha = \{g \in \,|gh = hg \text{ for all } h \in \mathbf{Z}_\alpha\}$$

acts on $\mathscr{C}^\alpha$ and $\mathscr{A}^\alpha$. We let $\mathscr{B}^\alpha = \mathscr{C}^\alpha/\mathscr{G}^\alpha$ and $\mathscr{M}^\alpha = \mathscr{A}^\alpha/\mathscr{G}^\alpha$.

There is an action of $\mathbf{Z}_\alpha$ on each $\Omega^k(\mathfrak{g}_E)$ defined by

$$(h\Phi)_{v_1,\ldots,v_k} = h\Phi_{h^{-1}*(v_1),\ldots,h^{-1}*(v_k)}$$

where $\mathbf{Z}_\alpha$ acts on $\mathfrak{g}_E$ by means of the isomorphism $E \cong \mathfrak{g}_E$ described above. We let $\Omega^k(\mathfrak{g}_E)^\alpha$ denote the invariant subspace of this action. As usual a connection ∇ on E induces a connection ∇ on $\mathfrak{g}_E$ by the formula

$$\nabla(\phi) = [\nabla, \phi] \quad \text{for } \phi \in \Omega^0(\mathfrak{g}_E).$$

This means that $(\nabla\phi)(\sigma) = \nabla(\phi(\sigma)) - \phi(\nabla\sigma)$ for any $\sigma \in \Omega^0(E)$. Under the isomorphism $E \cong \mathfrak{g}_E$ this rule corresponds to the differentiation law $\nabla(u \times \sigma) = (\nabla u \times \sigma) + (u \times \nabla\sigma)$ for $u \in \Omega^0(E)$. The connection $\nabla\colon \Omega^0(\mathfrak{g}_E) \to \Omega^1(\mathfrak{g}_E)$ can be extended to $d^\nabla\colon \Omega^k(\mathfrak{g}_E) \to \Omega^{k+1}(\mathfrak{g}_E)$ by defining for

$\Phi \in \Omega^k(\mathfrak{g}_E)$:

$$d^\nabla \Phi_{v_0,\ldots,v_k} = \sum_{j=0}^{k} (-1)^j \nabla_{v_j}\left(\Phi_{v_0,\ldots,\hat{v}_j,\ldots,v_k}\right) + \sum_{i<j} (-1)^{i+j} \Phi_{[v_i,v_j],v_0,\ldots,\hat{v}_i,\ldots,\hat{v}_j,\ldots,v_k}.$$

PROPOSITION 5.1. *If* $\nabla \in \mathscr{C}^\alpha$ *then* $d^\nabla(\Omega^k(\mathfrak{g}_E)^\alpha) \subset \Omega^{k+1}(\mathfrak{g}_E)^\alpha$.

Proof. The map d^∇: $\Omega^k(\mathfrak{g}_E) \to \Omega^{k+1}(\mathfrak{g}_E)$ is the composition of the induced connection:

$$\Omega^0\left(\Lambda^k T^* \otimes \mathfrak{g}_E\right) \xrightarrow{\nabla} \Omega^1\left(\Lambda^k T^* \otimes \mathfrak{g}_E\right) = \Omega^0\left(T^* \otimes \Lambda^k T^* \otimes \mathfrak{g}_E\right)$$

with antisymmetrization

$$T^* \otimes \Lambda^k T^* \to \Lambda^{k+1} T^*.$$

Since ∇ on E and M is fixed by $\mathbf{Z}_\alpha$, so is the induced connection above. Antisymmetrization is clearly $\mathbf{Z}_\alpha$-equivariant. □

Similar proofs yield:

PROPOSITION 5.2. *If* $\nabla \in \mathscr{C}^\alpha$ *then* $R^\nabla \in \Omega^2(\mathfrak{g}_E)^\alpha$. *Furthermore, for any connection* $\nabla, \nabla' \in \mathscr{C}^\alpha$ *if and only if* $\nabla - \nabla' \in \Omega^1(\mathfrak{g}_E)^\alpha$. □

Throughout this work we shall need to use various Sobolev spaces. For the space $\Omega^0(E)$ of sections of a smooth Riemannian bundle over a Riemannian manifold we let $L^p_k(\Omega^0(E))$ denote the completion of $\Omega^0(E)$ in the Sobolev norm $\| \ \|_{L^p_k}$. (If the manifold is compact, different metrics give equivalent norms.) If we fix a base connection $\nabla_0 \in \mathscr{C}$, we define the Sobolev space of connections $\mathscr{C}_k$ to be $\mathscr{C}_k = \{\nabla_0 + A \mid A \in L^2_k(\Omega^1(\mathfrak{g}_E))\}$. Because of the affine structure this does not depend upon the base connection. Similarly if $\nabla_0 \in \mathscr{A}$ we define $\mathscr{A}_k = \{\nabla_0 + A \mid A \in L^2_k(\Omega^1(\mathfrak{g}_E)),\ d^\nabla_-(A) + [A, A]_- = 0\}$, where $d^\nabla_-(A) + [A, A]_-$ is the orthogonal projection of $d^\nabla(A) + [A, A]$ onto the (-1)-eigenspace $\Omega^2_-(\mathfrak{g}_E)$ of the action of the $*$-operator.

Since $\mathscr{G} = \Omega^0(\mathrm{Aut}_{\mathrm{SO}(3)}(E))$, the gauge transformation group of $\mathscr{C}_k$ is defined to be $\mathscr{G}_{k+1} = L^2_{k+1}(\Omega^0(\mathrm{Aut}_{\mathrm{SO}(3)}(E)))$. It is known that $\mathscr{G}_4$ is a Hilbert Lie group with Lie algebra $L^2_4(\Omega^0(\mathfrak{g}_E))$, and that the action of $\mathscr{G}$ on $\mathscr{C}$ extends to a smooth action of $\mathscr{G}_4$ on $\mathscr{C}_3$.

Since the structure group of E reduces to SO(2) there are connections on E which split as direct sum connections for some splitting $E \cong L \oplus \varepsilon$ for some rank 2 vector bundle L and trivial line bundle ε. Such connections are called *reducible*. The restriction of the orbit map of the action of $\mathscr{G}_4$ on $\mathscr{C}_3$ to the irreducible connections induces a principal bundle projection π: $\mathscr{C}_3^* \to \mathscr{B}_3^*$, where $\mathscr{B}_3^*$ is a smooth Hausdorff-Hilbert manifold. Local charts of this manifold are given by $\pi\colon \mathscr{O}_{\nabla,\varepsilon} \to \mathscr{B}^*$ where

$$\mathscr{O}_{\nabla,\varepsilon} = \left\{\nabla + A \mid A \in L_3^2(\Omega^1(\mathfrak{g}_E)),\ \delta^\nabla A = 0, \text{ and } \|A\|_{L_3^2} < \varepsilon\right\}$$

for ε sufficiently small [L; p. 65]. Here δ^∇ denotes the formal adjoint of d^∇.

If $\nabla \in \mathscr{C}_3 - \mathscr{C}_3^*$, i.e. if ∇ is a reducible connection, then Γ_∇, the isotropy group of the action of $\mathscr{G}_4$ at ∇, is isomorphic to SO(2). (Compare [L, Prop. II, 8.10] or [FS2, Prop. 3.1].) In this case Γ_∇ preserves $\mathscr{O}_{\nabla,\varepsilon}$ and the map $\pi'\colon \mathscr{O}_{\nabla,\varepsilon}/\Gamma_\nabla \to \mathscr{B}$ is a homeomorphism onto a neighborhood of $\pi(\nabla)$ and is a diffeomorphism off the fixed point set of Γ_∇.

We define $\mathscr{C}_k^\alpha$ by fixing a base connection $\nabla_0 \in \mathscr{C}^\alpha$ and letting $\mathscr{C}_k^\alpha = \{\nabla_0 + A \mid A \in L_k^2(\Omega^1(\mathfrak{g}_E)^\alpha)\}$. Similarly we can define $\mathscr{A}_k^\alpha$. The gauge transformation group in this context is $\mathscr{G}_{k+1}^\alpha = L_{k+1}^2(\Omega^0(\mathrm{Aut}_{SO(3)}(E))^\alpha)$. The same proofs as in the nonequivariant case (see [L; II.10]) show that $\mathscr{G}_4^\alpha$ is a Hilbert-Lie group with Lie algebra $L_4^2(\Omega^0(\mathfrak{g}_E)^\alpha)$ and that $\mathscr{G}_4^\alpha$ acts smoothly and freely on $\mathscr{C}_3^{\alpha *}$ such that the space $\mathscr{B}_3^\alpha$ is Hausdorff and such that $\mathscr{B}_3^{\alpha *} = \mathscr{C}_3^{\alpha *}/\mathscr{G}_4^\alpha$ is a smooth Hausdorff-Hilbert manifold and $\mathscr{C}_3^{\alpha *} \to \mathscr{B}_3^{\alpha *}$ is a principal bundle projection. (In this regard one should keep in mind Corollary 5.6 below.) For $\nabla \in \mathscr{C}_3^\alpha$ we have slices $\mathscr{O}_{\nabla,\varepsilon}^\alpha = \mathscr{O}_{\nabla,\varepsilon} \cap \mathscr{C}^\alpha$ as in the nonequivariant case.

From now on we shall write $\mathscr{C}$ to mean $\mathscr{C}_3$, $\mathscr{G}$ for $\mathscr{G}_4$ and drop the Sobolev subscripts. We also do this for $\mathscr{C}^\alpha$, $\mathscr{G}^\alpha$, etc.

Next we shall discuss the reducible self-dual connections on E.

PROPOSITION 5.3. *Suppose $M/\mathbf{Z}_\alpha$ has positive definite intersection form and $H^1(M/\mathbf{Z}_\alpha; \mathbf{R}) = 0$. Then each $\mathbf{Z}_\alpha$-equivariant SO(2)-vector bundle L over M has a unique $\mathbf{Z}_\alpha$-equivariant gauge equivalence class of $\mathbf{Z}_\alpha$-invariant self-dual connections.*

Proof. Let ∇ be an arbitrary SO(2) connection on L. By averaging ∇ over the group, we obtain a $\mathbf{Z}_\alpha$-invariant connection D on L which locally is $D = d + iw$ where iw is a real-valued 1-form. The curvature of D is $R^D = idw \in \Omega^2(M)^\alpha$, and the de Rham class $[(1/2\pi i)R^D] \in H^2(M; \mathbf{R})$ is the real Euler class e of L. It follows from the uniqueness of harmonic representatives that there is a unique harmonic form $\bar{e} \in \Omega^2(M)^\alpha$ such that $[\bar{e}] = e \in H^2(M; \mathbf{R})$. So $(1/2\pi i)R^D - \bar{e} = dA$, and by averaging we may assume $A \in \Omega^1(M)^\alpha$. So $D' = D - 2\pi A$ is a $\mathbf{Z}_\alpha$-invariant connection on L, and $R^{D'} = R^D - id(2\pi A) = 2\pi i\bar{e}$ is harmonic in $\Omega^2(M)^\alpha$. Thus $R^{D'}$ represents an element $[R^{D'}]$ of the $\mathbf{Z}_\alpha$-invariant de Rham cohomology of M, therefore of $H^2(M/\mathbf{Z}_\alpha; \mathbf{R})$.

Let $\Omega_\pm^2(M)^\alpha$ denote the $\pm$-eigenspaces of the $*$ operator; then $\Omega^2(M)^\alpha = \Omega_+^2(M)^\alpha \oplus \Omega_-^2(M)^\alpha$. Identifying $H^2(M/\mathbf{Z}_\alpha; \mathbf{R})$ with the harmonic space of $\Omega^2(M)^\alpha$ we obtain the corresponding splitting

$$H^2(M/\mathbf{Z}_\alpha;\mathbf{R}) = H^2_+(M/\mathbf{Z}_\alpha;\mathbf{R}) \oplus H^2_-(M/\mathbf{Z}_\alpha;\mathbf{R}).$$

If $[\phi] \in H^2_\pm(M/\mathbf{Z}_\alpha;\mathbf{R})$ then $[\phi]^2 = \int_M \phi \wedge \phi = \pm \int_M \phi \wedge {*\phi} = \pm(\|\phi\|_0)^2$. Thus the intersection form on $H^2(M/\mathbf{Z}_\alpha;\mathbf{R})$ is positive definite on $H^2_+(M/\mathbf{Z}_\alpha;\mathbf{R})$ and negative definite on $H^2_-(M/\mathbf{Z}_\alpha;\mathbf{R})$. Since $M/\mathbf{Z}_\alpha$ has a positive definite intersection pairing, $H^2_-(M/\mathbf{Z}_\alpha;\mathbf{R}) = 0$; so $R^{D'} \in H^2_+(M/\mathbf{Z}_\alpha;\mathbf{R})$ and D' is self-dual.

Furthermore, if ∇ is any other $\mathbf{Z}_\alpha$-invariant connection on L with $R^\nabla = R^{D'}$ then $\nabla = D' + iA'$ where $A' \in \Omega^1(M)^\alpha$ and $dA' = 0$. However $H^1(M/\mathbf{Z}_\alpha;\mathbf{R}) = 0$ so that $A' = -df$ for $f \in \Omega^0(M)^\alpha$. Thus $\nabla = D' - idf$. Multiplication by the function $g = \exp(if)$ gives a $\mathbf{Z}_\alpha$-equivariant gauge transformation, and

$$g(D') = \exp(if)D'\exp(-if) = D' - idf = \nabla. \qquad \square$$

Recall that $X = M/\mathbf{Z}_\alpha$ and $D(X) = X - \bigcup_{i=1}^n \operatorname{int}(cL(a_i, b_i))$.

Proposition 5.4. *Suppose X has positive definite intersection form, $H_1(D(X);\mathbf{Z}_2) = 0$ and $i^*\operatorname{Tor} H^2(D(X),\mathbf{Z})) = 0$. Then there are exactly $\mu(e)\cdot|H_1(D(X);\mathbf{Z})|$ $\mathbf{Z}_\alpha$-equivariant gauge equivalence classes of reducible $\mathbf{Z}_\alpha$-invariant self-dual connections on E.*

Proof. By Proposition 4.2 $m = \mu(e)\cdot|H_1(D(X);\mathbf{Z})|$ is just the number, up to orientation, of reductions of $E = L_e \oplus \varepsilon$.

A reducible $\mathbf{Z}_\alpha$-invariant connection ∇ gives a $\mathbf{Z}_\alpha$-invariant splitting $E \cong L \oplus \varepsilon$ and $\nabla = D \oplus d$. But $E \cong (-L) \oplus (-\varepsilon)$ and D also defines a connection on $-L$, so that by Proposition 5.3, there are at most m $\mathbf{Z}_\alpha$-equivariant gauge equivalence classes of reducible $\mathbf{Z}_\alpha$-invariant self-dual connections on E.

Any gauge equivalence of reducible SO(3)-connections must preserve parallel sections of E and hence induce a gauge equivalence of the corresponding SO(2)-connections. Thus there are exactly m $\mathbf{Z}_\alpha$-equivariant gauge equivalence classes of reducible $\mathbf{Z}_\alpha$-invariant self-dual connections on E. $\square$

Proposition 5.5. *Suppose $\nabla, \nabla' \in \mathscr{C}^{\alpha*}$ (i.e. are irreducible) and $g \in \mathscr{G}$ with $\nabla' = g(\nabla)$. Then $g \in \mathscr{G}^\alpha$.*

Proof. If ∇ is irreducible and $\nabla' = g(\nabla)$ then for $h \in \mathbf{Z}_\alpha$, $h(g(\nabla)) = h(\nabla') = \nabla' = g(\nabla) = g(h(\nabla))$; i.e. $[g,h](\nabla) = \nabla$ and $[g,h] \in \mathscr{G}$. So $[g,h] \in \Gamma_\nabla$ the isotropy group of the action of $\mathscr{G}$ at ∇. However a connection has a nontrivial isotropy group if and only if it is reducible (see [D, page 287] or Proposition 3.1 of [FS2]). So $\Gamma_\nabla = 1$ and $g \in \mathscr{G}^\alpha$. $\square$

Note that for reducible $\mathbf{Z}_\alpha$-invariant connections as above with $g(\nabla) = \nabla$, we can only conclude that $[g,h] \in \Gamma_\nabla \cong \mathrm{SO}(2)$ for each $h \in \mathbf{Z}_\alpha$; so g need not be equivariant.

COROLLARY 5.6. *Let* $\pi\colon \mathscr{C} \to \mathscr{B}$ *be the projection. Then* $\mathscr{B}^{\alpha*}(\overset{\text{def}}{=} \mathscr{C}^{\alpha*}/\mathscr{G}^{\alpha}) = \pi(\mathscr{C}^{\alpha*})$ *and* $\mathscr{M}^{\alpha*}(\overset{\text{def}}{=} \mathscr{A}^{\alpha*}/\mathscr{G}^{\alpha}) = \pi(\mathscr{A}^{\alpha*})$. □

COROLLARY 5.7. *The induced maps* $\mathscr{B}^{\alpha} \to \pi(\mathscr{C}^{\alpha})$ *and* $\mathscr{M}^{\alpha} \to \pi(\mathscr{A}^{\alpha})$ *are quotient maps topologically.*

In fact the only difference between $\mathscr{M}^{\alpha}$ and $\pi(\mathscr{A}^{\alpha})$ is that some identifications may have to be made among the finite number of gauge equivalence classes of reducible connections of $\mathscr{M}^{\alpha}$ in order to obtain $\pi(\mathscr{A}^{\alpha})$. □

6. The invariant fundamental elliptic complex

Consider a connection $\nabla \in \mathscr{A}$. We then have the fundamental complex

$$0 \to \Omega^0(\mathfrak{g}_E) \xrightarrow{\nabla} \Omega^1(\mathfrak{g}_E) \xrightarrow{d_-^{\nabla}} \Omega_-^2(\mathfrak{g}_E) \to 0$$

where d_-^{∇} is the orthogonal projection of d^{∇} on the anti-self-dual 2-forms. Since ∇ is self-dual, $d_-^{\nabla} \circ \nabla = R_-^{\nabla} = 0$, and this complex is elliptic [AHS, §6]. Thus the complex has cohomology groups $H_{\nabla}^0, H_{\nabla}^1, H_{\nabla}^2$ which we identify with spaces of harmonic forms.

We have seen that the linear action of $\mathbf{Z}_\alpha$ on $\mathfrak{g}_E$ induces an action on the above complex and the $\mathbf{Z}_\alpha$-invariant subcomplex

$$0 \to \Omega^0(\mathfrak{g}_E)^{\alpha} \xrightarrow{\nabla} \Omega^1(\mathfrak{g}_E)^{\alpha} \xrightarrow{d_-^{\nabla}} \Omega_-^2(\mathfrak{g}_E)^{\alpha} \to 0.$$

(Compare Proposition 5.1, and note that the $\mathbf{Z}_\alpha$-action on M commutes with the * operator.) This is an elliptic complex with cohomology groups $H_{\nabla}^{0,\alpha}$, $H_{\nabla}^{1,\alpha}$, and $H_{\nabla}^{2,\alpha}$ which may be identified with spaces of invariant harmonic forms.

Our goal is to compute

$$-\dim H_{\nabla}^{0,\alpha} + \dim H_{\nabla}^{1,\alpha} - \dim H_{\nabla}^{2,\alpha}.$$

Following [AHS; § 6] we may replace the invariant fundamental elliptic complex by a single elliptic operator

$$\delta^{\nabla} + d_-^{\nabla} : \Omega^1(\mathfrak{g}_E)^{\alpha} \to \Omega^0(\mathfrak{g}_E)^{\alpha} \oplus \Omega_-^2(\mathfrak{g}_E)^{\alpha}$$

where δ^{∇} is the formal adjoint of d^{∇}. And as in [AHS] this has the same index as the Dirac operator

$$D\colon \Gamma(V_+(M) \otimes V_-(M) \otimes \mathfrak{g}_E)^{\alpha} \to \Gamma(V_-(M) \otimes V_-(M) \otimes \mathfrak{g}_E)^{\alpha}$$

where $V_+(M)$ and $V_-(M)$ are the complex spinor bundles of $\pm\frac{1}{2}$-spinors on M. Now the index of $\delta^{\nabla} + d_-^{\nabla}$ is the average of the Lefschetz numbers:

$$\operatorname{Ind}(\delta^{\nabla} + d_-^{\nabla}) = \frac{1}{\alpha} \sum_{g \in \mathbf{Z}_\alpha} L(g, \delta^{\nabla} + d_-^{\nabla}) = \frac{1}{\alpha} \sum_{g \in \mathbf{Z}_\alpha} L(g, D).$$

However by the Lefschetz Theorem of Atiyah and Segal [ASII], $L(g, D)$ can be computed in terms of the index of associated elliptic symbol classes on fixed point sets.

As in [AHS] for $g = 1$ we have:

$$L(1, D) = \operatorname{ch}(\mathfrak{g}_E \otimes \mathbf{C})\operatorname{ch}(V_-)\hat{A}(M)[M]$$
$$= p_1(\mathfrak{g}_E \otimes \mathbf{C})[M] + 3(\operatorname{ind}\Delta)$$

where $\Delta\colon \Gamma(V_+ \otimes V_-)^\alpha \to \Gamma(V_- \otimes V_-)^\alpha$ and

$$\operatorname{ind}\Delta = \frac{-\alpha}{2}\left[\left(\chi(X) - d_\chi\right) - \left(\sigma(X) - d_\sigma\right)\right]$$

where d_χ and d_σ are the defect terms for multiplicativity of Euler characteristic and signature under branched covers. These defect terms may be computed in terms of the Lefschetz numbers:

$$-\frac{\alpha}{2}(d_\chi - d_\sigma) = \sum_{\substack{g \in \mathbf{Z}_\alpha \\ g \neq 1}} L(g, \Delta).$$

We have already computed in the proof of (4.1) that $p_1(\mathfrak{g}_E)[M] = p_1(E)[M] = \alpha e^2$. Hence $p_1(\mathfrak{g}_E \otimes \mathbf{C})[M] = 2\alpha e^2$. Also, under the hypothesis of Theorem 2.1, the oriented rational homology manifold $X = M/\mathbf{Z}_\alpha$ has $H^1(X; \mathbf{Q}) = 0$ and has positive definite intersection form; hence

$$\chi(X) - \sigma(X) = 2.$$

So we have

$$L(1, D) = 2\alpha e^2 - 3\alpha + \frac{3\alpha}{2}(d_\chi - d_\sigma).$$

To compute the Lefschetz numbers $L(g, D)$ for $g \neq 1$ we must consider restrictions to the isolated fixed points I_j of the $\mathbf{Z}_{\alpha_j}$ and to the fixed surface $\tilde{F} = \lambda^{-1}(F)$ of $\mathbf{Z}_\alpha$. Consider first an isolated fixed point y of $\mathbf{Z}_{a_j}$; there are α/aj such points in M. Let $g = e^{2\pi i k/a_j} \in \mathbf{Z}_{a_j}$ and let $\theta_1 = 2\pi r_j k/a_j$ and $\theta_2 = 2\pi s_j k/a_j$. The contribution to the Lefschetz number of g at y is:

$$L(g, D)_y = \frac{\operatorname{ch}_g(V_+ - V_-)\operatorname{ch}_g(V_-)}{\operatorname{ch}_g(\Lambda_{-1})}(T \otimes \mathbf{C}) \cdot \operatorname{ch}_g(\mathfrak{g}_E \otimes \mathbf{C})[\text{point}]$$

where T is the tangent space at the point y. We have

$$\operatorname{ch}_g(\mathfrak{g}_E \otimes \mathbf{C}) = \operatorname{ch}_g((L \otimes \varepsilon) \otimes \mathbf{C}) = \operatorname{ch}_g L + \operatorname{ch}_g \bar{L} + 1 = g + g^{-1} + 1$$
$$= 1 + 2\cosh\left(\frac{2\pi i k}{a_j}\right) = 1 + 2\cos\frac{2\pi k}{a_j} = 3 - 4\sin^2\left(\frac{\pi k}{a_j}\right).$$

Hence

$$L(g,D)_y = 3L(g,\Delta)_y - 4\sin^2\left(\frac{\pi k}{a_j}\right)\frac{\mathrm{ch}_g(V_+ - V_-)\mathrm{ch}_g(V_-)}{\mathrm{ch}_g(\Lambda_{-1})}(T\otimes \mathbf{C})[\text{point}].$$

Now compute the second term on the right.

$$
\begin{aligned}
&- 4\sin^2\left(\frac{\pi k}{a_j}\right)\frac{\mathrm{ch}_g(V_+ - V_-)\mathrm{ch}_g(V_-)}{\mathrm{ch}_g(\Lambda_{-1})}(T\otimes \mathbf{C})[\text{point}]\\
&= -4\left[\prod_{p=1}^{2}\frac{(e^{i\theta_p/2}-e^{-i\theta_p/2})e^{-i\theta_p/2}}{(1-e^{i\theta_p})(1-e^{-i\theta_p})}\right](e^{i\theta_1}+e^{i\theta_2})\sin^2\left(\frac{\pi k}{a_j}\right)\\
&= -4\left[\prod_{p=1}^{2}\frac{(e^{i\theta_p/2}-e^{-i\theta_p/2})}{(1-e^{i\theta_p})(1-e^{-i\theta_p})}\right](e^{i(\theta_1-\theta_2)/2}+e^{-i(\theta_1-\theta_2)/2})\sin^2\left(\frac{\pi k}{a_j}\right)\\
&= -4\left[\prod_{p=1}^{2} - \frac{1}{e^{i\theta_p/2}-e^{-i\theta_p/2}}\right](e^{i(\theta_1-\theta_2)/2}+e^{-i(\theta_1-\theta_2)/2})\sin^2\left(\frac{\pi k}{a_j}\right)\\
&= -4\left[\prod_{p=1}^{2} - \frac{1/2}{\sinh(i\theta_p/2)}\right]\left[2\cosh\left(\frac{i\theta_1 - i\theta_2}{2}\right)\right]\sin^2\left(\frac{\pi k}{a_j}\right)\\
&= -2\,\mathrm{csch}\left(\frac{i\theta_1}{2}\right)\mathrm{csch}\left(\frac{i\theta_2}{2}\right)\left[\cosh\left(\frac{i\theta_1}{2}\right)\cosh\left(\frac{i\theta_2}{2}\right)\right.\\
&\qquad \left. - \sinh\left(\frac{i\theta_1}{2}\right)\sinh\left(\frac{i\theta_2}{2}\right)\right]\sin^2\left(\frac{\pi k}{a_j}\right)\\
&= -2\left[\coth\left(\frac{i\theta_1}{2}\right)\coth\left(\frac{i\theta_2}{2}\right) - 1\right]\sin^2\left(\frac{\pi k}{a_j}\right)\\
&= -2\left[-\cot\left(\frac{\theta_1}{2}\right)\cot\left(\frac{\theta_2}{2}\right) - 1\right]\sin^2\left(\frac{\pi k}{a_j}\right)\\
&= 2\left(1+\cot\left(\frac{\pi k r_j}{a_j}\right)\cot\left(\frac{\pi k s_j}{a_j}\right)\right)\sin^2\left(\frac{\pi k}{a_j}\right).
\end{aligned}
$$

Summing over all $g \in \mathbf{Z}_{a_j}$ we obtain the contribution:

$$2\sum_{k=1}^{a_j-1}\left(1+\cot\left(\frac{\pi k r_j}{a_j}\right)\cot\left(\frac{\pi k s_j}{a_j}\right)\right)\sin^2\left(\frac{\pi k}{a_j}\right).$$

But

$$\sum_{k=1}^{a_j-1} \sin^2\left(\frac{\pi k}{a_j}\right) = \sum_{k=1}^{a_j-1} \frac{1-\cos(2\pi k/a_j)}{2} = \frac{1}{2}(a_j-1) - \frac{1}{2}\sum_{k=1}^{a_j-1}\cos(2\pi k/a_j)$$

$$= \frac{1}{2}(a_j-1) + \frac{1}{2} = \frac{1}{2}a_j.$$

So the contribution from $\mathbf{Z}_{a_j}$ obtained by summing over the α/a_j points in I_j is:

$$3\sum_{\substack{g\in \mathbf{Z}_\alpha\\ g\neq 1}}\sum_{y\in I_j} L(g,\Delta)_y + \alpha + \frac{2\alpha}{a_j}\sum_{k=1}^{a_j-1}\cot\left(\frac{\pi k r_j}{a_j}\right)\cot\left(\frac{\pi k s_j}{a_j}\right)\sin^2\left(\frac{\pi k}{a_j}\right).$$

The surface $\tilde{F} = \lambda^{-1}(F)$ is the fixed point set of $\mathbf{Z}_\alpha$. Since $\mathbf{Z}_\alpha$ acts trivially on the fibers of $\mathfrak{g}_E$ over F we have

$$\mathrm{ch}_g(\mathfrak{g}_{E|\tilde{F}}\otimes \mathbf{C}) = \mathrm{ch}(L\oplus\varepsilon)|_{\tilde{F}}\otimes\mathbf{C} = \mathrm{ch}(L|_{\tilde{F}}) + \mathrm{ch}(\bar{L}|_{\tilde{F}}) + 1 = 3.$$

So computing as above, for any $g\neq 1$ in $\mathbf{Z}_\alpha$ we have

$$L(g,D)_{\tilde{F}} = 3L(g,\Delta)_{\tilde{F}}.$$

THEOREM 6.1. $-\dim H^{0,\alpha}_\nabla + \dim H^{1,\alpha}_\nabla - \dim H^{2,\alpha}_\nabla = R(X,e)$.

Proof. Using our computations, we have

$$\mathrm{Ind}(\delta^\nabla + d^\nabla_-) = \frac{1}{\alpha}\sum_{g\in\mathbf{Z}_\alpha} L(g,D)$$

$$= 2e^2 - 3 + \left[\frac{3}{2}(d_X - d_\sigma) + \frac{3}{\alpha}\sum_{\substack{g\in\mathbf{Z}_\alpha\\ g\neq 1}} L(g,\Delta)\right] + n$$

$$+ \sum_{i=1}^{n}\frac{2}{a_i}\sum_{k=1}^{a_i-1}\cot\left(\frac{\pi k r_i}{a_i}\right)\cot\left(\frac{\pi k s_i}{a_i}\right)\sin^2\left(\frac{\pi k}{a_i}\right).$$

Since the term in the brackets is 0, we have our result. □

Next, note that $H^0_\nabla = \ker\nabla$ consists of the covariant constant sections of ∇. Thus

$$\dim H^0_\nabla = \begin{cases} 0 & \text{if } \nabla \text{ is irreducible} \\ 1 & \text{if } \nabla \text{ is reducible.}\end{cases}$$

Similarly if ∇ is $\mathbf{Z}_\alpha$-invariant then $H^{0,\alpha}_\nabla$ consists of covariant constant $\mathbf{Z}_\alpha$-invariant sections of ∇ and

$$\dim H^{0,\alpha}_\nabla = \begin{cases} 0 & \text{if } \nabla \text{ is irreducible} \\ 1 & \text{if } \nabla \text{ is reducible.}\end{cases}$$

At a reducible $\mathbf{Z}_\alpha$-invariant self-dual connection ∇, $E \cong \mathfrak{g}_E \cong L \oplus \varepsilon$, and the isotropy group of $\mathscr{G}^\alpha$, Γ^α_∇, at ∇ consists of the SO(2) which rotates L and acts trivially on ε. So we have $\Omega^p(\mathfrak{g}_E)^\alpha \cong \Omega^p(L)^\alpha \oplus \Omega^p(\varepsilon)^\alpha$. The isotropy group Γ^α_∇ acts by the standard action of SO(2) on the vector space $\Omega^p(L)^\alpha$ and acts trivially on $\Omega^p(\varepsilon)^\alpha$. So the fixed point set of the Γ^α_∇ action on $\Omega^p(E)^\alpha$ is $\Omega^p(\varepsilon)^\alpha$. For the harmonic spaces we have

$$H^{1,\alpha}_\nabla = H^{1,\alpha}_\nabla(L) \oplus H^{1,\alpha}_\nabla(\varepsilon) \cong H^{1,\alpha}_\nabla(L) \oplus H^1(M/\mathbf{Z}_\alpha;\mathbf{R})$$
$$H^{2,\alpha}_\nabla = H^{2,\alpha}_\nabla(L) \oplus H^{2,\alpha}_\nabla(\varepsilon) \cong H^{2,\alpha}_\nabla(L) \oplus H^2_-(M/\mathbf{Z}_\alpha;\mathbf{R}).$$

However by hypothesis $H^1(M/\mathbf{Z}_\alpha;\mathbf{R}) = 0 = H^2_-(M/\mathbf{Z}_\alpha;\mathbf{R})$ (since the intersection form of $M/\mathbf{Z}_\alpha$ must be negative definite on $H^2_-(M/\mathbf{Z}_\alpha;\mathbf{R})$). We thus have the following description of the harmonic spaces at a reducible $\nabla \in \mathscr{A}^\alpha$.

PROPOSITION 6.2. *For a reducible* $\nabla \in \mathscr{A}^\alpha$, $\dim H^{0,\alpha}_\nabla = 1$, *and* $H^{1,\alpha}_\nabla$ *and* $H^{2,\alpha}_\nabla$ *are even dimensional vector spaces with* Γ^α_∇ *acting via the standard action of* SO(2) (*leaving only* 0 *fixed*). □

COROLLARY 6.3. *$R(X,e)$ is an odd integer.* □

Proof. By Proposition (5.3), $E \cong L_e \oplus \varepsilon$ always admits a reducible self-dual $\mathbf{Z}_\alpha$-equivariant connection. □

7. The compactness theorem

We now turn to the key idea that will allow us to prove Theorem 2.1, namely to determine when $\mathscr{M}$ and $\mathscr{M}^\alpha$ are compact spaces. If ∇ is a connection on E its *Yang-Mills action* is defined by:

$$\mathscr{YM}(\nabla) = \frac{1}{2}\int_M \|R^\nabla\|^2.$$

(View $R^\nabla \in \Omega^2(\mathfrak{g}_E)$, and think of $\mathfrak{g}_E \subset \operatorname{Hom}(E,E)$; then use the inner product $\langle A, B\rangle = \operatorname{tr}(A^t \circ B)$ to induce the norm.) Two theorems of Karen Uhlenbeck are fundamental.

PROPOSITION 7.1 (The Bubble Theorem [U_1]). *Let $\{\nabla_i\}$ be any sequence of self-dual connections on E. Either*

(1) *There are a subsequence $\{\nabla_{i'}\}$ and gauge equivalent connections $\{\tilde\nabla_{i'}\}$ such that $\tilde\nabla_{i'} \to \nabla_\infty$, a self-dual connection on E, in the C^∞-topology (so that $[\nabla_{i'}] \to [\nabla_\infty]$ in $\mathscr{M}$), or*

(2) *There are a finite number of points $x_1 \dots, x_k$ in M and a subsequence $\{\nabla_{i'}\}$ and gauge equivalent connections $\{\tilde\nabla_{i'}\}$ such that $\tilde\nabla_{i'} \to \nabla_\infty$ a self-dual connection on $E|_{M_0} = M_0 = M - \{x_1,\dots,x_k\})$ in the C^∞-topology.*

PROPOSITION 7.2 (Removability of Singularities [U_2]). *Let* ∇ *be a self-dual* SO(3)*-connection on a bundle* E_0 *defined over* $M_0 = M - \{x_1, \ldots x_k\}$. *Suppose* $\mathscr{YM}(\nabla) < \infty$. *Then* (E_0, ∇) *extend smoothly over* M.

An important consequence is that the moduli space is compact when $p_1(E)$ is small enough.

THEOREM 7.3 (Compactness Theorem). *Let* E *be an* SO(3)*-vector bundle over an oriented 4-manifold* M. *Suppose that* $0 \leq p_1(E) \leq 3$; *then* $\mathscr{M}$ *is compact. This also holds if* $p_1(E) = 4$ *if we also assume that* $w_2(E) \neq 0$ *and* $H^2(\pi_1 M; \mathbf{Z}_2) = 0$.

Proof. Consider a sequence $\{[\nabla_i]\}$ in $\mathscr{M}$. If $\{[\nabla_i]\}$ has no convergent subsequence then the Bubble Theorem implies that there are some subsequence $\{\nabla_{i'}\}$ and gauge equivalent connections $\{\tilde{\nabla}_{i'}\}$ such that $\tilde{\nabla}_{i'} \to \nabla_\infty$, a self-dual connection on $E|_{M_0}$ where $M_0 = M - \{x_1, \ldots, x_k\}$ for some finite number of points $x_i \in M$. Since each $\tilde{\nabla}_i$ is a self-dual connection on E we have

$$\mathscr{YM}(\tilde{\nabla}_{i'}) = 2\pi^2 p_1(E)$$

(see [L], [U_3], or [FS_2]). So from Fatou's Lemma

$$\begin{aligned} 0 \leq \mathscr{YM}(\nabla_\infty) &= \frac{1}{2}\int_{M_0} \|R^{\nabla_\infty}\|^2 \leq \frac{1}{2}\liminf \int_{M_0} \|R^{\tilde{\nabla}_i}\|^2 \\ &= \mathscr{YM}(\tilde{\nabla}_i) = 2\pi^2 p_1(E) < \infty. \end{aligned}$$

Thus the Removability of Singularities Theorem applies; so ∇_∞ extends to a self-dual connection on a bundle E_∞ over all of M. Since $E_\infty|_{M_0} = E|_{M_0}$ we have $w_2(E_\infty) = w_2(E)$; it follows that $p_1(E_\infty) \equiv p_1(E) \pmod 4$ (for example see [DW, Thm. 2]). However since

$$0 \leq \frac{1}{2\pi^2}\mathscr{YM}(\nabla_\infty) = p_1(E_\infty)$$

we have $0 \leq p_1(E_\infty) \leq p_1(E)$. If $0 \leq p_1(E) \leq 3$ this implies that $p_1(E_\infty) = p_1(E)$. So by the classification of SO(3)-bundles [DW], $E_\infty \cong E$ and $\{[\nabla_i]\}$ actually has a convergent subsequence in $\mathscr{M}$.

In case $p_1(E) = 4$, $w_2(E) \neq 0$, and $H^2(\pi_1 M; \mathbf{Z}_2) = 0$, we might have $p_1(E_\infty) = 0$. However this would imply that the bundle E_∞ was flat and so had finite structure group G. So the classifying space $BG = K(G, 1)$, and the classifying map for the vector bundle E_∞ must factor through $K(\pi_1 M, 1)$. Thus the second Stiefel-Whitney class $w_2(E_\infty)$ was pulled back through $H^2(\pi_1 M; \mathbf{Z}_2) = 0$; i.e. $w_2(E_\infty) = 0$, a contradiction. Hence $E_\infty \cong E$ in this case as well. □

COROLLARY 7.4. *Let X be a pseudofree orbifold with pseudofree Euler class e. Suppose $e^2 \leq 4/\alpha$ and, if $e^2 = 4/\alpha$, also assume $e \not\equiv 0 (\mathrm{mod}\, 2)$ and $H^2(\pi_1(D(X)); \mathbf{Z}_2) = 0$. Then the moduli space of self-dual connections on the* SO(3)*-vector bundle $E = L_e \oplus \varepsilon$ over $M(X)$ is compact.*

Proof. This follows immediately from (7.3) since $p_1(E) = \alpha e^2 \leq 4$. In case $\alpha e^2 = 4$, note that $E|_N$ flat implies $E/\mathbf{Z}_\alpha|_{D(X)}$ is flat; so since $w_2(E/\mathbf{Z}_\alpha|_{D(X)}) \equiv e (\mathrm{mod}\, 2)$ the proof goes through just as in (7.3). □

THEOREM 7.5. *With the hypothesis of* (7.4), *$\mathscr{M}^\alpha$ is compact.*

Proof. Since it is the fixed point set of the action of $\mathbf{Z}_\alpha$ on $\mathscr{M}$, $\pi(\mathscr{A}^\alpha)$ is closed and therefore compact. Now by (5.7) the map $\mathscr{M}^\alpha \to \pi(\mathscr{A}^\alpha)$ is a topological quotient map and is one-to-one off a finite set. Thus any open cover of $\mathscr{M}^\alpha$ clearly has a finite subcover. Also, as we have pointed out before, $\mathscr{M}^\alpha \subset \mathscr{B}^\alpha$ is Hausdorff. Thus $\mathscr{M}^\alpha$ is compact. □

8. The perturbed moduli space

Let X be a pseudofree orbifold with pseudofree Euler class e. Suppose we are in the situation where $R(X, e) > 0$ and the moduli space $\mathscr{M}^\alpha$ of equivariant self-dual connections on the bundle $E = L_e \oplus \varepsilon$ over $M = M(X)$ is compact. We will now shamelessly follow the perturbation argument of [D] as described in [L] to perturb $\mathscr{M}^\alpha$ to obtain a compact manifold of dimension $R(X, e)$ which has for each equivariant gauge equivalence class of reducible $\mathbf{Z}_\alpha$-invariant self-dual connections a singular point which has as a neighborhood a cone on a complex projective space.

First, we have a theorem of Atiyah, Hitchin, and Singer

PROPOSITION 8.1 [AHS]. *Let $\nabla \in \mathscr{C}^\alpha$; then there are a neighborhood $\mathscr{O}^\alpha$ of $0 \in H^{1,\alpha}_\nabla$ and a differentiable map*

$$\Phi\colon \mathscr{O}^\alpha \to H^{2,\alpha}_\nabla$$

with $\Phi(0) = 0$ which is $\mathrm{SO}(2) = \Gamma^\alpha_\nabla$*-equivariant when ∇ is reducible. Furthermore,*

(1) $\mathscr{M}^\alpha \cap \mathscr{O}^\alpha_{\nabla, \varepsilon} \simeq \Phi^{-1}(0)$ *if ∇ is irreducible,*
(2) $\mathscr{M}^\alpha \cap (\mathscr{O}^\alpha_{\nabla, \varepsilon}/\mathrm{SO}(2)) \cong \Phi^{-1}(0)/\mathrm{SO}(2)$ *if ∇ is reducible.*

Recall that $\mathscr{O}^\alpha_{\nabla, \varepsilon}$ is the slice of the $\mathscr{G}^\alpha$ action at ∇ which is described in Section 5. This proposition is proved by applying the Kuranishi technique to the map

$$\psi\colon \ker \delta^\nabla \to L^2_2(\Omega^2_-(\mathfrak{g}_E)^\alpha)$$

given by $\psi(A) = d^\nabla_- A + [A, A]_-$ where $\ker \delta^\nabla \subset L^2_3(\Omega^1(\mathfrak{g}_E)^\alpha)$. See [L; Theorem IV. 2.1] for an excellent exposition.

It follows from Section 6 that

$$\dim H^{1,\alpha}_{\nabla} - \dim H^{2,\alpha}_{\nabla} = R(X,e) \quad \text{if } \nabla \text{ is irreducible,}$$

$$\dim H^{1,\alpha}_{\nabla} - \dim H^{2,\alpha}_{\nabla} = R(X,e) + 1 \quad \text{if } \nabla \text{ is reducible,}$$

provided that the intersection form on X is positive definite and $H^1(X;\mathbf{R}) = 0$.

We now want to deal with the possibility that there are $\mathbf{Z}_\alpha$-invariant self dual connections ∇ on E such that $H^{2,\alpha}_{\nabla} \neq 0$. Assume that we are in a situation where $\mathscr{M}$ and $\mathscr{M}^\alpha$ are compact.

Let $\mathscr{F}^2 = \mathscr{C}^\alpha \times_{\mathscr{G}^\alpha} L^2_2(\Omega^2_-(\mathfrak{g}_E)^\alpha)$. Outside the reducible connections we have a principal fiber bundle $\mathscr{C}^{\alpha *} \to \mathscr{B}^{\alpha *}$ with fiber $\mathscr{G}^\alpha$ and $\mathscr{F}^2 \to \mathscr{B}^{\alpha *}$ is a smooth associated vector bundle. The assignment

$$\nabla \to R^{\nabla}_-$$

is a cross-section of this bundle, and $\mathscr{M}^{\alpha *} \subset \mathscr{B}^{\alpha *}$ is the zero set of this cross-section. As in Donaldson's proof we shall change R^{∇}_- by adding on a compact perturbation term so that the new section $R^{\nabla}_- + \sigma(\nabla)$ cuts across the zero section of $\mathscr{F}^2$ transversely. We shall choose σ to have values in the subbundle

$$\mathscr{F}^2_0 = \mathscr{C}^\alpha \times_{\mathscr{G}^\alpha} L^2_3\big(\Omega^2_-(\mathfrak{G}_E)^\alpha\big).$$

Note that $\mathscr{F}^2_0 \subset \mathscr{F}^2$ is compact on each fiber.

Consider a reducible connection $\nabla_0 \in \mathscr{M}^\alpha$ and let Φ be the function given in Proposition 8.1. We now review the arguments of [L, §IV.4] with minor modifications. Since $H^{1,\alpha}_{\nabla_0}$ and $H^{2,\alpha}_{\nabla_0}$ are even dimensional vector spaces with SO(2) acting standardly, we may identify them with complex vector spaces and identify SO(2) with U(1) acting in the usual way. After a smooth change of coordinates in $\ker \delta^{\nabla_0}$, the map R_- on $\mathscr{O}^\alpha_{\nabla_0,\varepsilon}$ decomposes as

$$(d_-,\Phi)\colon V_1 \times \mathbf{C}^{k+(1/2)(R(X,e)-1)} \to W_1 \times \mathbf{C}^k$$

where $d_- = d\psi_0$ is an isomorphism $V_1 \cong W_1$, and (d_-,Φ) commutes with the action of $\Gamma^\alpha_{\nabla_0} \cong \mathrm{U}(1)$. Let L be a $\mathbf{C}$-linear surjective map

$$L\colon \mathbf{C}^{k+(1/2)(R(X,e)-1)} \to \mathbf{C}^k$$

and let $\rho\colon \mathscr{O}^\alpha_{\nabla_0,\varepsilon} \to \mathbf{R}$ be a smooth cutoff function such that $\rho \equiv 1$ near 0. Define a new section

$$(d_-,(1-\rho)\Phi + \rho L) = (d_-,\Phi) + (0,\rho(L-\Phi)) = R_- + \sigma$$

which is $\mathbf{C}$-linear and surjective in a neighborhood of 0. So the new zero-set modulo U(1) is a cone on $\mathbf{CP}(\tfrac{1}{2}(R(X,e)-1)$ in a neighborhood of 0. Furthermore, the new section

$$\Psi = R_- + \sigma$$

meets the zero section of $\mathscr{F}^2$ transversely near ∇_0. Since transition functions of $\mathscr{F}^2$ are smooth and uniformly bounded, σ remains in $L^2_3(\Omega^2_-(\mathfrak{g}_E)^\alpha)$ after being transformed by a transition function for the bundle $\mathscr{F}^2$; i.e. σ is a section of the compactly embedded bundle $\mathscr{F}^2_0 \subset \mathscr{F}^2$.

Since d_- is an isomorphism, inside $\mathscr{O}^\alpha_{\nabla_0, \varepsilon}$ the new zero set is contained in $0 \times \mathbf{C}^{k+(1/2)(R(X,e)+1)}$ and is just $\{\Psi = 0\} \cap \operatorname{supp} \rho$ which is closed and bounded, hence compact. Outside of $\mathscr{O}^\alpha_{\nabla_0, \varepsilon}$ the zero set is unchanged. Thus $\mathscr{M}'$ is compact. Furthermore since $d\Psi$ and $d(R_-)$ differ by an operator with finite dimensional range, $d\Psi$ is Fredholm and its index is the index of $d(R_-)$, viz. $R(X, e)$.

By Proposition 5.4 there are a finite number (namely $\mu(e)|H_1(D(X); \mathbf{Z})|$) of reducible connections in $\mathscr{M}^\alpha$. Perform the above perturbation corresponding to each of these, and continue to call the resulting compact perturbed moduli space

$$\mathscr{M}' = \{\nabla \in \mathscr{B}^\alpha | \Psi(\nabla) = 0\}.$$

Then $\mathscr{M}'$ arees with $\mathscr{M}^\alpha$ outside neighborhoods of the reducible connections.

Now work in $\mathscr{B}^{\alpha *}$, the complement of the reducible connections. If $\Psi(\nabla) = 0$ then using the Kuranishi argument write Ψ as

$$\ker \delta^\nabla = V_0 \oplus V_1 \overset{(Q,L)}{\to} W_0 \oplus W_1 = L^2_3(\Omega^2_-(\mathfrak{g}_E)^\alpha),$$

where $L = (d\Psi)_\nabla$, $V_0 = \ker L \cong \mathbf{R}^{k+R(X,e)}$, $W_0 = (\operatorname{Im} L)^\perp \cong \mathbf{R}^k$, and $L\colon V_1 \to W_1$ is a Hilbert space isomorphism. Then $(w, 0) \in W_0 \oplus W_1$ is a regular value of Ψ if w is a regular value of the finite dimensional smooth map Q, and by Sard's Theorem regular values of Q are dense in $\mathbf{R}^k$.

Cover $\mathscr{M}'$ with finitely many slices $\{\mathscr{O}^\alpha_{\nabla_i, \varepsilon_i} | i = 1, \ldots, m\}$ each contained in $\mathscr{B}^{\alpha *}$ along with the finitely many open cones on projective spaces which are neighborhoods of the reducible connections. We may suppose that the ∇_i's are smooth connections.

We obtain a family of perturbations

$$\Psi_w = \Psi + \sigma_{w_1} + \cdots + \sigma_{w_m}$$

for each $w = (w_1, \ldots, w_m) \in \mathbf{R}^{k_1} \times \cdots \times \mathbf{R}^{k_m} = \mathbf{R}^N$ where $\sigma_{w_i} = \rho_i \cdot w_i$ for $w_i \in W_0 = \mathbf{R}^{k_i}$ as above and for a cutoff function ρ_i on $\mathscr{O}^\alpha_{\nabla_i, \varepsilon_i}$.

Following Lawson, view Ψ_w as a smooth map

$$\mathscr{B}^\alpha \times B^N(\eta) \to \mathscr{F}^2$$

for $B^N(\eta) = \{w \in \mathbf{R}^N | \|w\| < \eta\}$. For η small this map is transverse to the zero section of $\mathscr{F}^2$ [L; p. 101]. Hence Ψ_w is transverse to the zero section of $\mathscr{F}^2$ for almost all $w \in B^N(\eta)$. Applying the same argument as in the reducible case to each of the m new perturbations we see that the new moduli space is still compact.

THEOREM 8.2. *Under the hypotheses of Theorem* 2.1 *there is a compact perturbation* $\Psi = R_- + \sigma$ *of the self-duality equations on* $\mathscr{C}^\alpha$ *so that the new moduli space* $\mathscr{M}' = \{\nabla \in \mathscr{C}^\alpha | \Psi(\nabla) = 0\}$ *is a compact smooth* $R(X, e)$-*dimensional manifold with* $\mu(e)|H_1(D(X); \mathbf{Z})|$ *singular points such that each has a neighborhood which is the cone on the complex projective space* $\mathbf{CP}(\frac{1}{2}R(X, e) - 1))$. □

It is important to note that by construction the equation $\Psi(\nabla) = 0$ is invariant under $\mathscr{G}^\alpha$.

9. Proof of Theorem 2.1

We now are in a position to prove our main theorem.

THEOREM 2.1. *Let X be a pseudofree orbifold with pseudofree Euler class e. Suppose that*:

(i) *The intersection form on X is positive definite*;

(ii) $i^*(\operatorname{Tor} H^2(D(X); \mathbf{Z}) = 0$;

(iii) $H_1(D(X); \mathbf{Z}_2) = 0$;

(iv) $e^2 \leq 4/\alpha$.

(*If* $e^2 = 4/\alpha$ *assume* $e \not\equiv 0 (\operatorname{mod} 2)$ *and* $H^2(\pi_1(D(X); \mathbf{Z}_2) = 0)$. *If* $R(X, e) > 0$, *then* $\mu(e) \equiv 0 (\operatorname{mod} 2)$.

Proof. Let $E = L_e \oplus \varepsilon$ be the $\mathbf{Z}_\alpha$-equivariant bundle over $M = M(X)$ given in Section 3. By Proposition 5.4 there are exactly $m = \mu(e) \cdot |H_1(D(X); \mathbf{Z})| \equiv \mu(e) \pmod 2$ $\mathbf{Z}_\alpha$-equivariant gauge equivalence classes of reducible $\mathbf{Z}_\alpha$-invariant self-dual connections to E. If $R(X, e) > 0$, Proposition 8.2 yields a compact smooth $R(X, e)$-dimensional manifold $\mathscr{M}'$ with m singular points each having a neighborhood which is the cone on the complex projective space $\mathbf{CP}(\frac{1}{2}(R(X, e) - 1))$. If we remove the interiors of these cones from $\mathscr{M}'$ we then obtain a compact manifold whose boundary consists of m disjoint copies of $\mathbf{CP}(\frac{1}{2}(R(X, e) - 1))$. If $\frac{1}{2}(R(X, e) - 1)$ is even, this implies that m is even (hence $\mu(e) \equiv 0 (\operatorname{mod} 2)$), for an odd number of $\mathbf{CP}(2k)$'s cannot bound a smooth manifold. ($\mathbf{CP}(2k)$ has odd Euler characteristic.)

In any case we shall show that m must be even. Fix a regular point x of the branched cover $M \to M/\mathbf{Z}_\alpha = X$ and consider the reduced equivariant gauge group $\mathscr{G}_0^\alpha = \{g \in \mathscr{G}^\alpha | g_x = \mathrm{id}_x\}$. The normal subgroup $\mathscr{G}_0^\alpha$ acts freely on $\mathscr{C}^\alpha$

and $\mathcal{G}^\alpha/\mathcal{G}_0^\alpha \cong \mathrm{Aut}(E_x) \cong \mathrm{SO}(3)$. The fibration $\pi\colon \mathcal{C}^{\alpha*} \to \mathcal{B}^{\alpha*}$ now factors into two fibrations, $\pi_0\colon \mathcal{C}^{\alpha*} \to \mathcal{C}^{\alpha*}/\mathcal{G}_0^\alpha$, a principal $\mathcal{G}_0^\alpha$-fibration, and $\pi_1\colon \mathcal{C}^{\alpha*}/\mathcal{G}_0^{\alpha*} \to \mathcal{B}^{\alpha*}$, a principal SO(3)-fibration.

Let $\nabla_1, \ldots, \nabla_m \in \mathcal{A}^\alpha$ be representatives of the distinct $\mathbf{Z}_\alpha$-equivariant gauge equivalence classes of reducible connections in $\mathcal{M}^\alpha$. Each ∇_i has a slice $\mathcal{O}^\alpha_{\nabla_i,\varepsilon_i}$ in $\mathcal{C}^\alpha$, and in $\mathcal{O}^\alpha_{\nabla_i,\varepsilon_i}$ there is a complex space $\mathcal{O}_i$ on which $\Gamma^\alpha_{\nabla_i} \cong \mathrm{U}(1)$ acts in the usual manner and such that $\mathcal{O}_i/\mathrm{U}(1)$ is a neighborhood of $[\nabla_i]$ in $\mathcal{M}'$. (See §8.) Let S_i denote the unit sphere in the complex space $\mathcal{O}_i$. If $g \in \mathcal{G}^\alpha$ moves a connection in S_i to another connection in S_i then (since $\mathcal{O}^\alpha_{\nabla_i,\varepsilon_i}$ is actually a slice [L; II.10.13, 14]), $g \in \Gamma^\alpha_{\nabla_i}$. So $g(\mathcal{O}^\alpha_{\nabla_i,\varepsilon_i}) = \mathcal{O}^\alpha_{\nabla_i,\varepsilon_i}$, and since $\mathcal{M}'$ is the moduli space of solutions of the $\mathcal{G}^\alpha$-invariant equation $\Psi(\nabla) = 0$, $g(S_i) = S_i$.

Recall that $\Gamma^\alpha_{\nabla_i}$ acts on $E \cong L_i \oplus \varepsilon$ by acting as SO(2) on L_i and trivially on ε. Thus $\Gamma^\alpha_{\nabla_i} \cap \mathcal{G}_0^\alpha = \{\mathrm{id}\}$. So each $g \in \mathcal{G}_0^\alpha$ moves S_i off itself, and the projection $\pi_0\colon \mathcal{C}^\alpha \to \mathcal{C}^\alpha/\mathcal{G}_0^\alpha$ maps S_i and, in fact, $\mathcal{O}^\alpha_{\nabla_i,\varepsilon_i}$ isomorphically. In $\mathcal{C}^\alpha/\mathcal{G}_0^\alpha$; $g \in \mathrm{SO}(3)$ takes a connection in $\pi_0(S_i)$ to another connection in $\pi_0(S_i)$ if and only if $g \in \Gamma^\alpha_{\nabla_i}$ (as above). Let $\nabla \in S_i$ and consider the SO(3)-orbit $\mathrm{SO}(3)(\pi_0(\nabla))$. The intersection $\mathrm{SO}(3)(\pi_0(\nabla)) \cap \pi_0(S_i) = \Gamma^\alpha_{\nabla_i}(\pi_0(\nabla))$ is a circle. Hence the SO(3) bundle

$$\pi_1^{-1}(\pi(S_i)) \to \pi(S_i) \cong \mathbf{CP}\left(\frac{1}{2}(R(X,e) - 1)\right)$$

reduces to an S^1-bundle

$$S_i \cong \pi_0(S_i) \to \pi(S_i) \cong \mathbf{CP}\left(\frac{1}{2}(R(X,e) - 1)\right).$$

This is just the Hopf bundle. Hence this SO(3)-bundle over $\pi(S_i)$ has $w_2 = w_2$ (Hopf bundle) $\neq 0 \in H^2(\pi(S_i); \mathbf{Z}_2) \cong H^2(\mathbf{CP}(\frac{1}{2}(R(X,e) - 1)); \mathbf{Z}_2)$; so $w_2^{(1/2)(R(X,e)-1)} \neq 0$. Thus an odd number of these SO(3)-bundles cannot bound a smooth SO(3)-bundle. Hence, by removing the interiors of the cones on the projective spaces in $\mathcal{M}'$, we see that m must be even. □

Remark 9.1. If the conjecture stated in [FS2] is true, then Theorem 2.1 holds without hypothesis (iv).

We shall now restate our theorem avoiding some of the technical hypotheses of (2.1). For a pseudofree orbifold X with pseudofree Euler class e, consider as above the $\mathbf{Z}_\alpha$-invariant SO(3) vector bundle $E = L_e \oplus \varepsilon$ over M, an α-fold branched cover desingularizing X. Define $\rho(e)$ to be the number, up to orientation, of $\mathbf{Z}_\alpha$-invariant reductions of E.

THEOREM 9.2. *Let X be a positive definite pseudofree orbifold with $H_1(D(X);\mathbf{Z}_2) = 0$, and let e be a pseudofree Euler class with $e^2 \leq 4/\alpha$. (If $e^2 = 4/\alpha$ also assume $e \not\equiv 0 \pmod 2$ and $H^2(\pi_1(DX),\mathbf{Z}_2) = 0$). If $R(X,e) > 0$ then $\rho(e) \equiv 0 \pmod 2$.* □

10. Applications

In this section we will give some applications of Theorem 2.1 mentioned in the introduction. Throughout this section we shall use $\Sigma(a_1,\dots a_n)$ to denote the Seifert integral homology sphere with exceptional fibers of (pairwise relatively prime) order $a_1,\dots,a_n$, and its orientation as the link of an algebraic singularity. In Section 1 we defined the invariant

$$R(a_1,\dots,a_n) = \frac{2}{\alpha} - 3 + n + \sum_{i=1}^{n} \frac{2}{a_i} \sum_{k=1}^{a_i-1} \cot(\pi\alpha k/a_i^2)\cot(\pi k/a_i)\sin^2(\pi k/a_i)$$

where $\alpha = a_1 \dots a_n$. We showed in Section 2 that the following theorem is a corollary of Theorem 2.1.

THEOREM 10.1. *If $R(a_1,\dots a_n) > 0$ then $\Sigma(a_1,\dots,a_n)$ does not bound an oriented smooth 4-manifold W whose intersection pairing is positive definite and whose first homology $H_1(W;\mathbf{Z})$ has no 2-torsion.* □

THEOREM 10.2. *If $R(a_1,\dots,a_n) > 0$ then $\Sigma(a_1,\dots,a_n)$ is not oriented cobordant to any $-\bigcup_{j=1}^{r}\Sigma(b_{j,1},\dots,b_{j,m(j)})$ by an oriented positive definite cobordism W whose first homology $H_1(W;\mathbf{Z})$ has no 2-torsion.*

Proof. If such a W exists, cap off with the positive definite simply-connected 4-manifolds that the $-\Sigma(b_{j,1},\dots,b_{j,m(j)})$ bound to obtain a manifold contradicting Theorem 10.1. □

The terms $\delta(p;a,b) = (2/p)\Sigma\cot(\pi ak/p)\cot(\pi bk/p)\sin^2(\pi k/p)$ which occur in the formula for $R(a_1,\dots,a_n)$ are very closely related to the Casson-Gordon invariant for slice knots [CG1]. The following computational device explained in [CG1] is quite useful. Let $\Delta(x,y)$ be the triangle whose vertices have coordinates $(0,0)$, $(x,0)$, and (x,y). Let $\operatorname{int}\Delta(x,y)$ be the number of integer lattice points in $\Delta(x,y)$, where boundary points count $1/2$ ($(0,0)$ is not counted) and other vertices count $1/4$.

LEMMA 10.3 [CG1].

$$\delta(p;a,b) = 4\{\operatorname{int}\Delta(b^*,a^*bb^*/p)\text{-area }\Delta(b^*,a^*bb^*/p)\}$$

where $aa^* \equiv 1 \pmod p$ and $bb^* = 1 \pmod p$. □

Corollary 10.4. $\delta(p; 1, 1) = (p - 2)/p = - \delta(p; p - 1, 1)$. □

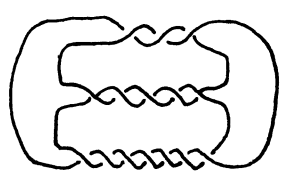
By (10.3) and (10.4) it is easy to check that $R(2, 3, 6k - 1) = +1$ for $k = 1, 2, \ldots$. Hence we have the following result (cf. [K; Problem 4.2]).

Proposition 10.5. *The μ-invariant zero Brieskorn homology spheres $\Sigma(2, 3, 12k - 1)$ do not bound $\mathbf{Z}_2$-acyclic 4-manifolds.* □

Also, (10.2) shows that $\Sigma(2, 3, 5)$ has infinite order in θ_3^H, the integral homology cobordism group of integral homology 3-spheres. Furthermore, the Brieskorn sphere $\Sigma(2, 3, 7)$ has nonzero μ-invariant and bounds both positive and negative definite simply connected 4-manifolds (namely ± 1 surgery on the figure 8 knot). Let $\mathbf{Z}_{2k}$ denote the subgroup of θ_3^H generated by $\Sigma(2, 3, 7)$. If any multiple $n\Sigma(2, 3, 5)$ is integral homology cobordant to a multiple $m\Sigma(2, 3, 7)$ where $m, n \in \mathbf{Z}$, then by gluing the appropriate definite manifolds that $\Sigma(2, 3, 7)$ bounds we obtain a positive definite manifold that $n\Sigma(2, 3, 5)$ bounds. This contradicts (10.2). Thus we have:

Theorem 10.6. *The integral homology cobordism group θ_3^H is infinite, and in fact contains $\mathbf{Z} \oplus \mathbf{Z}_{2k}$ for some $k = 0, 1, 2, \ldots$.* □

Next we consider the pretzel knots $K(p, q, r)$:



(where $K(-3, 5, 7)$ is pictured above). The Alexander polynomial of $K(p, q, r)$ is 1 when $pq + pr + qr = -1$ and p, q, r are odd. Casson pointed this out in Kirby's problem list [K; Problem 1.37] and asked whether these $K(p, q, r)$ are slice knots. He also pointed out that the 2-fold branched covers of these knots are $\Sigma(|p|, |q|, |r|)$ which must bound $\mathbf{Z}_2$-acyclic 4-manifolds if the knots $K(p, q, r)$ are slice. However using (10.4) one computes that $R(|p|, |q|, |r|) = +1$ if $pq + pr + qr = -1$ and p, q, r have absolute value > 1. Thus:

Theorem 10.7. *Let p, q, r be odd integers of absolute value greater than 1. If $pq + pr + qr = -1$ then $K(p, q, r)$ is an Alexander polynomial 1 knot that is not slice.* □

As a consequence of (10.5) and a theorem of Galewski-Stern [GS], we have:

THEOREM 10.8. *For each $n \geq 5$, there exist closed topological n-manifolds with infinitely many nonconcordant simplicial triangulations.* □

As a final application, we point out how K. Kuga's theorem [KK] concerning the nonrepresentability of homology classes in $S^2 \times S^2$ by spheres can be obtained from our invariant R.

THEOREM 10.9. *Let M be a closed smooth 4-manifold having the integral homology of $S^2 \times S^2$. Let x, y be the standard generators of $H_2(M; \mathbf{Z})$. If $(p, q) = 1$, $p \neq \pm 1$ and $q \neq \pm 1$, then the homology class $px + qy$ cannot be represented by a smoothly embedded 2-sphere.*

Proof. Consider $u = px + qy$ with $(p, q) = 1$. Then set $v = ax + by$ where

$$\begin{vmatrix} p & q \\ a & b \end{vmatrix} = 1.$$

Then u, v generate $H_2(M; \mathbf{Z})$. Suppose that u is represented by a smoothly embedded 2-sphere; let N be its tubular neighborhood. Since $u^2 = 2pq$, $\partial N = L(2pq, 1)$.

Set $Y = M - \operatorname{int} N$. Then $\partial Y = L(2pq, -1)$. We easily compute $H_1(Y; \mathbf{Z}) \cong H_3(Y; \mathbf{Z}) \cong 0$ and $H_2(Y; \mathbf{Z}) \cong \mathbf{Z}$. Furthermore, $H_2(Y, \partial Y; \mathbf{Z}) \cong \mathbf{Z}$ is generated by the class w represented by $F_v \cap Y$ where $[F_v] = -v$.

Let $e \in H^2(Y; \mathbf{Z})$ be the Poincaré dual of w. If

$$\mathbf{Z} = H^2(Y; \mathbf{Z}) \xrightarrow{i^*} H^2(\partial Y; \mathbf{Z}) = \mathbf{Z}_{|2pq|},$$

then $i^*(e)$ is the Poincaré dual of ∂w. However, since $v \cdot u = bp + aq$, $\partial w \in H_1(L(2pq, -1); \mathbf{Z})$ consists of $bp + aq$ times the ∂D^2-fiber of the appropriate disk bundle over the S^2 representing u. Since $bp - aq = 1$, it follows that $4pqab - (bp + aq)^2 = -1$; so $(bp + aq)^2 \equiv 1 \pmod{2pq}$; hence $(bp + aq, 2pq) = 1$. This means that ∂w is a unit in $H_1(L(2pq, -1); \mathbf{Z})$; i.e. $i^*(e)$ is a unit. Thus we can form the pseudofree orbifold $Y \cup c(\partial Y) = X$ with pseudofree Euler class e. With an appropriate orientation X has positive definite intersection form with $H_1(X, \mathbf{Z}) = 0$, $e^2 = (1/|2pq|)$ and $\mu(e) = 1$. To conclude the proof we show that $R(X, e) > 0$ if and only if $p \neq \pm 1$ and $q \neq \pm 1$.

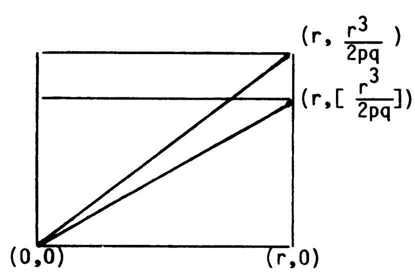
We have

$$R(X,e) = \frac{2}{|2pq|} - 2 + \frac{2}{|2pq|}\sum_{k=1}^{|2pq|-1} \cot^2\left(\frac{\pi(bp+aq)k}{|2pq|}\right)\sin^2\frac{\pi k}{|2pq|}$$
$$= \frac{1}{|pq|} - 2 + \delta(|2pq|; bp+aq, bp+aq).$$

By changing the orientation of M and of x or y, we may assume that $p \geq 0$, $q \geq 0$. Then, after perhaps interchanging the roles of x and y we may assume $0 \leq a < p$, $0 \leq b < q$. Further if we assume that $p \geq 2$, $q \geq 2$ then $1 \leq a < p$, $1 \leq b < q$. Let $r = bp + aq$. Since $r^* = r$, Lemma 10.3 says

$$\delta(2pq; r, r) = 4\left\{\operatorname{int}\Delta\left(r, \frac{r^3}{2pq}\right) - \operatorname{area}\Delta\left(r, \frac{r^3}{2pq}\right)\right\}.$$

If β is a rational number, let $[\beta]$ denote the largest integer $\leq \beta$. To get an estimate on $\delta(2pq; r, r)$, let Γ be the smaller rectangle in the figure



Then $\operatorname{int}\Delta(r, (r^3/2pq)) \geq A + (1/2)B + (1/2)D$ where A is the contribution from the lattice points on $\operatorname{bdry}\Delta(r, (r^3/2pq)) \cap \operatorname{bdry}\Gamma$, B is the number of interior lattice points of Γ, and D is the number of interior lattice points on the diagonal of Γ. Now

$$A = \frac{1}{4} + \frac{1}{2}\left(r - 1 + \left[\frac{r^3}{2pq}\right]\right),$$

$$B = (r+1)\left(\left[\frac{r^3}{2pq}\right] + 1\right) - 2\left(r + \left[\frac{r^3}{2pq}\right]\right) = (r-1)\left[\frac{r^3}{2pq}\right] - r + 1.$$

However

$$r^2 = (bp + aq)^2 = 4pqab + 1 \quad \text{and} \quad r^3 = 4p^2qab^2 + 4pq^2a^2b + r,$$

and $r = bp + aq < 2pq$. Hence $[r^3/2pq] = 2pab^2 + 2qa^2b = 2abr$; then

$D = \#\{x | 0 < x < r \text{ and } r | [r^3/2pq]x\} = r - 1$. Thus

$$\begin{aligned}\delta(2pq; r, r) &\geq 4\left\{\frac{1}{4} + \frac{1}{2}\left(r - 1 + \left[\frac{r^3}{2pq}\right]\right) + \frac{1}{2}\left((r-1)\left[\frac{r^3}{2pq}\right] - r + 1\right)\right. \\ &\qquad \left. + \frac{1}{2}(r-1) - \frac{r^4}{4pq}\right\} \\ &= 2r - 1 + 2r\left(\left[\frac{r^3}{2pq}\right] - \frac{r^3}{2pq}\right) = 2r - 1 - \frac{r^2}{pq} \\ &= 2r - 1 - 4ab - \frac{1}{pq}.\end{aligned}$$

Thus

$$\begin{aligned}R(X, e) &\geq \frac{1}{pq} - 2 + 2r - 1 - 4ab - \frac{1}{pq} = \left(2\sqrt{4pqab + 1} - 4ab\right) - 3 \\ &\geq 4\left(\sqrt{pqab} - ab\right) - 3.\end{aligned}$$

When $p \geq 2$, $q \geq 2$, $1 \leq a \leq p - 1$, $1 \leq b \leq q - 1$; then

$$\sqrt{pqab} \geq \sqrt{(a+1)(b+1)ab} \geq ab + 1; \quad \text{so } R(X, e) \geq 1. \qquad \square$$

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(Received August 29, 1984)